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EXTRINSIC RADIUS PINCHING IN SPACE FORMS OF NONNEGATIVE SECTIONAL CURAVTURE

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Abstract. We first give new estimates for the extrinsic radius of compact hypersurfaces of the Euclidean space $\mathbb{R}^{n+1}$ and the open hemisphere in terms of high order mean curvatures. Then we prove pinching results corresponding to theses estimates. We show that under a suitable pinching condition, M is diffeomorphic and almost isometric to an n-dimensional sphere.

Key words: Extrinsic radius, pinching, hypersurfaces, r-th mean curvature
Mathematics Subject Classification: 53A07, 53C20, 53C21

1. Introduction

Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into the $(n+1)$-dimensional simply connected space-form $(\mathbb{M}^{n+1}(\delta), g_{can})$ of sectional curvature δ with $n \geqslant 2$. The extrinsic radius of (M, g) is the real number defined as follows

$$R = R(M) = \inf \left\{ r > 0 \middle| \ \exists x \in \mathbb{M}^{n+1}(\delta) \text{ s.t. } \phi(M) \subset B(x, r) \right\},$$

where $B(x, r)$ $\big($*resp.* $\overline{B}(x, r)$ and $S(x, r)\big)$ is the open ball $\big($*resp.* the closed ball and the sphere$\big)$ of center x and radius r in $\mathbb{M}^{n+1}(\delta)$. An immediate consequence of the above definition is that there exists $p_0 \in \mathbb{M}^{n+1}(\delta)$ such that

$$\phi(M) \subset \overline{B}(p_0, R) \quad \text{and} \quad \phi(M) \cap S(p_0, R) \neq \emptyset.$$

Moreover, the extrinsic radius is bounded from below in terms of the mean curvature. More precisely, we have the following estimate obtained by comparing the shape operators for hypersurfaces that have a contact point and where one is outside the other (see [2, 4, 14] for

details)

$$t_\delta(R) \geqslant \frac{1}{||H||_\infty}, \tag{1}$$

where $t_\delta(s) = \begin{cases} \frac{1}{\sqrt{\delta}}\tan(\sqrt{\delta}s) & \text{if } \delta > 0 \\ s & \text{if } \delta = 0 \\ \frac{1}{\sqrt{-\delta}}\tanh(\sqrt{-\delta}s) & \text{if } \delta < 0 \end{cases}$

and H is the mean curvature of the immersion. Note that if $\delta > 0$, the image $\phi(M)$ is assumed to be contained in a ball of radius less than $\frac{\pi}{2\sqrt{\delta}}$ which is equivalent to the fact that M lies in the open hemisphere $\mathbb{S}^{n+1}_+(\delta)$.

Since the equality case is characterized by the geodesic spheres of radius R, the question of the associated pinching problem was asked, *i.e.*, what happens in the case of almost equality.

Many pinching results are known for intrinsic geometric invariants defined on Riemannian manifold with positive Ricci curvature, as the intrinsic diameter, the volume, the radius or the first eigenvalue of the Laplacian ([17, 19, 9, 8, 10, 23, 3]).

Nethertheless, few results are known about pinching problems in the extrinsic case. In [7], B. Colbois and J.F. Grosjean give a first result about the first eigenvalue $\lambda_1(M)$ of the Laplacian. More precisely, they proved that there exists a constant C depending on n and the L^∞-norm of the second fundamental form such that if

$$\frac{n}{V(M)^{1/p}}||H||^2_{2p} - C < \lambda_1(M)$$

then M is diffeomorphic to an n-dimensional sphere.

We gave a second extrinsic pinching result in [21]. We proved the pinching result associated with Inequality (1), that is, there exists a constant C depending on the L^∞-norm of the second fundamental form so that if

$$t_\delta(R) < \frac{1}{||H||_\infty} + C,$$

then M is diffeomorphic and almost isometric to a geodesic sphere of radius R.

In this paper, we extend the results in [21] to high order mean curvatures H_k, which are the natural generalization of the mean curvature H. They are defined to be the k-th elementary symmetric polynomial in the principal curvature of M (see Section 2). For instance, for hypersurfaces of $\mathbb{R}^{n+1}$, up to a multiplicative constant, H_1 is the mean curvature, H_2 is the scalar curvature and H_n is the Gauss-Kronecker curvature. These curvatures give, in general, better inequalities than those involving the mean curvature H. Indeed, R.C. Reilly gave in [20] a sharper upper bound for the first eigenvalue of the Laplacian for hypersurfaces of $\mathbb{R}^{n+1}$. The analogue of this upper bound was proved

by J.F. Grosjean for hypersurfaces of $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ (see [12]).

Regarding the extrinsic radius, T. Vlachos improved Inequality (1) in terms of H_k, *i.e.*,

$$t_\delta(R)^k \geqslant \frac{1}{||H_k||_\infty}, \tag{2}$$

with equality only for the geodesic spheres of radius R (see [22]). This is an improvement of Inequality (1) since we have the following sequence of inequalities

$$H_k^{1/k} \leqslant \cdots \leqslant H_2^{1/2} \leqslant H.$$

A first question is to know if Inequality (2) can be improved, replacing the L^∞-norm by an L^p-norm, as for (1) (see [21] for details).

We prove, in Section 3, that such L^p-lower bounds are true for $\delta \geqslant 0$ and any $k \in \{1, \cdots, n\}$ if H_k is a positive function. Moreover, the equality is characterized by geodesic hyperspheres. Then, another natural question, is to know if there exists a pinching result for these inequalities. That is, is there a constant C such that if the pinching condition

$$t_\delta(R)^k < \frac{1}{||H_k||_\infty} + C \tag{P_C}$$

holds, then M is closed to a geodesic hypersphere?

In what follows, we denote by $\mathcal{M}(n, \delta, R, k)$ *the family of all compact, connected and oriented n-dimensional Riemannian manifolds without boundary isometrically immersed into $\mathbb{R}^{n+1}$ if $\delta = 0$ or into the open hemisphere of $\mathbb{S}^{n+1}(\delta)$ if $\delta > 0$, of extrinsic radius R, volume equal to 1 and positive H_k*. We prove the following

Theorem 1. *Let $(M^n, g) \in \mathcal{M}(n, \delta, R, k)$, $\delta \geqslant 0$ and p_0 be the center of the ball of radius R containing M. Then for any $\varepsilon > 0$, there exists a positive constant C_ε depending only on n, δ, the L^∞-norm of the mean curvature and the L^{2p}-norm of the k-th mean curvature H_k such that if*

$$t_\delta(R)^k < \frac{1}{||H_k||_{2p}} + C_\varepsilon \tag{P_{C_ε}}$$

then

i) $\phi(M) \subset B\big(p_0, R\big) \setminus B\big(p_0, R - \varepsilon\big)$.
ii) $\forall x \in S\big(p_0, R\big), \quad B(x, \varepsilon) \cap \phi(M) \neq \emptyset$.

Remarks

1) We will see in the proof that $C_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow \infty$ or $\varepsilon \longrightarrow 0$.
2) An immediate consequence of *i)* and *ii)* of Theorem 1 is that the Haussdorff-distance between M and $S(p_0, R)$ satisfies

$$d_H\left(M, S(p_0, R)\right) \leqslant \varepsilon,$$

If the pinching condition is strong enough, with a control on the L^∞-norm of the second fundamental form B instead of the L^∞-norm of the mean curvature, we obtain that M is diffeomorphic and almost isometric to a geodesic sphere in the following sense:

Theorem 2. *Let $(M^n, g) \in \mathcal{M}(n, \delta, R, k)$, $\delta \geqslant 0$ and p_0 be the center of the ball of radius R containing M. Then there exists a constant C depending only on n, δ, the L^∞-norm of the second fundamental form and the L^{2p}-norm of the k-th mean curvature H_k such that if*

$$(P_C) \qquad t_\delta(R)^k < \frac{1}{||H_k||_{2p}} + C$$

then M is diffeomorphic to $S(p_0, R)$.

More precisely, there exists a diffeomorphism F from M into the geodesic hypersphere $S(p_0, R)$ which is a quasi-isometry. That is, for all $\theta \in]0, 1[$, there exist a constant C_θ depending on n, δ, $||B||_\infty$, $||H_k||_{2p}$ and θ such that the pinching condition (P_C) implies

$$\big| |dF_x(u)|^2 - 1 \big| \leqslant \theta,$$

for all unit vector $u \in T_xM$.

Remarks.

1) It is obvious that the pinching condition

$$(\widetilde{P_C}) \qquad t_\delta(R)^k < \frac{1}{||H_k||_\infty} + C$$

implies (P_C). So we deduce immediately from Theorems 1 and 2 the same results with the second pinching condition $(\widetilde{P_C})$.

2) In general, the constants C_ε and C_θ of Theorems 1 and 2 depend on $||H_k||_{2p}$. In fact, the constant C_ε of Theorem 1 does not depend on $||H_k||_{2p}$ when
   (a) $\delta > 0$
   (b) $\delta = 0$ and $k \geqslant 4$,
   (c) $\delta = 0$ and $p \geqslant \frac{n}{2k}$,
   and the constant C_θ of Theorem 2 does not depend on $||H_k||_{2p}$ when $\delta = 0$ and $p \geqslant \frac{n}{2k}$.
3) Our approach does not work in the case $\delta < 0$. As we will see, it is due to the fact that the function c_δ is increasing if $\delta < 0$.
4) By homothety, we can deduce the same results for manifolds with arbitrary volume. Indeed, $(M, g') \in \mathcal{M}(n, \delta', R')$, with $g' = V(M)^{-2/n} g$, $\delta' = V(M)^{2/n}\delta$ and $R' = V(M)^{-1/n} R$.

These results are of special interest for $k = 2$. Indeed, in that case, up to a constant, H_2 is the scalar curvature. So we obtain a relation between the extrinsic radius and the scalar curvature, which is an intrinsic geometric invariant. In particular, we have the following corollary

Corollary 1. *Let $(M^n, g) \in \mathcal{M}(n, 0, R)$ and p_0 be the center of the ball of radius R containing M. We assume that (M^n, g) has positive scalar curvature. Then for any $p \geqslant \frac{n}{4}$, there exists a constant C depending only on n and $||B||_\infty$ such that if*

$$R^2 < \frac{n(n-1)}{||\mathrm{Scal}||_{2p}} + C$$

then M is diffeomorphic and almost isometric to $S(p_0, R)$ in the sense of Theorem 2

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2. Preliminraries

First, let's introduce the following functions:

$$s_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}\, t) & \text{if } \delta > 0 \\ t & \text{if } \delta = 0 \end{cases}$$

and

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}\, t) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \end{cases}$$

Throughout this paper, we consider a manifold $(M^n, g) \in \mathcal{M}(n, \delta, R, k)$, $\delta \geqslant 0$. For simplicity, we assume that $\delta = 0$ or 1. By homothety, we can deduce the results for any $\delta \geqslant 0$. Let ν be the outward unit vector field. The second fundamental form B of the immersion is defined by

$$B(X, Y) = \left\langle \overline{\nabla}_X \nu, Y \right\rangle,$$

where $< \cdot, \cdot >$ and $\overline{\nabla}$ are respectively the Riemannian metric and the Riemannian connection of $\mathbb{M}^{n+1}(\delta)$. The mean curvature of the immersion is

$$H = \frac{1}{n} \mathrm{tr}\,(B).$$

Now let's recall the definition of the high order mean curvature H_k. Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame of $T_x M$. For all $k \in \{1, \cdots, n\}$, the k-th mean curvature of the immersion is

$$H_k = \begin{pmatrix} n \\ k \end{pmatrix}^{-1} \sum_{\substack{1 \leqslant i_1, \cdots, i_k \leqslant n \\ 1 \leqslant j_1, \cdots, j_k \leqslant n}} \epsilon \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix} B_{i_1 j_1} \cdots B_{i_k j_k},$$

where the B_{ij} are the coefficients of the real second fundamental form. The symbols $\epsilon \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}$ are the usual premutation symbols which are zero if the sets $\{i_1, \cdots, i_k\}$ and $\{j_1, \cdots, j_k\}$ are different or if there

exist distinct p and q with $i_p = i_q$. For all other cases, $\epsilon \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}$ is the signature of the permutation $\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}$. By convention, we set $H_0 = 1$ et $H_{n+1} = 0$.

For $k \in \{1, \cdots, n\}$, the symmetric $(1,1)$-tensor associated to H_k is

$$T_k = \frac{1}{k!} \sum_{\substack{1 \leqslant i, i_1, \cdots, i_k \leqslant n \\ 1 \leqslant j, j_1, \cdots, j_k \leqslant n}} \epsilon \begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix} B_{i_1 j_1} \cdots B_{i_k j_k} e_i^* \otimes e_j^*.$$

This tensor is divergence free, symmetric $(1,1)$-tensor. For any symmetric $(1,1)$-tensor, we define the following function

$$H_T(x) = \sum_{i=1}^{n} B_x(Te_i, e_i), \tag{3}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of T_xM. Then, we have the following relations

Lemma 2.1. *For $k \in \{1, \cdots, n\}$, we have:*

(1) $\mathrm{tr}\,(T_k) = m(k) H_k$,
(2) $H_{T_k} = m(k) H_{k+1}$,

where $m(k) = (n-k) \begin{pmatrix} n \\ k \end{pmatrix}$ *and* H_{T_k} *is given by (3).*

Let $p_0 \in \mathbb{M}^{n+1}(\delta)$. We denote by $r(x) = d(p_0, x)$ the geodesic distance from p_0 to x on $(\mathbb{M}^{n+1}(\delta), g_{can})$. We denote by ∇ (*resp.* $\overline{\nabla}$) the gradient associated with (M, g) (*resp.* $(\mathbb{M}^{n+1}(\delta), g_{can})$). Let $Z := s_\delta(r)\overline{\nabla} r$ be the postion vector field and $Z^T = s_\delta(r)\nabla r$ its tangential projection on the tangent bundle of $\phi(M)$. We have the following lemma (see [15] or [13] for a proof):

Lemma 2.2. *Let T be a positive definite $(1,1)$-tensor with* $\mathrm{div}\,(T) = 0$. *Then we have*

$$\mathrm{div}\,(TZ^T) \geqslant c_\delta(r)\mathrm{tr}\,(T) - \langle Z, H_T \rangle .$$

If $\delta = 0$ and T is the identity, then equality holds.

Remark. Note that, by integration, in the case $\delta = 0$ and $T = \mathrm{Id}$, this lemma is nothing else but the Hsiung-Minkowski formula (see [16]).

Finally, we recall the following lemma (see [7] or [21])

Lemma 2.3. *Let (M^n, g) be a compact, connected, oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ*

into $\mathbb{R}^{n+1}$ *or an open hemisphere of* $\mathbb{S}^{n+1}(\delta)$. *Let* ξ *be a nonnegative continuous function on* M *such that* ξ^k *is smooth for* $k \geqslant 2$. *Let* $0 \leqslant l < m \leqslant 2$ *such that*

$$\frac{1}{2}\xi^{2k-2}\Delta\xi^2 \leqslant \operatorname{div}\omega + (\alpha_1 + k\alpha_2)\xi^{2k-l} + (\beta_1 + k\beta_2)\xi^{2k-m},$$

where ω *is a 1-form and* α_1, α_2, β_1 *and* β_2 *some nonnegative constants. Then for all* $\eta > 0$, *there exists a constant* L *depending only on* α_1, α_2, β_1, β_2, $||H||_\infty$ *and* η *such that if* $||\xi||_\infty > \eta$, *then*

$$||\xi||_\infty \leqslant L||\xi||_2.$$

Moreover, L *is bounded when* $\eta \longrightarrow \infty$ *and if* $\beta_1 > 0$, $L \longrightarrow \infty$ *when* $||H||_\infty \longrightarrow \infty$ *or* $\eta \longrightarrow 0$.

3. New estimates for the extrinsic radius

In this section, we give new lower bounds for the extrinsic radius of hypersurfaces of the Euclidean space and the open hemisphere $\mathbb{S}^{n+1}_+(\delta)$. Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ in $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}_+(\delta)$. An immediate consequence of Lemma 2.2 is the following

Proposition 3.1. *Let* $k \in \{1, \cdots, n\}$, *if* H_k *is positive, then for all* $j \in \{1, \cdots, k\}$, *the function* H_j *is positive and*

$$\int_M H_{j-1}\, c_\delta(r) dv_g \leqslant \int_M H_j\, s_\delta(r) dv_g.$$

Proof: By Lemma 2.2, we have

$$\begin{aligned}\int_M c_\delta(r)\,\mathrm{tr}\,(T) dv_g &\leqslant \int_M \big|\,\langle Z, H_T\rangle\,\big| dv_g \\ &\leqslant \int_M |H_T|\, s_\delta(r) dv_g\end{aligned}$$

If $T = T_{j-1}$ is the $(1,1)$-tensor associated with H_{j-1}, we have

$$\int_M c_\delta(r)\, H_{j-1} dv_g \leqslant \int_M |H_j|\, s_\delta(r) dv_g$$

Moreover, Barbosa and Colares proved in [5] that if H_k is positive, then for all $j \in \{1, \cdots, k\}$, H_j is positive. So for all $j \in \{1, \cdots, k\}$, we have

$$\int_M H_{j-1}\, c_\delta(r) dv_g \leqslant \int_M H_j\, s_\delta(r) dv_g.$$

□

From this proposition, we deduce the following estimates

Theorem 3. *Let $(M^n, g) \in \mathcal{M}(n, \delta, R, k)$ (we do not assume that $V(M) = 1$). Then for all $p \geqslant 1$, we have*

$$t_\delta(R)^k \geqslant \frac{V(M)^{1/p}}{||H_k||_p}. \tag{4}$$

Equality occurs if and only if M is a geodesic hypersphere.

Remarks.

i) These lower bounds improve Inequalities (1) and (2).

ii) In the case $k = 2$, this inequality 4 translates to

$$R^2 \geqslant \frac{n(n-1)}{||\mathrm{Scal}\,||_1}, \text{ if } \delta = 0 \tag{5}$$

$$\tan^2(R) \geqslant \frac{n(n-1)}{||\mathrm{Scal}\, - n(n-1)||_1}, \text{ if } \delta = 1. \tag{6}$$

Inequaliy (5) was proved by S. Deshmukh ([11]). In the spherical case, our inequality improves the following one due to L.J Alias ([1])

$$\sin^2(R) \geqslant \frac{n(n-1)}{||\mathrm{Scal}\,||_1}. \tag{7}$$

Indeed, Alias proved Inequality (7) with the assumption that $\mathrm{Ric}(M) \geqslant (n+2)(n-1)$. With such an assumption, Inequalities (7) and (6) are exactly the same, but Inequality (6) is valid without any assumption on the Ricci curvature of M.

Proof: We use Proposition 3.1 and the fact that the functions s_δ and c_δ are respectively increasing and decreasing.

$$\begin{aligned}
c_\delta(R)^k V(M) &\leqslant c_\delta(R)^{k-1} \int_M c_\delta(r) dv_g \\
&\leqslant c_\delta(R)^{k-1} \int_M H\, s_\delta(r) dv_g \\
&\leqslant c_\delta(R)^{k-2} s_\delta(R) \int_M H\, c_\delta(r) dv_g \\
&\leqslant c_\delta(R)^{k-2} s_\delta(R) \int_M H_2\, s_\delta(r) dv_g \\
&\leqslant \cdots \\
&\leqslant s_\delta(R)^k \int_M H_k dv_g.
\end{aligned}$$

So we have

$$t_\delta(R)^k \geqslant \frac{V(M)}{||H_k||_1},$$

and equality occurs if and only if $r = R$ for all $x \in M$. Finally, the Hölder inequality gives the result with the L^p-norm, for all $p \geqslant 1$. □

4. An L^2-approach to pinching

Let $(M^n, g) \in \mathcal{M}(n, \delta, R, k)$. A first step in the proof of the pinching restults is to prove that the pinching condition

$$(P_C) \qquad t_\delta(R)^k < \frac{1}{||H_k||_{2p}} + C$$

implies that M is closed to a geodesic hypersphere in an L^2-sense. For this, let's introduce the following functions:

$$\begin{cases} \varphi(x) = s_\delta^2(R) - s_\delta^2(r), \\ \psi(x) = c_\delta(r)|Z^T|. \end{cases}$$

In what follows, we assume that the pinching constant satisfies $C < 1$. We prove the following lemma

Lemma 4.1. *The pinching condition (P_C) with $C < 1$ implies*

$$||\varphi||_2^2 \leqslant A_1 C,$$

where A_1 is a positive explicit constant depending only on the dimension n, δ and $||H_k||_{2p}$. Moreover in certain cases, the dependence on $||H_k||_{2p}$ can be replaced by a dependence on $||H||_\infty$, precisely when

(1) $\delta > 0$,
(2) $\delta = 0$ and $k \geqslant 4$,
(3) $\delta = 0$ and $p \geqslant \frac{n}{2k}$.

Proof: Since s_δ and c_δ are respectively increasing and decreasing, we have

$$\begin{aligned} ||\varphi||_2^2 &\leqslant s_\delta^2(R) \int_M \left(s_\delta^2(R) - s_\delta^2(r)\right) \\ &\leqslant s_\delta^2(R) \left[t_\delta^2(R) \left(\int_M c_\delta(r) \right)^2 - \int_M s_\delta^2(r) \right] \end{aligned}$$

By the Hölder inequality, we get

$$||\varphi||_2^2 \leqslant s_\delta^2(R) \left[t_\delta^2(R) \left(\int_M c_\delta(r) \right)^2 - \frac{1}{||H_k||_{2p}^2} \left(\int_M H_k s_\delta(r) \right)^2 \right]$$

Now, using Proposition 3.1, we have

$$\begin{aligned} ||\varphi||_2^2 &\leqslant s_\delta^2(R) \left[t_\delta^2(R) \left(\int_M c_\delta(r) \right)^2 - \frac{1}{||H_k||_{2p}^2} \left(\int_M H_{k-1} c_\delta(r) \right)^2 \right] \\ &\leqslant s_\delta^2(R) \left[t_\delta^2(R) \left(\int_M c_\delta(r) \right)^2 - \frac{1}{||H_k||_{2p}^2 t_\delta^2(R)} \left(\int_M H_{k-1} s_\delta(r) \right)^2 \right] \end{aligned}$$

Proposition 3.1 applied $(k-2)$ more times, yields

$$\begin{aligned}||\varphi||_2^2 &\leqslant s_\delta^2(R)\left[t_\delta^2(R)\left(\int_M c_\delta(r)\right)^2 - \frac{1}{||H_k||_{2p}^2 t_\delta^{2k-2}(R)}\left(\int_M c_\delta(r)\right)^2\right]\\ &\leqslant \frac{s_\delta^2(R)}{t_\delta^{2k-2}(R)}\left(\int_M c_\delta(r)\right)^2\left[t_\delta^{2k}(R) - \frac{1}{||H_k||_{2p}^2}\right]\end{aligned}$$

Since we assume (P_C) is true with $C<1$, if $\delta>0$, we have

$$||\varphi||_2^2 \leqslant \frac{C}{\delta t_\delta^{2k-2}(R)}\left[1+\frac{2}{||H_k||_{2p}}\right]$$

Moreover, we have

$$\frac{1}{t_\delta(R)} \leqslant ||H_k||_{2p}^{1/k},$$

so

$$||\varphi||_2^2 \leqslant \frac{C}{\delta}\left[||H_k||_{2p}^{\frac{2k-2}{k}} + 2||H_k||_{2p}^{\frac{k-2}{k}}\right]$$

Since $H_k>0$, we have $H \geqslant H_k^{1/k}$ and then $||H_k||_{2p} \leqslant ||H||_\infty^k$. Finally, for $k \geqslant 2$, we have

$$||\varphi||_2^2 \leqslant \frac{C}{\delta}\left[||H||_\infty^{2k-2} + 2||H||_\infty^{k-2}\right] := A_1 C,$$

If $\delta = 0$, then

$$\begin{aligned}||\varphi||_2^2 &\leqslant \frac{C}{R^{2k-4}}\left[1+\frac{2}{||H_k||_{2p}}\right]\\ &\leqslant C\left[||H_k||_{2p}^{\frac{2k-4}{k}} + 2||H_k||_{2p}^{\frac{k-4}{k}}\right] := A_1 C,\end{aligned}$$

If, in addition, $4 \leqslant k \leqslant n$, by the same argument as above, A_1 depends on n and $||H||_\infty$.
Moreover, for $\delta = 0$, we have the following lower bound for the k-th mean curvature (see [6] page 221)

$$\int_M H_k^{n/k} dv_g \geqslant \omega_n,$$

where ω_n is the volume of the n-dimensional Euclidean sphere. Then, for $p \geqslant \frac{n}{2k}$, we have

$$\omega_n^{k/n} \leqslant ||H_k||_{2p} \leqslant ||H||_\infty^k.$$

So the dependence on $||H_k||_{2p}$ can be replaced by a dependence on $||H||_\infty$. □

Lemma 4.2. *The pinching condition (P_C) implies*

$$||\psi||_2^2 \leqslant A_2 C,$$

where A_2 is a positive explicit constant depending only on n and $||H||_\infty$.

Proof: Since $c_\delta(r) \leqslant 1$, we have

$$\begin{aligned} ||\psi||_2^2 &\leqslant \int_M |Z^T|^2 \leqslant \int_M |Z|^2 - \langle Z, \nu\rangle^2 \\ &\leqslant \int_M s_\delta^2(r) - \frac{1}{||H_k||_{2p}^2}\left(\int_M H_k \langle Z, \nu\rangle\right)^2 \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned} ||\psi||_2^2 &\leqslant t_\delta^2(R)\left(\int_M c_\delta(r)\right)^2 - \frac{1}{||H_k||_{2p}^2}\left(\int_M c_\delta(r) H_{k-1}\right)^2 \\ &\leqslant t_\delta^2(R)\left(\int_M c_\delta(r)\right)^2 - \frac{1}{t_\delta^2(R)||H_k||_{2p}^2}\left(\int_M s_\delta(r) H_{k-1}\right)^2 \end{aligned}$$

Using Proposition 3.1, we get

$$\begin{aligned} ||\psi||_2^2 &\leqslant t_\delta^2(R)\left(\int_M c_\delta(r)\right)^2 - \frac{1}{t_\delta^2(R)||H_k||_{2p}^2}\left(\int_M c_\delta(r) H_{k-2}\right)^2 \\ &\leqslant t_\delta^2(R)\left(\int_M c_\delta(r)\right)^2 - \frac{1}{t_\delta^4(R)||H_k||_{2p}^2}\left(\int_M c_\delta(r) H_{k-2}\right)^2 \\ &\leqslant \cdots \\ &\leqslant t_\delta^2(R)\left(\int_M c_\delta(r)\right)^2 - \frac{1}{t_\delta^{2k-2}(R)||H_k||_{2p}^2}\left(\int_m c_\delta(r)\right)^2 \\ &\leqslant \frac{1}{t_\delta^{2k-2}(R)}\left(t_\delta^2(R) - \frac{1}{||H_k||_{2p}^2}\right) \end{aligned}$$

So the pinching condition (P_C) with $C < 1$ implies

$$\begin{aligned} ||\psi||_2^2 &\leqslant \frac{C}{t_\delta^{2k-2}(R)}\left(1 + \frac{2}{||H_k||_{2p}}\right) \\ &\leqslant C||H_k||_{2p}^{\frac{2k-2}{k}}\left(1 + \frac{2}{||H_k||_{2p}}\right) \\ &\leqslant C\left(||H_k||_{2p}^{\frac{2k-2}{k}} + 2||H_k||_{2p}^{\frac{k-2}{k}}\right) \\ &\leqslant C\left(||H||_\infty^{2k-2} + 2||H||_\infty^{k-2}\right) := A_2 C, \end{aligned}$$

□

The next step to prove Theorems 1 and 2 is to get L^∞-estimates from these L^2-estimates. For this, we will use Lemma 2.3.

5. Proof of Therorem 1

The proof of Theorem 1 is an immediate consequence of the following three lemmas

Lemma 5.1. *For any $\varepsilon > 0$, there exists K_ε depending on n and $||H||_\infty$ so that if (P_{K_ε}) is true, then*

$$\phi(M) \subset B(p_0, R) \setminus B(p_0, R - \varepsilon).$$

Moreover, $K_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow +\infty$ or $\varepsilon \longrightarrow 0$.

Proof: We showed in [21] that the function φ satisfies

$$\varphi^{2k-2}\Delta\varphi^2 \leqslant \operatorname{div}(\omega) + (\alpha_1 + k\alpha_2)\varphi^{2k-1} + (\beta_1 + k\beta_2)\varphi^{2k-2}, \tag{8}$$

where ω is a 1-form, α_1, α_2, β_1 and β_2 some nonnegative constants depending on n, δ and $||H||_\infty$. We can apply Lemma 2.3 to the function φ with $l = 1$ and $m = 2$. We deduce that if $||\varphi||_\infty > \varepsilon$ then there exists a constant L such that

$$||\varphi||_\infty \leqslant L||\varphi||_2.$$

On the other hand, by Lemma 4.1, we know that if the pinching condition (P_C) is satisfied for $C \leqslant 1$, then

$$||\varphi||_2^2 \leqslant A_1 C.$$

Take $C = K_\varepsilon = \inf\left\{1, \frac{\varepsilon^2}{L^2 A_1}\right\}$. This choice implies

$$||\varphi||_\infty \leqslant \varepsilon,$$

that is, $t_\delta^2(R) - t_\delta^2(r) \leqslant \varepsilon$. Finally, we can choose K_ε smaller in order to have $R - r \leqslant \varepsilon$. □

The second lemma is due to B. Colbois and J.F. Grosjean (see [7]).

Lemma 5.2. *Let x_0 be a point of the sphere $S(0, R)$ of $\mathbb{R}^{n+1}$. Assume that $x_0 = Ru$ where $u \in \mathbb{S}^n$. Now let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into $\mathbb{R}^{n+1}$ so that*

$$\phi(M) \subset \Big(B(p_0, R + \eta) \setminus B(p_0, R - \eta)\Big) \setminus B(x_0, \rho)$$

with $\rho = 4(2n - 1)\eta$ and suppose there exists a point $p \in M$ so that $\langle Z, u\rangle(p) \geqslant 0$. Then there exists $y_0 \in M$ so that the mean curvature satisfies $|H(y_0)| > \frac{1}{4n\eta}$.

Remark. Note that in [7], it is supposed that $\langle Z, u\rangle(p) > 0$, but the condition $\langle Z, u\rangle(p) \geqslant 0$ is sufficient.

In [21], we give a corresponding lemma for the spherical case.

Lemma 5.3. *Let x_0 be a point of the sphere $S(p_0, R)$ of an open hemisphere of $\mathbb{S}^{n+1}(\delta)$. Let (M^n, g) be a compact, connected and oriented n-dimensional Riemannian manifold without boundary isometrically immersed by ϕ into this open hemisphere of $\mathbb{S}^{n+1}(\delta)$ so that*

$$\phi(M) \subset \Big(B(p_0, R) \setminus B(p_0, R - \eta)\Big) \setminus B(x_0, \rho)$$

with ρ such that

$$t_\delta(R/2) - t_\delta\left((R-\rho)/2\right) = 4(2n-1)\eta$$

Then there exists two constants D and E depending on n, δ and R such that if $\eta \leqslant D$, then there exists $y_0 \in M$ so that

$$|H(y_0)| \geqslant \frac{E}{8n\eta}.$$

Proof of Theorem 1:
The proof for $\delta = 0$ is an immediate consequence of Lemmas 5.1 and 5.2. Indeed, let $\varepsilon > 0$. By Lemma 5.1, there exists K_ε such that if (P_{K_ε}) is true, then

$$\phi(M) \subset B(p_0, R) \setminus B(p_0, R-\varepsilon).$$

Let $x = Ru \in S(0,R)$ and assume that $\phi(M) \cap B(x,\varepsilon) = \emptyset$. Since R is the extrinsic radius, there exists a point $p \in M$ such that $\langle Z, u\rangle(p) \geqslant 0$. If (P_{C_ε}) is true with $C_\varepsilon = K_{\frac{\varepsilon}{4(2n-1)}}$ and $\varepsilon < \frac{2}{3||H||_\infty}$, then by Lemma 5.2, there exists $y_0 \in M$ so that

$$|H(y_0)| \geqslant \frac{1}{4n(\frac{\varepsilon}{4(2n-1)})} \geqslant \frac{2n-1}{n\varepsilon} > ||H||_\infty,$$

which is a contradiction. Finally, $\phi(M) \cap B(x,\varepsilon) \neq \emptyset$ which completes the proof for $\delta = 0$.
For $\delta \neq 0$, let $\varepsilon > 0$. We set $0 < \eta := \inf\left\{D, \varepsilon, \frac{\gamma(\varepsilon)}{8(2n-1)}\right\}$, where

$$\gamma(\varepsilon) = t_\delta\left(\frac{R}{2}\right) - t_\delta\left(\frac{R-\varepsilon}{2}\right).$$

Note that γ is an increasing smooth function with $\gamma(0) = 0$. From Lemma 5.1, there exists $C_\varepsilon = K_\eta$ such that (P_{C_ε}) implies

$$R - r \leqslant \eta \leqslant \varepsilon.$$

That's the first point of Theorem 1. Assume that $\varepsilon < \gamma^{-1}\left(\frac{2E}{3||H||_\infty}\right)$. Suppose there exists $x \in S(p_0, R)$ such that $B(x,\varepsilon) \cap M = \emptyset$. Since $\gamma(\varepsilon) \geqslant 4(2n-1)\eta$, by Lemma 5.3, there exists a point $y_0 \in M$ so that

$$|H(y_0)| \geqslant \frac{E}{8n\eta} \geqslant \frac{(2n-1)E}{n\gamma(\varepsilon)} > ||H||_\infty.$$

Hence a contradiction and $B(x,\varepsilon) \cap M \neq \emptyset$.
Moreover, for $\delta = 0$ or $\delta > 0$, by Lemma 5.1, $C_\varepsilon \longrightarrow 0$ when $||H||_\infty \longrightarrow 0$ or $\varepsilon \longrightarrow 0$. □

6. Proof of Theorem 2

We first need the following lemma

Lemma 6.1. *For any $\varepsilon > 0$, there exists K_ε depending on n and $||B||_\infty$ so that if (P_{K_ε}) is true, then*

$$||\psi||_\infty \leqslant \varepsilon.$$

Moreover, $K_\varepsilon \longrightarrow 0$ when $||B||_\infty \longrightarrow +\infty$ or $\varepsilon \longrightarrow 0$.

Proof: We proved in [21] that the function ψ satisfies

$$\psi^{2k-2}\Delta\psi^2 \leqslant \operatorname{div}(\omega) + (\alpha_3 + k\alpha_4)\psi^{2k-1} + (\beta_3 + k\beta_4)\psi^{2k-2}. \tag{9}$$

Now applying Lemma 2.3 with $l = 1$ and $m = 2$, we get that for any $\eta > 0$, there exists L depending on n and $||B||_\infty$ so that if $||\psi||_\infty > \eta$ then

$$||\psi||_\infty \leqslant L||\psi||_2.$$

From Lemma 4.1, we know that if (P_C) holds, then

$$||\psi||_2^2 \leqslant A_2 C.$$

Let $\varepsilon > 0$, we set $C = K_\varepsilon = \inf\left\{1, \frac{\varepsilon^2}{L^2 A_2}\right\}$. For this choice of C we get

$$||\psi||_\infty \leqslant \varepsilon.$$

This completes the proof. □

Proof of Theorem2:
We consider the following map

$$\begin{array}{rccl} F: & M & \longrightarrow & S(p_0, R) \\ & x & \longmapsto & \exp_{p_0}\left(R\left(\mathrm{d}\exp_{p_0}\right)^{-1}(\overline{\nabla} r)\right). \end{array}$$

We proved in [21] that for any uniatry vector $u \in T_xM$

$$\left|\left|\mathrm{d}_x F(u)\right|^2 - 1\right| \leqslant \frac{1}{s_\delta^2(r)}\left|s_\delta^2(R) - s_\delta^2(r)\right| + \frac{s_\delta^2(R)}{c_\delta(r)s_\delta^3(r)}||\psi||_\infty.$$

From Lemma 5.2, we know that for any $\eta > 0$, there exists a constant K_η so that (P_{K_η}) implies $||\psi||_\infty \leqslant \eta$. Moreover, by Theorem 1, there exist C_η depending on n, δ, $||H||_\infty$ and η so that (P_{C_η}) implies $R - r \leqslant \eta$. We set $C'_\eta = \inf\{C_\eta, K_\eta\}$. Then, since R is bounded by a constant depending only on n, δ, $||H_k||_{2p}$ and $||H||_\infty$ there exist three positive constants A_3, A_4 and A_5 depending on n, δ, $||H_k||_{2p}$ and $||H||_\infty$ so that

$$\begin{aligned} \left|\left|\mathrm{d}_x F(u)\right|^2 - 1\right| &\leqslant A_3||R - r||_\infty + A_4||\psi||_\infty \\ &\leqslant A_3\eta + A_4\eta \leqslant A_5\eta \end{aligned}$$

Now, choosing $\eta = \frac{\theta}{A_5}$, we get

$$\left|\left|\mathrm{d}_x F(u)\right|^2 - 1\right| \leqslant \theta. \tag{10}$$

For $\theta \in]0,1[$, by (10), F is a local diffeomorphism from M to $S(p_0, R)$. Since for $n \geqslant 2$, $S(p_0, R)$ is simply connected, F is a diffeomorphism. Moreover, the relation (10) says that F is a quasi-isometry. □

Remark. If $\delta = 0$ and $p \geqslant \frac{n}{2k}$, we saw that

$$\omega_n^{k/n} \leqslant ||H_k||_{2p} \leqslant ||H||_\infty^k \leqslant \frac{1}{n^{k/2}} ||B||_\infty^k,$$

so there is no dependence on $||H_k||_{2p}$.

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