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A PINCHING THEOREM FOR THE FIRST EIGENVALUE OF THE LAPLACIAN ON HYPERSURFACES OF THE EUCLIDEAN SPACE

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Abstract

In this paper, we give pinching Theorems for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian on the compact hypersurfaces of the Euclidean space. Indeed, we prove that if the volume of $M$ is 1 then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending on the dimension $n$ of $M$ and the $L_\infty$-norm of the mean curvature $H$, so that if the $L_2^p$-norm $\|H\|_{2^p}$ ($p \geq 2$) of $H$ satisfies $n\|H\|_{2^p}^2 - C_\varepsilon < \lambda_1(M)$, then the Hausdorff-distance between $M$ and a round sphere of radius $(n/\lambda_1(M))^{1/2}$ is smaller than $\varepsilon$. Furthermore, we prove that if $C$ is a small enough constant depending on $n$ and the $L_\infty$-norm of the second fundamental form, then the pinching condition $n\|H\|_{2^p}^2 - C < \lambda_1(M)$ implies that $M$ is diffeomorphic to an $n$-dimensional sphere.

Key words: Spectrum, Laplacian, pinching results, hypersurfaces.

1 Introduction and preliminaries

Let \((M^n, g)\) be a compact, connected and oriented \(n\)-dimensional Riemannian manifold without boundary isometrically immersed by \(\phi\) into the \(n+1\)-dimensional euclidean space \((\mathbb{R}^{n+1}, \text{can})\) (i.e. \(\phi^* \text{can} = g\)). A well known inequality due to Reilly ([11]) gives an extrinsic upper bound for the first nonzero eigenvalue \(\lambda_1(M)\) of the Laplacian of \((M^n, g)\) in terms of the square of the length of the mean curvature. Indeed, we have

\[
\lambda_1(M) \leq \frac{n}{V(M)} \int_M |H|^2 dv
\]

where \(dv\) and \(V(M)\) denote respectively the Riemannian volume element and the volume of \((M^n, g)\). Moreover the equality holds if and only if \((M^n, g)\) is a geodesic hypersphere of \(\mathbb{R}^{n+1}\).

By using Hölder inequality, we obtain some other similar estimates for the \(L^{2p}\)-norm \((p \geq 1)\) with \(H\) denoted by \(\|H\|_{2p}^2\)

\[
\lambda_1(M) \leq \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2, \tag{2}
\]

and as for the inequality (1), the equality case is characterized by the geodesic hyperspheres of \(\mathbb{R}^{n+1}\).

A first natural question is to know if there exists a pinching result as the one we state now: does a constant \(C\) depending on minimum geometric invariants exist so that if we have the pinching condition

\[(P_C) \quad \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2 - C < \lambda_1(M)\]

then \(M\) is close to a sphere in a certain sense?

Such questions are known for the intrinsic lower bound of Lichnerowicz-Obata ([3]) of \(\lambda_1(M)\) in terms of the lower bound of the Ricci curvature (see [1], [8], [10]). Other pinching results have been proved for Riemannian manifolds with positive Ricci curvature, with a pinching condition on the \(n+1\)-st eigenvalue ([14]), the diameter ([3], [8], [15]), the volume or the radius (see for instance [4] and [5]).

For instance, S. Ilias proved in [8] that there exists \(\varepsilon\) depending on \(n\) and an upper bound of the sectional curvature so that if the Ricci curvature \(\text{Ric}\) of \(M\) satisfies \(\text{Ric} \geq n - 1\) and \(\lambda_1(M) \leq \lambda_1(\mathbb{S}^n) + \varepsilon\), then \(M\) is homeomorphic to \(\mathbb{S}^n\).

In this article, we investigate the case of hypersurfaces where, as far as we know, very little is known about pinching and stability results (see however [12] and [13]).

More precisely, in our paper, the hypothesis made in [8] that \(M\) has a positive Ricci curvature is replaced by the fact that \(M\) is isometrically immersed as a hypersurface in \(\mathbb{R}^{n+1}\), and the bound on the sectional curvature by an \(L^\infty\)-bound on the mean curvature.
or on the second fundamental form. Note that we do not know if such bounds are sharp, or if a bound on the $L^q$-norm (for some $q$) of the mean curvature would be enough.

We get the following results

**Theorem 1.1** Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold without boundary isometrically immersed by $\phi$ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$ and let $x_0$ be the center of mass of $M$. Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending only on $n$, $\varepsilon > 0$ and on the $L^\infty$-norm of $H$ so that if

$$(P_{C_\varepsilon}) \quad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then the Hausdorff-distance $d_H$ of $M$ to the sphere $S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ of center $x_0$ and radius $\sqrt{\frac{n}{\lambda_1(M)}}$ satisfies $d_H\left(\phi(M), S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)\right) < \varepsilon$.

We recall that the Hausdorff-distance between two compact subsets $A$ and $B$ of a metric space is given by

$$d_H(A, B) = \inf\{\eta|V_\eta(A) \supset B \text{ and } V_\eta(B) \supset A\}$$

where for any subset $A$, $V_\eta(A)$ is the tubular neighborhood of $A$ defined by $V_\eta(A) = \{x|\text{dist}(x, A) < \eta\}$.

**Remark** We will see in the proof that $C_\varepsilon(n, \|H\|_\infty) \to 0$ when $\|H\|_\infty \to \infty$ or $\varepsilon \to 0$.

In fact the previous Theorem is a consequence of the above definition and the following Theorem

**Theorem 1.2** Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold without boundary isometrically immersed by $\phi$ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$ and let $x_0$ be the center of mass of $M$. Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending only on $n$, $\varepsilon > 0$ and on the $L^\infty$-norm of $H$ so that if

$$(P_{C_\varepsilon}) \quad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

then

1. $\phi(M) \subset B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right) \setminus B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)$.

2. $\forall x \in S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right), B(x, \varepsilon) \cap \phi(M) \neq \emptyset$.
In the following Theorem, if the pinching is strong enough, with a control on \( n \) and the \( L_\infty \)-norm of the second fundamental form, we obtain that \( M \) is diffeomorphic to a sphere and even almost isometric with a round sphere in a sense we will make precise.

**Theorem 1.3** Let \( (M^n, g) \) be a compact, connected and oriented \( n \)-dimensional Riemannian manifold \( (n \geq 2) \) without boundary isometrically immersed by \( \phi \) in \( \mathbb{R}^{n+1} \). Assume that \( V(M) = 1 \). Then for any \( p \geq 2 \), there exists a constant \( C \) depending only on \( n \) and the \( L_\infty \)-norm of the second fundamental form \( B \) so that if

\[
(P_C) \quad n\|H\|_{L^p}^2 - C < \lambda_1(M)
\]

Then \( M \) is diffeomorphic to \( S^n \).

More precisely, there exists a diffeomorphism \( F \) from \( M \) into the sphere \( S^n \left( \sqrt{\frac{n}{\lambda_1(M)}} \right) \) of radius \( \sqrt{\frac{n}{\lambda_1(M)}} \) which is a quasi-isometry. Namely, for any \( \theta, 0 < \theta < 1 \), there exists a constant \( C \) depending only on \( n \), the \( L_\infty \)-norm of \( B \) and \( \theta \), so that the pinching condition \( (P_C) \) implies

\[
|dF_x(u)|^2 - 1 \leq \theta
\]

for any \( x \in M \) and \( u \in T_xM \) so that \( |u| = 1 \).

Now we will give some preliminaries for the proof of these Theorems. Throughout the paper, we consider a compact, connected and oriented \( n \)-dimensional Riemannian manifold \( (M^n, g) \) without boundary isometrically immersed by \( \phi \) into \( (\mathbb{R}^{n+1}, \text{can}) \) (i.e. \( \phi^*\text{can} = g \)). Let \( \nu \) be the outward normal vector field. Then the second fundamental form of the immersion will be defined by \( B(X, Y) = \langle \nabla^0 X, Y \rangle \), where \( \nabla^0 \) and \( \langle \ , \ \rangle \) are respectively the Riemannian connection and the inner product of \( \mathbb{R}^{n+1} \). Moreover the mean curvature \( H \) will be given by \( H = \frac{1}{n}\text{trace}(B) \).

Now let \( \partial_i \) be an orthonormal frame of \( \mathbb{R}^{n+1} \) and let \( x_i : \mathbb{R}^{n+1} \to \mathbb{R} \) be the associated component functions. Putting \( X_i = x_i \circ \phi \), a straightforward calculation shows us that

\[
B \otimes \nu = - \sum_{i \leq n+1} \nabla dX_i \otimes \partial_i
\]

and

\[
nH \nu = \sum_{i \leq n+1} \Delta X_i \partial_i
\]

where \( \nabla \) and \( \Delta \) denote respectively the Riemannian connection and the Laplace-Beltrami operator of \( (M^n, g) \). On the other hand, we have the well known formula

\[
\frac{1}{2} \Delta |X|^2 = nH \langle \nu, X \rangle - n
\]

(3)
where \( X \) is the position vector given by \( X = \sum_{i \leq n+1} X_i \partial_i \).

We recall that to prove the Reilly inequality, we use the functions \( X_i \) as test functions (cf [11]). Indeed, doing a translation if necessary, we can assume that \( \int_M X_i \mathrm{d}v = 0 \) for all \( i \leq n+1 \) and we can apply the variational characterization of \( \lambda_1(M) \) to \( X_i \). If the equality holds in (1) or (2), then the functions are nothing but eigenfunctions of \( \lambda_1(M) \) and from the Takahashi’s Theorem (cf [14]) \( M \) is immersed isometrically in \( \mathbb{R}^{n+1} \) as a geodesic sphere of radius \( \sqrt{\frac{n}{\lambda_1(M)}} \).

Throughout the paper we use some notations. From now on, the inner product and the norm induced by \( g \) and \( \cdots \) on a tensor \( T \) will be denoted respectively by \( \langle \cdots, \cdots \rangle \) and \( \| \cdot \| \), and the \( L^p \)-norm will be given by

\[
\|T\|_p = \left( \int_M |T|^p \mathrm{d}v \right)^{1/p}
\]

and

\[
\|T\|_\infty = \sup_M |T|
\]

We end these preliminaries by a convenient result

**Lemma 1.1** Let \((M^n, g)\) be a compact, connected and oriented \( n \)-dimensional Riemannian manifold \((n \geq 2)\) without boundary isometrically immersed by \( \phi \) in \( \mathbb{R}^{n+1} \). Assume that \( V(M) = 1 \). Then there exist constants \( c_n \) and \( d_n \) depending only on \( n \) so that for any \( p \geq 2 \), if \((P_C)\) is true with \( C < c_n \) then

\[
\frac{n}{\lambda_1(M)} \leq d_n
\]  

**Proof:** We recall the standard Sobolev inequality (cf [8], [7], [16] and p 216 in [1]). If \( f \) is a smooth function and \( f \geq 0 \), then

\[
\left( \int_M f^{n/(1-n)} \mathrm{d}v \right) \leq K(n) \int_M (|df| + |H f|) \mathrm{d}v
\]

where \( K(n) \) is a constant depending on \( n \) and the volume of the unit ball in \( \mathbb{R}^n \). Taking \( f = 1 \) on \( M \), and using the fact that \( V(M) = 1 \), we deduce that

\[
\|H\|_{2p} \geq \frac{1}{K(n)}
\]

and if \((P_C)\) is satisfied and \( C \leq \frac{n}{2K(n)^2} = c_n \), then
\[ \frac{n}{\lambda_1(M)} \leq \frac{1}{n\|H\|_{2p}^2 - C} \leq 2K(n)^2 = d_n \]

Throughout the paper, we will assume that \( V(M) = 1 \) and \( \int_M X_i dv = 0 \) for all \( i \leq n + 1 \). The last assertion implies that the center of mass of \( M \) is the origin of \( \mathbb{R}^{n+1} \).

## 2 An \( L^2 \)-approach of the problem

A first step in the proof of the Theorem 1.2 is to prove that if the pinching condition \((P_C)\) is satisfied, then \( M \) is close to a sphere in an \( L^2 \)-sense.

In the following Lemma, we prove that the \( L^2 \)-norm of the position vector is close to \( \sqrt{n\lambda_1(M)} \).

**Lemma 2.1** If we have the pinching condition \((P_C)\) with \( C < c_n \), then

\[ \frac{n\lambda_1(M)}{(C + \lambda_1(M))^2} \leq \|X\|_2^2 \leq \frac{n}{\lambda_1(M)} \leq d_n \]

**Proof:** Since \( \int_M X_i dv = 0 \), we can apply the variational characterization of the eigenvalues to obtain

\[ \lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 dv = n \]

which gives the inequality of the right-hand side.

Let us prove now the inequality of the left-hand side.

\[ \lambda_1(M) \int_M |X|^2 dv \leq \left( \frac{\int_M \sum_{i \leq n+1} |dX_i|^2 dv}{n^3} \right)^4 \leq \left( \frac{\int_M \sum_{i \leq n+1} (\Delta X_i)X_i dv}{n^3} \right)^4 \]

\[ \leq \frac{\left( \int_M \sum_{i \leq n+1} (\Delta X_i)^2 dv \right)^2}{n^3} \left( \int_M |X|^2 dv \right)^2 \]
= \left( \int_M H^2 dv \right)^2 \left( \int_M |X|^2 dv \right)^2
\]
then using again the Hölder inequality, we get
\[
\lambda_1(M) \leq \frac{1}{n} (n \| H \|^2_{2p})^2 \int_M |X|^2 dv \leq \frac{(C + \lambda_1(M))^2}{n} \int_M |X|^2 dv
\]
This completes the proof.

From now on, we will denote by $X^T$ the orthogonal tangential projection on $M$. In fact, at $x \in M$, $X^T$ is nothing but the vector of $T_x M$ defined by $X^T = \sum_{1 \leq i \leq n} \langle X, e_i \rangle e_i$ where $(e_i)_{1 \leq i \leq n}$ is an orthonormal basis of $T_x M$. In the following Lemma, we will show that the condition $(P_C)$ implies that the $L^2$-norm of $X^T$ of $X$ on $M$ is close to 0.

**Lemma 2.2** If we have the pinching condition $(P_C)$, then
\[
\| X^T \|^2_2 \leq A(n) C
\]

**Proof:** From the lemma 2.1 and the relation (3), we have
\[
\lambda_1(M) \int_M |X|^2 dv \leq n = n \left( \int_M H \langle X, \nu \rangle dv \right)^2 \leq \left( \int_M |H| \langle X, \nu \rangle dv \right)^2 \leq n \| H \|^2_{2p} \left( \int_M |\langle X, \nu \rangle|^{\frac{2p}{p-1}} dv \right)^{\frac{2p-1}{p}} \leq n \| H \|^2_{2p} \left( \int_M |\langle X, \nu \rangle| dv \right) = n \| H \|^2_{2p} \int_M |X|^2 dv
\]
Then we deduce that
\[
n \| H \|^2_{2p} \| X^T \|^2_2 = n \| H \|^2_{2p} \left( \int_M (|X|^2 - |\langle X, \nu \rangle|^2) dv \right) \leq (n \| H \|^2_{2p} - \lambda_1(M)) \| X \|^2_2 \leq d_n C
\]
where in the last inequality we have used the pinching condition and the Lemma 2.1.

Now, we will show that the condition $(P_C)$ implies that the component functions are almost eigenfunctions in an $L^2$-sense. For this, let us consider the vector field $Y$ on $M$ defined by
\[
Y = \sum_{i \leq n+1} (\Delta X_i - \lambda_1(M) X_i) \partial_i = nH\nu - \lambda_1(M) X
\]
Lemma 2.3 If \((P_C)\) is satisfied, then

\[ \|Y\|_2^2 \leq nC \]

\textbf{Proof:} We have

\[ \int_M |Y|^2 dv = \int_M \left( n^2 H^2 - 2n\lambda_1(M) H \langle \nu, X \rangle + \lambda_1(M)^2 |X|^2 \right) dv \]

Now by integrating the relation \([3]\) we deduce that

\[ \int_M H \langle \nu, X \rangle dv = 1 \]

Furthermore, since \(\int_M X dv = 0\), we can apply the variational characterization of the eigenvalues to obtain

\[ \lambda_1(M) \int_M |X|^2 dv = \lambda_1(M) \int_M \sum_{i \leq n+1} |X_i|^2 dv \leq \int_M \sum_{i \leq n+1} |dX_i|^2 dv = n \]

Then

\[ \int_M |Y|^2 dv \leq n^2 \int_M |H|^2 dv - n\lambda_1(M) \leq n \left( n\|H\|_{2p}^2 - \lambda_1(M) \right) \leq nC \]

where in this last inequality we have used the Hölder inequality.

To prove Assertion 1 of Theorem 1.2, we will show that \(\|X - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} \nu \| \|_\infty \leq \varepsilon\). For this we need to have an \(L^2\)-upper bound on the function \(\varphi = |X| \left( |X| - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2\).

Before giving such estimate, we will introduce the vector field \(Z\) on \(M\) defined by

\[ Z = \left( \frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} \nu - \frac{X}{|X|^{1/2}} \]

We have

\textbf{Lemma 2.4} If \((P_C)\) is satisfied with \(C < c_n\), then

\[ \|Z\|_2^2 \leq B(n)C \]

\textbf{Proof:} We have

\[ \|Z\|_2^2 = \left\| \left( \frac{n}{\lambda_1(M)} \right)^{1/2} |X|^{1/2} \nu - \frac{X}{|X|^{1/2}} \right\|_2^2 \]
\[
\int_M \left( \frac{n}{\lambda_1(M)} |X| H^2 - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2} H \langle \nu, X \rangle + |X| \right) \, dv 
\]
\[
\leq \frac{n}{\lambda_1(M)} \left( \int_M |X|^2 \, dv \right)^{1/2} \left( \int_M H^4 \, dv \right)^{1/2} - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \left( \int_M |X|^2 \, dv \right)^{1/2}
\]

Note that we have used the relation (3). Finally for \( p \geq 2 \), we get

\[
\|Z\|_2^2 \leq \left( \int_M |X|^2 \, dv \right)^{1/2} \left( \frac{n}{\lambda_1(M)} \|H\|_{2^p} + 1 \right) - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2}
\]
\[
\leq \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \left( \frac{C}{\lambda_1(M)} + 2 \right) - 2 \left( \frac{n}{\lambda_1(M)} \right)^{1/2}
\]
\[
= \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{C}{\lambda_1(M)} \leq \frac{d_{3/2}^n}{n} C
\]

This concludes the proof of the Lemma.

Now we give an \( L^2 \)-upper bound of \( \varphi \)

**Lemma 2.5** Let \( p \geq 2 \) and \( C \leq c_n \). If we have the pinching condition \( (P_C) \), then

\[
\|\varphi\|_2 \leq D(n) \|\varphi\|_{\infty}^{3/4} C^{1/4}
\]

**PROOF:** We have

\[
\|\varphi\|_2 = \left( \int_M \varphi^{3/2} \varphi^{1/2} \, dv \right)^{1/2} \leq \|\varphi\|_{\infty}^{3/4} \|\varphi^{1/2}\|_1^{1/2}
\]

and noting that

\[
|X| \left( |X| - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2 = \left| |X|^{1/2} X - \left( \frac{n}{\lambda_1(M)} \right) \frac{X}{|X|^{1/2}} \right|^2
\]

we get

\[
\int_M \varphi^{1/2} \, dv = \left\| |X|^{1/2} X - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_1
\]
\[
= \left\| -\frac{|X|^{1/2}}{\lambda_1(M)} Y + \frac{n}{\lambda_1(M)} |X|^{1/2} H \nu - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \frac{X}{|X|^{1/2}} \right\|_1
\]

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\[ \leq \frac{|X|^{1/2}Y}{\lambda_1(M)} \|_1 + \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \|Z\|_1 \]  

From Lemmas 2.3 and 1.1 we get

\[ \left\| \frac{|X|^{1/2}Y}{\lambda_1(M)} \right\|_1 \leq \frac{1}{\lambda_1(M)} \left( \int_M |X| \, dv \right)^{1/2} \|Y\|_2 \leq \frac{1}{\lambda_1(M)} \left( \int_M |X|^2 \, dv \right)^{1/4} \|Y\|_2 \leq \frac{d_n^{2/4}}{n^{1/2}} C^{1/2} \]

Moreover, using Lemmas 2.4 and 1.1 again it is easy to see that the last term of (6) is bounded by \( d_1^{1/2} B(n)^{1/2} C^{1/2} \). Then \( \|\varphi^{1/2}\|_1 \leq D(n) C^{1/4} \).

\[ \square \]

### 3 Proof of Theorem 1.2

The proof of Theorem 1.2 is immediate from the two following technical Lemmas which we state below.

**Lemma 3.1** For \( p \geq 2 \) and for any \( \eta > 0 \), there exists \( K_\eta(n, \|H\|_\infty) \leq c_n \) so that if \( (P_{K_\eta}) \) is true, then \( \|\varphi\|_\infty \leq \eta \). Moreover, \( K_\eta \to 0 \) when \( \|H\|_\infty \to \infty \) or \( \eta \to 0 \).

and

**Lemma 3.2** Let \( x_0 \) be a point of the sphere \( S(O, R) \) of \( \mathbb{R}^{n+1} \) with the center at the origin and of radius \( R \). Assume that \( x_0 = Re \) where \( e \in \mathbb{S}^n \). Now let \( (M^n, g) \) be a compact oriented \( n \)-dimensional Riemannian manifold without boundary isometrically immersed by \( \phi \) in \( \mathbb{R}^{n+1} \) so that \( \phi(M) \subset (B(O, R + \eta) \setminus B(O, R - \eta)) \setminus B(x_0, \rho) \) with \( \rho = 4(2n - 1)\eta \) and suppose that there exists a point \( p \in M \) so that \( \langle X, e \rangle > 0 \). Then there exists \( y_0 \in M \) so that the mean curvature \( H(y_0) \) at \( y_0 \) satisfies \( |H(y_0)| \geq \frac{1}{4n} \).

Now, let us see how to use these Lemmas to prove Theorem 1.2.

**Proof of the Theorem 1.2:** Let \( \varepsilon > 0 \) and let us consider the function \( f(t) = t \left( t - \left( \frac{n}{\lambda_1(M)} \right)^{1/2} \right)^2 \). Let us put

\[ \eta(\varepsilon) = \min \left( \frac{1}{\|H\|_\infty} - \varepsilon, \frac{1}{\|H\|_\infty} + \varepsilon \right), \left( \left( \frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \right), \left( \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon \right) \]

\[ \leq \min \left( f \left( \left( \frac{n}{\lambda_1(M)} \right)^{1/2} - \varepsilon \right), f \left( \left( \frac{n}{\lambda_1(M)} \right)^{1/2} + \varepsilon \right) \right) \]
Then, as $\eta(\varepsilon) > 0$ and from Lemma 2.1, it follows that if the pinching condition ($P_{K_{\eta(\varepsilon)}}$) is satisfied with $K_{\eta(\varepsilon)} \leq c_n$, then for any $x \in M$, we have

$$f(|X|) \leq \eta(\varepsilon)$$

(7)

Now to prove Theorem 1.2, it is sufficient to assume $\varepsilon < \frac{2}{3\||H||_\infty}$. Let us show that either

$$\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}$$

(8)

By studying the function $f$, it is easy to see that $f$ has a unique local maximum in $\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and from the definition of $\eta(\varepsilon)$ we have $\eta(\varepsilon) < \frac{4}{27 \||H||_\infty^2} \leq \frac{4}{27} \left(\frac{n}{\lambda_1(M)}\right)^{3/2} = f \left(\frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)$.

Now since $\varepsilon < \frac{2}{3\||H||_\infty}$, we have $\varepsilon < \frac{4}{9} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}$, and $\frac{1}{3} \left(\frac{n}{\lambda_1(M)}\right)^{1/2} < \left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon$.

This and (7) yield (8).

Now, from Lemma 2.1 we deduce that there exists a point $y_0 \in M$ so that $|X(y_0)| \geq \frac{n^{1/2} \lambda_1(M)^{1/2}}{(K_{\eta(\varepsilon)} + \lambda_1(M))}$ and since $K_{\eta(\varepsilon)} \leq c_n = \frac{n}{\lambda_1(M)} \leq \lambda_1(M) \leq 2 \lambda_1(M)$ (see the proof of the Lemma 2.1), we obtain $|X(y_0)| \geq \frac{4}{27} \left(\frac{n}{\lambda_1(M)}\right)^{1/2}$.

By the connectedness of $M$, it follows that $\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon$ for any point of $M$ and Assertion 1 of Theorem 1.2 is shown for the condition ($P_{K_{\eta(\varepsilon)}}$).

In order to prove the second assertion, let us consider the pinching condition ($P_{C_\varepsilon}$) with $C_\varepsilon = K_{\eta(\sqrt{\frac{2n-1}{n\varepsilon}})}$. Then Assertion 1 is still valid. Let $x = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} e \in S \left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$, with $e \in S^n$ and suppose that $B(x, \varepsilon) \cap M = \emptyset$. Since $\int_M X_i dv = 0$ for any $i \leq n+1$, there exists a point $p \in M$ so that $\langle X, e \rangle > 0$ and we can apply Lemma 3.2. Therefore there is a point $y_0 \in M$ so that $H(y_0) \geq \frac{2n-1}{n\varepsilon} > \||H||_\infty$ since we have assumed $\varepsilon < \frac{2}{3\||H||_\infty} \leq \frac{2n-1}{2n\||H||_\infty}$.

Then we obtain a contradiction which implies $B(x, \varepsilon) \cap M \neq \emptyset$ and Assertion 2 is satisfied. Furthermore, $C_\varepsilon \to 0$ when $\||H||_\infty \to \infty$ or $\varepsilon \to 0$.

\section{Proof of Theorem 1.3}

From Theorem 1.2, we know that for any $\varepsilon > 0$, there exists $C_\varepsilon$ depending only on $n$ and $\||H||_\infty$ so that if ($P_{C_\varepsilon}$) is true then
\[ |X|_x - \sqrt{\frac{n}{\lambda_1(M)}} \leq \varepsilon \]

for any \( x \in M \). Now, since \( \sqrt{n}\|H\|_{\infty} \leq \|B\|_{\infty} \), it is easy to see from the previous proofs that we can assume that \( \|H\|_{\infty} \) is depending only on \( n \) and \( \|B\|_{\infty} \).

The proof of Theorem 1.3 is a consequence of the following Lemma on the \( L_\infty \)-norm of \( \psi = |X|' \)

**Lemma 4.1** For \( p \geq 2 \) and for any \( \eta > 0 \), there exists \( K_\eta(n, \|B\|_{\infty}) \) so that if \( (P_{K_\eta}) \) is true, then \( \|\psi\|_{\infty} \leq \eta \). Moreover, \( K_\eta \to 0 \) when \( \|B\|_{\infty} \to \infty \) or \( \eta \to 0 \).

This Lemma will be proved in the Section 5.

**Proof of Theorem 1.3** Let \( \varepsilon < \frac{1}{2} \sqrt{\frac{n}{\lambda_1(M)}} \leq \sqrt{\frac{n}{\lambda_1(M)}} \). From the choice of \( \varepsilon \), we deduce that the condition \( (P_{C_\varepsilon}) \) implies that \( |X|_x \) is nonzero for any \( x \in M \) (see the proof of Theorem 1.2) and we can consider the differential application

\[
F : M \rightarrow S \left( O, \sqrt{\frac{n}{\lambda_1(M)}} \right) \\
x \mapsto -\sqrt{\frac{n}{\lambda_1(M)}} X
\]

We will prove that \( F \) is a quasi isometry. Indeed, for any \( 0 < \theta < 1 \), we can choose a constant \( \varepsilon(n, \|B\|_{\infty}, \theta) \) so that for any \( x \in M \) and any unit vector \( u \in T_xM \), the pinching condition \( (P_{C_{\varepsilon(n, \|B\|_{\infty}, \theta})}) \) implies

\[
|\langle dF_x(u), u \rangle |^2 - 1 | \leq \theta
\]

For this, let us compute \( dF_x(u) \). We have

\[
dF_x(u) = \sqrt{\frac{n}{\lambda_1(M)}} \nabla_u^0 \left( \frac{X}{|X|} \right) |_x = \sqrt{\frac{n}{\lambda_1(M)}} u \left( \frac{1}{|X|} \right) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} \nabla_u^0 X
\]

\[
= -\frac{1}{2} \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|^3} u(|X|^2) X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} u
\]

\[
= -\sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|^3} \langle u, X \rangle X + \sqrt{\frac{n}{\lambda_1(M)}} \frac{1}{|X|} u
\]

By a straightforward computation, we obtain

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\[ ||dF_x(u)||^2 - 1\]
\[ = \left| \frac{n}{\lambda_1(M)} |X|^2 \left(1 - \frac{(u,X)^2}{|X|^2}\right) - 1 \right| \]
\[ \leq \left| \frac{n}{\lambda_1(M)} |X|^2 - 1 \right| + \frac{n}{\lambda_1(M)} |X|^4 (u,X)^2 \] (9)

Now
\[ \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| = \frac{1}{|X|^2} \frac{n}{\lambda_1(M)} - |X|^2 \leq \varepsilon \frac{\sqrt{n/\lambda_1(M)} + |X|}{|X|^2} \leq \varepsilon \frac{2 \sqrt{n/\lambda_1(M)} + \varepsilon}{\varepsilon (\lambda_1/M - \varepsilon)^2} \]

Let us recall that \( \frac{d\varepsilon}{d\varepsilon} \leq \lambda_1(M) \leq ||B||^2 \) (see (1)) for the first inequality). Since we assume \( \varepsilon < \frac{1}{2} \sqrt{\frac{n}{||B||^2}} \), the right-hand side is bounded above by a constant depending only on \( n \) and \( ||B||^2 \) and we have
\[ \left| \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} - 1 \right| \leq \varepsilon g(n, ||B||^2) \] (10)

On the other hand, since \( C_\varepsilon(n, ||B||^2) \to 0 \) when \( \varepsilon \to 0 \), there exists \( \varepsilon(n, ||B||^2, \eta) \) so that \( C_{\varepsilon(n, ||B||^2, \eta)} \leq K_\eta(n, ||B||^2) \) (where \( K_\eta \) is the constant of the Lemma) and then by Lemma 4.1, \( ||\psi||^2 \leq \eta^2 \). Thus, there exists a constant \( \delta \) depending only on \( n \) and \( ||B||^2 \) so that
\[ \frac{n}{\lambda_1(M)} \frac{1}{|X|^2} (u,X)^2 \leq \frac{n}{\lambda_1(M)} \frac{1}{|X|^4} ||\psi||^2_\infty \leq \eta^2 \delta(n, ||B||^2) \] (11)

then from (4), (9) and (10) we deduce that the condition \( (P_{\varepsilon(n, ||B||^2, \eta)}) \) implies
\[ ||dF_x(u)||^2 - 1 \leq \varepsilon g(n, ||B||^2) + \eta^2 \delta(n, ||B||^2) \]

Now let us choose \( \eta = \left( \frac{\theta}{2n} \right)^{1/2} \). Then we can assume that \( \varepsilon(n, ||B||^2, \eta) \) is small enough in order to have \( \varepsilon(n, ||B||^2, \eta) \gamma(n, ||B||^2) \leq \frac{\theta}{2} \). In this case we have
\[ ||dF_x(u)||^2 - 1 \leq \theta \]

Now let us fix \( \theta, 0 < \theta < 1 \). It follows that \( F \) is a local diffeomorphism from \( M \) to \( S \left( O, \sqrt{\frac{n}{\lambda_1(M)}} \right) \). Since \( S \left( O, \sqrt{\frac{n}{\lambda_1(M)}} \right) \) is simply connected for \( n \geq 2 \), \( F \) is a diffeomorphism.

\[ \square \]
5 Proof of the technical Lemmas

The proofs of Lemmas 3.1 and 4.1 are providing from a result stated in the following Proposition using a Nirenberg-Moser type of proof.

**Proposition 5.1** Let \((M^n, g)\) be a compact, connected and oriented \(n\)-dimensional Riemannian manifold without boundary isometrically immersed into the \(n + 1\)-dimensional euclidean space \((\mathbb{R}^{n+1}, \text{can})\). Let \(\xi\) be a nonnegative continuous function so that \(\xi^k\) is smooth for \(k \geq 2\). Let \(0 \leq r < s \leq 2\) so that

\[
\frac{1}{2} \Delta \xi^2 \xi^{2k-2} \leq \Delta \omega + (A_1 + kA_2)\xi^{2k-r} + (B_1 + kB_2)\xi^{2k-s}
\]

where \(\Delta \omega\) is the codifferential of a 1-form and \(A_1, A_2, B_1, B_2\) are nonnegative constants. Then for any \(\eta > 0\), there exists a constant \(L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)\) depending only on \(n, A_1, A_2, B_1, B_2, \|H\|_\infty\) and \(\eta\) so that if \(\|\xi\|_\infty > \eta\) then

\[
\|\xi\|_\infty \leq L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)\|\xi\|_2
\]

Moreover, \(L\) is bounded when \(\eta \to \infty\), and if \(B_1 > 0\), \(L \to \infty\) when \(\|H\|_\infty \to \infty\) or \(\eta \to 0\).

This Proposition will be proved at the end of the paper.

Before giving the proofs of Lemmas 3.1 and 4.1 we will show that under the pinching condition \((P_C)\) with \(C\) small enough, the \(L_\infty\)-norm of \(X\) is bounded by a constant depending only on \(n\) and \(\|H\|_\infty\).

**Lemma 5.1** If we have the pinching condition \((P_C)\) with \(C < c_n\), then there exists \(E(n, \|H\|_\infty)\) depending only on \(n\) and \(\|H\|_\infty\) so that \(\|X\|_\infty \leq E(n, \|H\|_\infty)\).

**Proof:** From the relation (3), we have

\[
\frac{1}{2} \Delta |X|^2 |X|^{2k-2} \leq n\|H\|_\infty |X|^{2k-1}
\]

Then applying Proposition 5.1 to the function \(\xi = |X|\) with \(r = 0\) and \(s = 1\), we obtain that if \(\|X\|_\infty > E\), then there exists a constant \(L(n, \|H\|_\infty, E)\) depending only on \(n, \|H\|_\infty\) and \(E\) so that

\[
\|X\|_\infty \leq L(n, \|H\|_\infty, E)\|X\|_2
\]

and under the pinching condition \((P_C)\) with \(C < c_n\) we have from Lemma 2.1

\[
\|X\|_\infty \leq L(n, \|H\|_\infty, E)\frac{d_n^{1/2}}{2}
\]
Now since $L$ is bounded when $E \to \infty$, we can choose $E = E(n, \|H\|_\infty)$ great enough so that

$$L(n, \|H\|_\infty, E)d_1^{1/2} < E$$

In this case, we have $\|X\|_\infty \leq E(n, \|H\|_\infty)$.

**Proof of Lemma 3.1:** First we compute the Laplacian of the square of $\varphi^2$. We have

$$\Delta \varphi^2 = \Delta \left(|X|^4 - 2 \left(\frac{n}{\lambda_1(M)}\right)^{1/2} |X|^3 + \frac{n}{\lambda_1(M)} |X|^2\right)$$

$$= -2|X|^2|d|X|^{2} + 2|X|^2 \Delta |X|^2$$

$$- 2 \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \left(-\frac{3}{4} |X|^{-1} |d|X|^2 + \frac{3}{2} |X| \Delta |X|^2\right) + \frac{n}{\lambda_1(M)} \Delta |X|^2$$

Now by a direct computation one gets $|d|X|^2| \leq 4|X|^2$. Moreover by the relation (3) we have $|\Delta |X|^2| \leq 2n\|H\|_\infty |X| + n$. Then applying Lemmas 1.1 and 5.1 we get

$$\Delta \varphi^2 \leq \alpha(n, \|H\|_\infty)$$

and

$$\frac{1}{2} \Delta \varphi^2 \varphi^{2k-2} \leq \alpha(n, \|H\|_\infty) \varphi^{2k-2}$$

Now, we apply Proposition 5.1 with $r = 0$ and $s = 2$. Then if $\|\varphi\|_\infty > \eta$, there exists a constant $L(n, \|H\|_\infty)$ depending only on $n$ and $\|H\|_\infty$ so that

$$\|\varphi\|_\infty \leq L\|\varphi\|_2$$

From Lemma 2.3, if $C \leq c_n$ and $(PC)$ is true, we have $\|\varphi\|_2 \leq D(n)\|\varphi\|_{\infty}^{3/4} C^{1/4}$. Therefore

$$\|\varphi\|_\infty \leq (LD)^4 C$$

Consequently, if we choose $C = K_\eta = \inf \left(\eta (LD)^4, c_n\right)$, then we obtain that $\|\varphi\|_\infty \leq \eta$.

**Proof of Lemma 4.1:** First we will prove that for any $C < c_n$, if $(PC)$ is true, then

$$\frac{1}{2} (\Delta \psi^2) \psi^{2k-2} \leq \delta \omega + (\alpha_1(n, \|B\|_\infty) + k\alpha_2(n, \|B\|_\infty)) \psi^{2k-2}$$

(12)

where $\delta \omega$ is the codifferential of a 1-form $\omega$.  

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First observe that the gradient $\nabla^M|X|^2$ of $|X|^2$ satisfies $\nabla^M|X|^2 = 2X^T$. Then by the Bochner formula we get

$$\frac{1}{2}\Delta|X|^2 = \frac{1}{4}\langle \Delta d|X|^2, d|X|^2 \rangle - \frac{1}{4}\nabla \langle \nabla^M|X|^2, \nabla^M|X|^2 \rangle \leq \frac{1}{4}\langle \delta \omega, d|X|^2 \rangle - \frac{1}{4}\Ric(\nabla^M|X|^2, \nabla^M|X|^2)$$

and by the Gauss formula we obtain

$$\frac{1}{2}\Delta|X|^2 = \frac{1}{4}\langle \delta \omega, d|X|^2 \rangle - \frac{1}{4}\Ric(\nabla^M|X|^2, \nabla^M|X|^2) \leq \frac{1}{4}\langle \delta \omega, d|X|^2 \rangle - nH \langle B\nabla^M|X|^2, \nabla^M|X|^2 \rangle$$

By Lemma 5.1 we know that $\|X\|_\infty \leq E(n, \|B\|_\infty)$ (the dependance in $\|H\|_\infty$ can be replaced by $\|B\|_\infty$). Then it follows that

$$\frac{1}{2}(\Delta \psi^2)\psi^{2k-2} \leq \frac{1}{4}\langle \delta \omega, d|X|^2 \rangle \psi^{2k-2} + \alpha(n, \|B\|_\infty)\psi^{2k-2} \tag{13}$$

Now, let us compute the term $\langle \delta \omega, d|X|^2 \rangle \psi^{2k-2}$. We have

$$\langle \delta \omega, d|X|^2 \rangle \psi^{2k-2} = \delta \omega + (\Delta|X|^2)\psi^{2k-2} - (2k-2)\Delta|X|^2 \langle d|X|^2, d\psi \rangle \psi^{2k-3}$$

$$= \delta \omega + (\Delta|X|^2)\psi^{2k-2} - 2(2k-2)\Delta|X|^2 \langle X^T, \nabla^M\psi \rangle \psi^{2k-3}$$

where $\omega = -\Delta|X|^2\psi^{2k-2}d|X|^2$. Now,

$$e_i(\psi) = \frac{e_i|X|^2}{2|X|^2} = \frac{e_i|X|^2 - e_i\langle X, \nu \rangle^2}{2|X|^2} = \frac{\langle e_i, X \rangle - B_{ij}\langle X, e_j \rangle \langle X, \nu \rangle}{|X|^2}$$

Then

$$\langle \delta \omega, d|X|^2 \rangle \psi^{2k-2} = \delta \omega + (\Delta|X|^2)\psi^{2k-2} - 2(2k-2)\Delta|X|^2 \langle X^T, X^T \rangle \psi^{2k-3} + 2(2k-2)\Delta|X|^2|X^T|^2 \psi^{2k-3} \leq \delta \omega + (\Delta|X|^2)\psi^{2k-2} + 2(2k-2)|\Delta|X|^2|\psi^{2k-2} + 2(2k-2)|\Delta|X|^2||B||X|\psi^{2k-2}$$

Now by relation (3) and Lemma 5.1 we have
\begin{align*}
\langle d\Delta |X|^2, d|X|^2 \rangle \psi^{2k-2} \leq \delta \omega + (\alpha''(n, \|B\|_\infty) + k\alpha''(n, \|B\|_\infty)) \psi^{2k-2}
\end{align*}
Inserting this in (12), we obtain the desired inequality (12).
Now applying again Proposition 5.1, we get that there exists \( L(n, \|B\|_\infty, \eta) \) so that if \( \|\psi\|_\infty > \eta \) then
\[ \|\psi\|_\infty \leq L\|\psi\|_2 \]
From the Lemma 2.2 we deduce that if the pinching condition \((P_c)\) holds then \( \|\psi\|_2 \leq A(n)^{1/2}C^{1/2} \). Then taking \( C = K_\eta = \inf \left( \frac{\eta}{L, a} \right) \), then \( \|\psi\|_\infty \leq \eta \).

**Proof of Lemma 3.2:** The idea of the proof consists in foliating the region \( B(O, R + \eta) \setminus B(O, R - \eta) \) with hypersurfaces of large mean curvature and to show that one of these hypersurfaces is tangent to \( \phi(M) \). This will imply that \( \phi(M) \) has a large mean curvature at the contact point.

Consider \( S^{n-1} \subset \mathbb{R}^n \) and \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}e \). Let \( a, L > l > 0 \) and
\[ \Phi_{L,l,a} : S^{n-1} \times S^1 \longrightarrow \mathbb{R}^{n+1} \]
\[ (\xi, \theta) \longrightarrow L\xi - l \cos \theta \xi + l \sin \theta e = ae \]
Then \( \Phi_{L,l,a} \) is a family of embeddings from \( S^{n-1} \times S^1 \) in \( \mathbb{R}^{n+1} \). If we orient the family of hypersurfaces \( \Phi_{L,l,a}(S^{n-1} \times S^1) \) by the unit outward normal vector field, a straightforward computation shows that the mean curvature \( H(\theta) \) depends only on \( \theta \) and we have
\[ H(\theta) = \frac{1}{n} \left( \frac{1}{l} - \frac{(n - 1) \cos \theta}{L - l \cos \theta} \right) \geq \frac{1}{n} \left( \frac{1}{l} - \frac{n - 1}{L - l} \right) \tag{14} \]
Now, let us consider the hypotheses of the Lemma and for \( t_0 = 2 \arcsin \left( \frac{\rho}{2R} \right) \leq t \leq \frac{\pi}{2} \), put \( L = R \sin t, l = 2\eta \) and \( a = R \cos t \). Then \( L > l \) and we can consider for \( t_0 \leq t \leq \frac{\pi}{2} \) the family \( \mathcal{M}_{R,t} \) of hypersurfaces defined by \( \mathcal{M}_{R,t} = \Phi_{R \sin t, 2\eta, R \cos t}(S^{n-1} \times S^1) \).

From the relation (14), the mean curvature \( H_{R,t} \) of \( \mathcal{M}_{R,t} \) satisfies
\[ H_{R,t} \geq \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n - 1}{R \sin t - 2\eta} \right) \geq \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n - 1}{R \sin t_0 - 2\eta} \right) \geq \frac{1}{n} \left( \frac{1}{2\eta} - \frac{n - 1}{\frac{\rho}{2R} - 2\eta} \right) = \frac{1}{4n\eta} \]
where we have used in this last equality the fact that \( \rho = 4(2n - 1)\eta \).
Since there exists a point \( p \in M \) so that \( \langle X(p), e \rangle > 0 \), we can find \( t \in [t_0, \pi/2] \) and a point \( y_0 \in M \) which is a contact point with \( \mathcal{M}_{R,t} \). Therefore \( |H(y_0)| \geq \frac{1}{4n\eta} \).
Proof of Proposition 5.1: Integrating by parts we have

\[
\int_M \frac{1}{2} \Delta \xi^2 \xi^{2k-2} dv = \frac{1}{2} \int_M \langle d\xi^2, d\xi^{2k-2} \rangle dv = 2 \left( \frac{k-1}{k^2} \right) \int_M |d\xi^k|^2 dv
\]

\[
\leq (A_1 + kA_2) \int_M \xi^{2k-r} dv + (B_1 + kB_2) \int_M \xi^{2k-s} dv
\]

Now, given a smooth function \( f \) and applying the Sobolev inequality (5) to \( f^2 \), we get

\[
\left( \int_M f^{2n} dv \right)^{1-\frac{1}{n}} \leq K(n) \int_M (2|df| + |H|f^2) dv
\]

\[
\leq 2K(n) \left( \int_M f^2 dv \right)^{1/2} \left( \int_M |df|^2 dv \right)^{1/2} + K(n)\|H\|_\infty \int_M f^2 dv
\]

\[
= K(n) \left( \int_M f^2 dv \right)^{1/2} \left( 2 \left( \int_M |df|^2 dv \right)^{1/2} + \|H\|_\infty \left( \int_M f^2 dv \right)^{1/2} \right)
\]
where in the second inequality, we have used the Hölder inequality. Using it again, by assuming that \( V(M) = 1 \), we have

\[
\left( \int_M f^2 \text{d}v \right)^{1/2} \leq \left( \int_M f^{\frac{2n}{n-1}} \text{d}v \right)^{\frac{n-1}{2n}}
\]

And finally, we obtain

\[
\|f\|_{\frac{2n}{n-1}} \leq K(n) \left( 2\|df\|_2 + \|H\|_{\infty} \|f\|_2 \right)
\]

For \( k \geq 2 \), \( \xi^k \) is smooth and we apply the above inequality to \( f = \xi^k \). Then we get

\[
\|\xi\|_{\frac{2k}{n-1}} \leq K(n) \left[ 2 \left( \int_M |d\xi^k|^2 \text{d}v \right)^{1/2} + \|H\|_{\infty} \left( \int_M \xi^{2k} \text{d}v \right)^{1/2} \right]
\]

\[
\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( (A_1 + kA_2) \int_M \xi^{2k-r} \text{d}v + (B_1 + kB_2) \int_M \xi^{2k-s} \text{d}v \right)^{1/2} \right]
\]

\[
+ \|H\|_{\infty} \left( \int_M \xi^{2k} \text{d}v \right)^{1/2} \right]
\]

\[
\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( (A_1 + kA_2) \|\xi\|_\infty^{2-r} + (B_1 + kB_2) \|\xi\|_\infty^{2-s} \right)^{1/2} \left( \|\xi\|_{\frac{k-1}{2k-2}} \right) \right]
\]

\[
+ \|H\|_{\infty} \|\xi\|_\infty \left( \|\xi\|_{\frac{k-1}{2k-2}} \right) \right]
\]

\[
\leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( \frac{A_1}{\|\xi\|_\infty^{r/2}} + \frac{B_1 + kB_2}{\|\xi\|_\infty^{s/2}} \right) \right]
\]

\[
\|\xi\|_{\frac{k}{n-1}} \leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( \frac{A_1^{1/2} + k^{1/2} A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2} + k^{1/2} B_2^{1/2}}{\eta^{s/2}} \right) \right]
\]

Now if we assume that \( \|\xi\|_\infty > \eta \), the last inequality becomes

\[
\|\xi\|_{\frac{k}{n-1}} \leq K(n) \left[ 2 \left( \frac{k^2}{2(k-1)} \right)^{1/2} \left( \frac{A_1^{1/2} + k^{1/2} A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2} + k^{1/2} B_2^{1/2}}{\eta^{s/2}} \right) \right]
\]

\[
+ \|H\|_{\infty} \|\xi\|_\infty \|\xi\|_{\frac{k-1}{2k-2}} \right]
\]
\[
\left[ (K_1 + k^{1/2} K_2) \left( \frac{k^2}{k - 1} \right)^{1/2} + K' \right] \|\xi\|_\infty \|\xi\|_2^{k-2}.
\]

Now let \( q = \frac{n}{n-1} > 1 \) and for \( i \geq 0 \) let \( k = q^i + 1 \geq 2 \). Then

\[
\|\xi\|_{2(q^{i+1}+q)} \leq \left( (K_1 + (q^i + 1)^{1/2} K_2) \left( \frac{q^i + 1}{q^{i/2}} \right) + K'' \right)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_2^{1-q^{-i+1}}
\]

where \( K = 2K_1 + 2^{3/2} K_2 + K' \). We see that \( K \) has a finite limit when \( \eta \to \infty \) and if \( B_1 > 0 \), \( K \to \infty \) when \( \|H\|_\infty \to \infty \) or \( \eta \to 0 \). Moreover the Hölder inequality gives

\[
\|\xi\|_{2q^{i+1}} \leq \|\xi\|_{2(q^{i+1}+q)}
\]

which implies

\[
\|\xi\|_{2q^{i+1}} \leq \left( K q^i \right)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_2^{1-q^{i+1}}
\]

Now, by iterating from 0 to \( i \), we get

\[
\|\xi\|_{2q^{i+1}} \leq \left( K q^i \right)^{\frac{1}{q^{i+1}}} \|\xi\|_\infty^{\frac{1}{q^{i+1}}} \|\xi\|_2^{1-q^{i+1}}
\]

Let \( \alpha = \sum_{k=0}^{\infty} k q^{-k} \) and \( \beta = \prod_{k=0}^{\infty} \left( 1 - \frac{1}{q^{k+1}} \right) = \prod_{k=0}^{\infty} \left( \frac{1}{1+(1/q)^k} \right) \). Then

\[
\|\xi\|_\infty \leq \tilde{K}^{1-\beta} q^\alpha \|\xi\|_\infty^{(1-\beta)} \|\xi\|_2^\beta
\]

and finally

\[
\|\xi\|_\infty \leq L \|\xi\|_2
\]

where \( L = \tilde{K}^{1-\beta} q^\alpha \) is a constant depending only on \( n, A_1, A_2, B_1, B_2, \|H\|_\infty \) and \( \eta \). From classical methods we show that \( \beta \in [e^{-n}, e^{-n/2}] \). In particular, \( 0 < \beta < 1 \) and we deduce that \( L \) is bounded when \( \eta \to \infty \) and \( L \to \infty \) when \( \|H\|_\infty \to \infty \) or \( \eta \to 0 \) with \( B_1 > 0 \).

**Remark** In [12] and [13] Shihohama and Xu have proved that if \((M^n, g)\) is a compact \( n \)-dimensional Riemannian manifold without boundary isometrically immersed in \( \mathbb{R}^{n+1} \)
and if \( \int_M (|B|^2 - n|H|^2) < D_n \) where \( D_n \) is a constant depending on \( n \), then all Betti numbers are zero. For \( n = 2 \), \( D_2 = 4\pi \), and it follows that if

\[
\int_M |B|^2 dv - 4\pi < \lambda_1(M)V(M)
\]

then we deduce from the Reilly inequality \( \lambda_1(M)V(M) \leq 2 \int_M H^2 dv \) that \( \int_M (|B|^2 - 2|H|^2) dv < 4\pi \) and by the result of Shihohama and Xu \( M \) is diffeomorphic to \( S^2 \).

**References**


