-trees and laminations for free groups I: Algebraic laminations

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1 Introduction

This paper is the first of a sequence of three papers, where the concept of an $\mathbb{R}$-tree dual to (the lift to the universal covering of) a measured geodesic lamination $\mathfrak{L}$ in a hyperbolic surface S is generalized to arbitrary $\mathbb{R}$-trees provided with a (very small) action of the free group F_N of finite rank $N \geq 2$ by isometries.

In [CHL-II] to any such $\mathbb{R}$-tree T a *dual algebraic lamination* $L^2(T)$ is associated in a meaningful way, and in [CHL-III] we consider invariant measures (called *currents*) μ on $L^2(T)$ and investigate the induced *dual metric* d_μ on T.

In this first paper we define and study the basic tools for the two subsequent papers: *laminations* in the free group F_N. We will use three different approaches, *algebraic laminations* L^2, *symbolic laminations* $L_{\mathcal{A}}$, and *laminary languages* $\mathcal{L}$. Each of them will be explained in detail, and each has its own virtues. Algebraic laminations do not need a specified basis of F_N and are hence of conceptional superiority. The other two objects are concretely defined in terms of infinite words (for symbolic laminations) or of finite words (for laminary languages) in a fixed basis $\mathcal{A}$. They are more practical for many tasks: Symbolic laminations are more suited for dynamical and laminary languages more for combinatorial purposes. The set of each of these three objects come naturally with a topology, a partial order, and an action by homeomorphisms of the group $\mathrm{Out}(F_N)$ of outer automorphisms of F_N. We will prove that the three approaches are equivalent:

Theorem 1.1. *Let F_N denote the free group of finite rank $N \geq 2$, and let $\mathcal{A}$ be a basis of F_N. There are canonical $Out(F_N)$-equivariant, order preserving homeomorphisms*

$$\Lambda^2(F_N) \longleftrightarrow \Lambda_{\mathcal{A}} \longleftrightarrow \Lambda_{\mathcal{L}}(\mathcal{A})$$

between the space $\Lambda^2(F_N)$ of algebraic laminations in F_N, the space $\Lambda_{\mathcal{A}}$ of symbolic laminations in $\mathcal{A}^{\pm 1}$, and the space $\Lambda_{\mathcal{L}}(\mathcal{A})$ of laminary languages in $\mathcal{A}^{\pm 1}$.

Symbolic laminations are subshifts (= symbolic flows) as classically used in symbolic dynamics, except that we work with the free group $F_N = F(\mathcal{A})$ rather than with the free monoid $\mathcal{A}^*$. Similarly, laminary languages over the alphabet $\mathcal{A}$ rather than $\mathcal{A}^{\pm 1} = \mathcal{A} \cup \mathcal{A}^{-1}$ are already studied in combinatorics, compare for instance [Nar96].

As in the surface case, the subset $\Lambda_{\mathrm{rat}} \subset \Lambda^2(F_N)$ of rational laminations, each corresponding to a finite collection of non-trivial conjugacy classes in F_N (see §2), is of special interest. Contrary to the analogous statement for measured laminations on a surface, or for currents on F_N (compare [Mar95]), we obtain in the setting of algebraic laminations:

Theorem 1.2. *Rational laminations are not dense in $\Lambda^2(F_N)$. However, the closure $\overline{\Lambda}_{rat}$ contains all minimal laminations.*

Algebraic laminations, as defined and studied in this paper, have three direct "ancesters", all three of them inspired by geodesic laminations on surfaces: In [Lus92] *combinatorial laminations* are defined to study decomposable automorphisms of F_N, in [BFH00] an *attracting lamination* is associated to each exponential stratum of an automorphism of F_N (see §2), and in [LL03] a kind of laminations is associated to certain $\mathbb{R}$-tree actions of F_N.

This paper (as well as the subsequent ones [CHL-II] and [CHL-III]) is a further attempt to bridge the "cultural gap" between two mathematical communities: symbolic and combinatorial dynamics on one hand, and geometric group theory on the other. Notice that in geometric group theory the notion of an algebraic lamination extends naturally to the more general setting of word-hyperbolic groups.

We hope to have given enough detail to carry along the novice reader from the "other" mathematical subculture, and not too much to bore the expert reader from "this" one.

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2 Algebraic laminations

Let F_N denote the free group of finite rank $N \geq 2$, and let ∂F_N denote its Gromov boundary, as usual equipped with the action of F_N (from the left) and with Gromov's topology at infinity, which gives ∂F_N the topology of a Cantor set. The choice of a basis $\mathcal{A}$ of F_N allows us to identify the elements of F_N with reduced words $w = x_1 x_2 \ldots x_n$ (with $x_{i+1} \neq x_i^{-1}$) in $\mathcal{A} \cup \mathcal{A}^{-1}$, and thus defines in particular the length function $w \mapsto |w|_{\mathcal{A}} = n$ on F_N. This length function induces the *word metric* $d_{\mathcal{A}}(v, w) = |v^{-1}w|_{\mathcal{A}}$ on F_N, which in turn defines a metric on $\partial F_N = \{x_1 x_2 x_3 \ldots \mid x_i \in \mathcal{A}^{\pm 1}, x_{i+1} \neq x_i^{-1}\}$, stated explicitely in §6.

Choosing another basis gives rise to a Lipschitz-equivalent metric on F_N and to a Hölder-equivalent metric on ∂F_N (compare [GdlH90]). As a consequence, the topology on $F_N \cup \partial F_N$ induced by the word metric does not depend on the choice of the basis $\mathcal{A}$. More details are given below in §8. Note that $F_N \cup \partial F_N$ as well as ∂F_N are compact spaces, and that every F_N-orbit in ∂F_N is dense.

For any element $w \neq 1$ of F_N we denote by $w^{+\infty}$ the limit in ∂F_N of the sequence $(w^n)_{n\in\mathbb{N}}$ and by $w^{-\infty}$ that of $(w^{-n})_{n\in\mathbb{N}}$. If $w = x_1 \dots x_p \cdot y_1 \dots y_q \cdot x_p^{-1} \dots x_1^{-1}$ is a reduced word in $\mathcal{A}^{\pm 1}$, with $y_q \neq y_1^{-1}$, then

$$w^{+\infty} = x_1 \dots x_p \cdot y_1 \dots y_q \cdot y_1 \dots y_q \cdot y_1 \dots y_q \cdot \dots$$

Following standard notation (see for example [Kap04, Kap03]), we define

$$\partial^2 F_N = \partial F_N \times \partial F_N \smallsetminus \Delta\,,$$

where Δ denotes the diagonal in $\partial F_N \times \partial F_N$. It follows directly that $\partial^2 F_N$ inherits from ∂F_N a topology and an F_N-action, given by $w(X, Y) = (wX, wY)$ for any $w \in F_N$ and any $X, Y \in \partial F_N$ with $X \neq Y$. The set $\partial^2 F_N$ admits also the *flip* involution $(X, Y) \mapsto (Y, X)$, which is an F_N-equivariant homeomorphism. Note that $\partial^2 F_N$ is not compact.

Definition 2.1. An *algebraic lamination* is a subset L^2 of $\partial^2 F_N$ which is non-empty, closed, symmetric (= flip invariant) and F_N-invariant. The set of all algebraic laminations is denoted by $\Lambda^2 = \Lambda^2(F_N)$.

The set Λ^2 of algebraic laminations inherits naturally a Hausdorff topology from $\partial^2 F_N$ which we will discuss in §6.

In [BFH00], M. Bestvina, M. Feighn and M. Handel associate an attracting lamination to each exponential stratum of an automorphism of F_N. These laminations are laminations in our sense. However, in [BFH00] there is no topology introduced on the space of laminations but rather only on $\partial^2 F_N$, and even there, their topology differs slighty from ours.

An important special class of algebraic laminations are the *rational* laminations, which are finite unions of *minimal rational* laminations $L(w)$, defined for any $w \in F_N \smallsetminus \{1\}$ by:

$$L(w) = \{(vw^{-\infty}, vw^{+\infty}) \mid v \in F_N\} \cup \{(vw^{+\infty}, vw^{-\infty}) \mid v \in F_N\}$$

Note that the lamination $L(w)$ depends only on the conjugacy class of w. We denote by Λ_{rat} the subspace of rational laminations. The Hausdorff topology on Λ^2 is stronger than one might intuitively expect. In particular on obtains the following result, proved in §6:

Proposition 2.2. *The subset Λ_{rat} is not dense in Λ^2.*

We observe that there is a natural (left) action of $\mathrm{Out}(F_N)$ on Λ^2, induced by the action of $\mathrm{Aut}(F_N)$ on ∂F_N. Indeed, an automorphism of F_N is a bi-Lipschitz homeomorphism on F_N and extends continuously to the boundary. Inner automorphisms act by left-multiplication on the boundary and thus trivially on the space Λ^2 of algebraic laminations (as the latter are F_N-invariant subsets of $\partial^2 F_N$). More details about the $\mathrm{Out}(F_N)$-action on Λ^2 will be given in §8.

Note that this action restricts to an action of $\mathrm{Out}(F_N)$ on the space of rational laminations Λ_{rat}: If α is an automorphism of F_N and $\widehat{\alpha}$ its class in the outer automorphism group $\mathrm{Out}(F_N)$ and, if w is an element of F_N, $\widehat{\alpha}(L(w)) = L(\alpha(w))$.

To stimulate the interest of the reader in these rather delicate matters we would like to pose here a question which is inspired by the thesis of R. Martin [Mar95]:

Question 2.3. Let $\mathcal{A}$ be any basis of F_N, and fix $a \in \mathcal{A}$ arbitrarily. Is the closure $\overline{\mathrm{Out}(F_N)L(a)}$ of the $\mathrm{Out}(F_N)$-orbit of $L(a)$ a minimal closed $\mathrm{Out}(F_N)$-invariant non-empty subset of Λ^2 ? If so, is it the unique such *minimal* set?

An answer to this question will be given in Proposition 8.2. Note that if $N = 2$ and $\{a, b\}$ is a basis of F_2 and $[a, b] = a^{-1}b^{-1}ab$, then it is well known that for any automorphism α of F_N, $\alpha([a, b])$ is conjugated to either $[a, b]$ or its inverse. Therefore $L([a, b])$ is a global fixed point of the action of $\mathrm{Out}(F_N)$ on Λ.

3 Surface laminations

An important class of algebraic laminations comes from geodesic laminations on hyperbolic surfaces. The discussion started below, to compare algebraic laminations in general with laminations on surfaces, is carried further in [CHL-II] and [CHL-III]. Throughout this section we assume a certain familiarity of the reader with this subject; for background see for example [CB88] and [FLP91]. Note that this section can be skipped by the reader without loss on the intrinsic logics of the material presented in this paper.

Let S be a hyperbolic surface with non-empty boundary and negative Euler characteristic, and fix an identification $\pi_1 S = F_N$. The surface S is provided with a hyperbolic structure, given by an identification of the universal covering $\widetilde{S}$ with a convex part of the hyperbolic plane $\mathbb{H}^2$, which realizes the deck transformation action of $F_N = \pi_1 S$ on $\widetilde{S}$ by hyperbolic isometries. Let $\mathfrak{L}$ be a geodesic lamination on S and let $\widetilde{\mathfrak{L}}$ be the (full) lift of $\mathfrak{L}$ to the universal covering $\widetilde{S}$ of S. The induced identification (an F_N-equivariant homeomorphism!) between ∂F_N and the boundary at infinity $\partial\widetilde{S}$ of $\widetilde{S}$ defines for any leaf l of $\widetilde{\mathfrak{L}}$ a pair of endpoints $(X, Y) \in \partial^2 F_N$, as well as its flipped pair (Y, X). The set of all such pairs is easily seen to define (via the above identification $\partial F_N = \partial\widetilde{S}$) an algebraic lamination $L^2(\mathfrak{L}) \in \Lambda^2(F_N)$.

Definition 3.1. An algebraic lamination $L^2 \in \Lambda^2(F_N)$ is called an *algebraic surface lamination* if there exists a hyperbolic surface S and an identification $\pi_1 S = F_N$ such that for some geodesic lamination $\mathfrak{L}$ on S one has:

$$L^2 = L^2(\mathfrak{L})$$

At first guess it may seem that the space $\Lambda^2(F_N)$ is a rather weak analogue of the space of geodesic laminations in a surface. Notice however that, if $L^2 \in \Lambda^2(F_N)$ is an algebraic surface lamination with respect to an isomorphism $\pi_1 S_1 = F_N$ for some surface S_1, and if S_2 is a second surface with identification $\pi_1 S_2 = F_N$, then typically a biinfinite geodesic on S_2, which realises an element of L^2, will self-intersect: Thus L^2 does not admit a realization as geodesic lamination on S_2.

4 Symbolic laminations

To a basis $\mathcal{A}$ there is naturally associated the space $\Sigma_{\mathcal{A}}$ of biinfinite reduced words Z in $\mathcal{A} \cup \mathcal{A}^{-1}$ with letters indexed by $\mathbb{Z}$:

$$\Sigma_{\mathcal{A}} = \{Z = \ldots z_{i-1} z_i z_{i+1} \ldots \mid z_i \in \mathcal{A} \cup \mathcal{A}^{-1}, z_i \neq z_{i+1}^{-1} \text{ for all } i \in \mathbb{Z}\}.$$

We want to stress that in this paper a biinfinite word comes always with a $\mathbb{Z}$-indexing, i.e. formally speaking, a biinfinite word is a map $Z : \mathbb{Z} \to \mathcal{A} \cup \mathcal{A}^{-1}$. For example, the non-indexed "biinfinite word"

$$\ldots ababab \ldots$$

becomes a biinfinte word Z only after specifying $z_1 = a$ or $z_1 = b$, which we indicate notationally by writing $Z = \ldots bab \cdot aba \ldots$ or $Z = \ldots aba \cdot bab \ldots$ respectively.

As usual, $\Sigma_{\mathcal{A}}$ comes with a canonical infinite cartesian product topology that makes it a Cantor set, and with a shift operator $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, given by

$$\sigma(Z) = Z',$$

where $Z = \ldots z_{i-1} z_i z_{i+1} \ldots$ and $Z' = \ldots z'_{i-1} z'_i z'_{i+1} \ldots$ with $z'_i = z_{i+1}$. Of course, σ is a homeomorphism.

For each biinfinite word $Z = \ldots z_{i-1} z_i z_{i+1} \ldots$ we denote its *inverse* by

$$Z^{-1} = \ldots z'_{i-1} z'_i z'_{i+1} \ldots, \text{ where } z'_i = (z_{1-i})^{-1}.$$

Again, the inversion map $\Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$, $Z \mapsto Z^{-1}$ is easily seen to be a homeomorphism. A subset L of $\Sigma_{\mathcal{A}}$ is called *symmetric* if $L = L^{-1}$.

Definition 4.1. A *symbolic lamination* in $\mathcal{A}^{\pm 1}$ is a non-empty subset $L_{\mathcal{A}} \subset \Sigma_{\mathcal{A}}$ which is closed, symmetric and σ-invariant. Together with the restriction of σ to $L_{\mathcal{A}}$ (which we continue to call σ) it is a *symbolic flow*. The elements of a symbolic lamination are sometimes called the *leaves* of the lamination. We denote the set of symbolic laminations in $\mathcal{A}^{\pm 1}$ by $\Lambda_{\mathcal{A}}$.

In symbolic dynamist's terminology, any symbolic lamination is a subshift of the subshift of finite type on the alphabet $\mathcal{A} \cup \mathcal{A}^{-1}$ which consists of all biinfinite reduced words.

As $\Sigma_{\mathcal{A}}$ is compact and symbolic laminations are closed, we get:

Lemma 4.2. *The intersection of a decreasing sequence*

$$L_{\mathcal{A}} \supset L'_{\mathcal{A}} \supset L''_{\mathcal{A}} \supset \dots$$

of symbolic laminations is a symbolic lamination. In particular it is non-empty. $\square$

Once the basis $\mathcal{A}$ is fixed, every boundary point $X \in \partial F_N$ corresponds canonically to a reduced, (one-sided) infinite word $X = x_1 x_2 \dots$ with letters in $\mathcal{A}^{\pm 1}$. For such a (one-sided) infinite word X we denote by X_n its prefix (= initial subword) of length n. For every pair $(X, Y) \in \partial^2 F_N$ we define a biinfinite reduced word

$$X^{-1}Y = \dots x_{k+2}^{-1} x_{k+1}^{-1} \cdot y_{k+1} y_{k+2} \cdots ,$$

where $X_k = x_1 x_2 \dots x_k = y_1 y_2 \dots y_k = Y_k$ is the longest common prefix of X and Y.

There is a subtlety in the last definition which we would like to point out: Although for any $X \neq Y \in \partial F_N$ the biinfinite (indexed) word $X^{-1}Y$ is well defined by our above definition, this particular way to associate the indices from $\mathbb{Z}$ to the non-indexed "biinfinite word" $\dots x_{k+2}^{-1} x_{k+1}^{-1} y_{k+1} y_{k+2} \dots$ is really in no way canonical, and often it does not behave quite naturally, in particular with respect to the action of $\mathrm{Aut} F_N$. Indeed, a biinfinite symbol sequence, contrary to a finite or a one-sided infinite one, doesn't really come by nature with a canonical indexing, but rather corresponds to the whole σ-orbit of a biinfinite word in $\Sigma_{\mathcal{A}}$. Nevertheless one obtains as direct consequence of the definitions:

Remark 4.3. The map $\rho_{\mathcal{A}} : \begin{array}{ccc} \partial^2 F_N & \to & \Sigma_{\mathcal{A}} \\ (X, Y) & \mapsto & X^{-1}Y \end{array}$ is continuous.

We note that the biinfinite indexed word from $\Sigma_{\mathcal{A}}$ associated via $\rho_{\mathcal{A}}$ to $w(X, Y)$, for any $w \in F_N$, can differ from the indexed word $X^{-1}Y$ only by an index shift. Conversely, for the pair $(X, Y) \in \partial^2 F_N$ with maximal common initial subword $X_k = Y_k$ as above, the map $\rho_{\mathcal{A}}$ associates the biinfinite indexed word $\sigma^m(X^{-1}Y)$ to the pair $Y_{k+m}^{-1}(X, Y)$ for $m \geq 0$, and to $X_{k-m}^{-1}(X, Y)$ for $m \leq 0$.

Hence the map $\rho_{\mathcal{A}}$ maps every F_N-orbit in $\partial^2 F_N$ onto a σ-orbit in $\Sigma_{\mathcal{A}}$, and thus induces a well defined map from F_N-orbits in $\partial^2 F_N$ to σ-orbits in $\Sigma_{\mathcal{A}}$. It is easy to see that this map between orbits is bijective, and that, moreover, this bijection respects the topology on both sides: Closed sets of F_N-orbits are mapped to closed sets of σ-orbits, and conversely. Finally, we note that the flip on $\partial^2 F_N$ corresponds to the inversion of biinfinite words in $\Sigma_{\mathcal{A}}$.

Thus, given $L^2 \in \Lambda^2$, we can define a symbolic lamination $L_{\mathcal{A}}$ by

$$L_{\mathcal{A}} = \rho_{\mathcal{A}}(L^2) = \{X^{-1}Y \mid (X, Y) \in L^2\}.$$

Conversely, given a symbolic lamination $L_{\mathcal{A}}$ as above, one obtains an algebraic lamination $L^2 = \rho_{\mathcal{A}}^{-1}(L_{\mathcal{A}})$ which consists of all pairs $w(Z_-, Z_+)$, for all

$w \in F_N$, and all $Z = \ldots z_{i-1}z_iz_{i+1}\ldots \in L_{\mathcal{A}}$ with associated right-infinite words $Z_- = z_0^{-1}z_{-1}^{-1}z_{-2}^{-1}\ldots$ and $Z_+ = z_1z_2\ldots$.

We summarize the above discussion:

Proposition 4.4. *For any basis $\mathcal{A}$ of the free group F_N, the maps $L^2 \mapsto L_{\mathcal{A}} = \rho_{\mathcal{A}}(L^2)$ and $L_{\mathcal{A}} \mapsto L^2 = \rho_{\mathcal{A}}^{-1}(L_{\mathcal{A}})$ define a bijection*

$$\rho_{\mathcal{A}}^2 : \Lambda^2(F_N) \to \Lambda_{\mathcal{A}}$$

between the set $\Lambda^2(F_N)$ of algebraic laminations L^2 and the set $\Lambda_{\mathcal{A}}$ of symbolic laminations $L_{\mathcal{A}}$ in $\mathcal{A}^{\pm 1}$. □

The map $\rho_{\mathcal{A}}^2$ respects the partial order given on algebraic or symbolic laminations by the inclusion as subsets of $\partial^2 F_N$ or $\Sigma_{\mathcal{A}}$ respectively. In particular, a minimal lamination $L_{\mathcal{A}}$ (or L^2) with respect to this partial order is precisely given by the analogous property that characterizes classically *minimal* symbolic flows: Every $< \sigma, (\cdot)^{-1} >$-orbit (or $< F_N, \mathrm{flip} >$-orbit, respectively) is dense in the lamination. Moreover, we note that Lemma 4.2 holds for algebraic laminations.

In order to connect the content (and also the notations) introduced in this section to the already existing notions in symbolic dynamics, we note:

A symbolic flow $\sigma : \Sigma_0 \to \Sigma_0$ in the "classical sense", i.e. a symbolic flow only on the letters of $\mathcal{A}$ (and not of $\mathcal{A}^{-1}$), gives directly rise to a symbolic lamination $L_{\mathcal{A}}(\Sigma_0) = \Sigma_0 \cup {\Sigma_0}^{-1} \in \Lambda_{\mathcal{A}}$. Conversely, a symbolic lamination $L_{\mathcal{A}} \in \Lambda_{\mathcal{A}}$ or a symbolic flow $\sigma : L_{\mathcal{A}} \to L_{\mathcal{A}}$ is called *orientable* if L can be written as disjoint union $L_{\mathcal{A}} = L_+ \cup L_+^{-1}$ of two σ-invariant closed subsets L_+ and L_+^{-1} that are inverses of each other, and it is called *positive* if one of them, say L_+, only uses letters from $\mathcal{A}$ (and not from $\mathcal{A}^{-1}$).

Remark 4.5. The fact that the laminations considered are positive is crucial for many of the traditional approaches and methods of symbolic dynamics. Similarly, for laminations (or foliations) on surfaces, almost always one first considers the orientable case and later tries to pass to the general situation via branched coverings. Note that in the context of free groups considered here any such attempt would miss most of the typical phenomena, and that hence struggling with the general kind of non-orientable laminations seems unavoidable. For an interesting case of such an encounter of the free group environment with the "already existing culture" in the context of the Rauzy fractal see [ABHS05].

5 Laminary languages

As before, we fix a basis $\mathcal{A}$ of F_N, and we denote by $F(\mathcal{A})$ the set of reduced words in $\mathcal{A}^{\pm 1}$. Although there is a canonical identification between F_N and $F(\mathcal{A})$, it is helpful in the context of this section to think of the elements of $F(\mathcal{A})$ as words and not as group elements.

Definition 5.1. Let S be any (finite or infinite) set of finite, one-sided infinite or biinfinite reduced words in $\mathcal{A}^{\pm 1}$. We denote by $\mathcal{L}(S) \subset F(\mathcal{A})$ the *language* generated by S, i.e. the set of all finite subwords (= *factors*) of any element of S. Moreover, for any integer n we denote by $\mathcal{L}_n(S)$ the subset of $\mathcal{L}(S)$ consisting of words of length smaller or equal to n.

We specially have in mind the language associated to a (symbolic) lamination. We thus abstractly define laminary languages which are in one-to-one correspondence with (symbolic) laminations.

Definition 5.2. A non-empty set $\mathcal{L} \subset F(\mathcal{A})$ of finite reduced words in $\mathcal{A}^{\pm 1}$ is a *laminary language* if it is (i) symmetric, (ii) factorial and (iii) bi-extendable. By this we mean that it is closed with respect to (i) inversion, (ii) passing to subwords, and (iii) that for any word $u \in \mathcal{L}$ there exists a word $v \in \mathcal{L}$ in which u occurs as subword other than as prefix or as suffix: $v = wuw'$ is a reduced product, with nontrivial $w, w' \in F(\mathcal{A})$. We denote by $\Lambda_{\mathcal{L}} = \Lambda_{\mathcal{L}}(\mathcal{A})$ the set of laminary languages over a fixed basis $\mathcal{A}$.

It is obvious from the definition that the set $\Lambda_{\mathcal{L}}$ is closed under (possibly infinite) unions in $F(\mathcal{A})$, and also under nested intersections (compare with Lemma 4.2). Note that the analogy of the former statement, for symbolic laminations rather than laminary languages, is false: An infinite union of symbolic laminations will in general not be a symbolic lamination; one first needs to take again the closure in $\Sigma_{\mathcal{A}}$. Note also that for any symbolic lamination $L_{\mathcal{A}} \subset \Sigma_{\mathcal{A}}$ the language $\mathcal{L}(L_{\mathcal{A}})$ is laminary.

For an infinite language $\mathcal{L} \subset F(\mathcal{A})$, we denote by $L(\mathcal{L})$ the set of all biinfinite words from $\Sigma_{\mathcal{A}}$ whose finite subwords are subwords of elements from $\mathcal{L} \cup \mathcal{L}^{-1}$. As $\mathcal{L}$ is infinite (hence in particular, if $\mathcal{L}$ is a laminary language), the definition enforces that $L(\mathcal{L})$ is not empty. It follows directly that $L(\mathcal{L})$ is indeed a symbolic lamination. We thus obtain a one-to-one correspondence between symbolic laminations and laminary languages (always for a fixed basis $\mathcal{A}$ of F_N): For any symbolic lamination $L_{\mathcal{A}}$ one has

$$L(\mathcal{L}(L_{\mathcal{A}})) = L_{\mathcal{A}}\,,$$

and conversely, for any laminary language $\mathcal{L}$ one has

$$\mathcal{L}(L(\mathcal{L})) = \mathcal{L}\,.$$

Moreover, a language $\mathcal{L}$ is laminary if and only if it is infinite, and if the last equation holds. For any set S of finite, one-sided infinite or biinfinite reduced words in $\mathcal{A}^{\pm 1}$, where we assume that S is infinite in case $S \subset F(\mathcal{A})$, we observe that $\mathcal{L}(L(\mathcal{L}(S)))$ is the largest laminary language contained in $\mathcal{L}(S)$. We call $L(\mathcal{L}(S))$ the symbolic lamination and $\mathcal{L}(L(\mathcal{L}(S)))$ the laminary language *generated* by S. We summarize this discussion:

Proposition 5.3. *For any finite alphabet $\mathcal{A}$ the maps $L_{\mathcal{A}} \mapsto \mathcal{L}(L_{\mathcal{A}})$ and $\mathcal{L} \mapsto L(\mathcal{L})$ define a bijection*

$$\rho^{\mathcal{A}}_{\mathcal{L}} : \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{L}}$$

between the set $\Lambda_{\mathcal{A}}$ of symbolic laminations $L_{\mathcal{A}}$ and the set $\Lambda_{\mathcal{L}}$ of laminary languages $\mathcal{L}$ in $\mathcal{A}^{\pm 1}$. $\square$

As in Proposition 4.4, the bijection $\rho_{\mathcal{L}}^{\mathcal{A}} : \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{L}}$ respects the partial order given by the inclusion.

To enforce the link between symbolic laminations and their laminary languages we introduce the following notation and state the following lemma, which will be used in the sequel: For any integer $k \geq 0$ and any reduced word $w = x_1 x_2 \dots x_n \in F(\mathcal{A})$ denote by $w\dagger_k$ ("w chop k") the word

(a) $w\dagger_k = 1$, if $|w| \leq 2k$, and

(b) $w\dagger_k = x_{k+1} x_{k+2} \dots x_{n-k}$, if $|w| > 2k$.

Similarly, for any integer $k \geq 0$ and any language $\mathcal{L}$ we denote by $\mathcal{L}\dagger_k$ ("L chop k") the language obtained from $\mathcal{L}$ by performing, in the given order:

1. replace every $w \in \mathcal{L}$ by $w\dagger_k$, and
2. add all subwords (= factors) to the language.

The following properties of (laminary) languages are rather useful; they follow directly from the definition.

Lemma 5.4. *(a) Every laminary language $\mathcal{L}$ satisfies, for every integer $k \geq 0$, the equality $\mathcal{L} = \mathcal{L}\dagger_k$.*
(b) For every infinite language $\mathcal{L}$ and for every integer k, $L(\mathcal{L}\dagger_k) = L(\mathcal{L})$ and $\mathcal{L}(L(\mathcal{L})) = \cap_{k \in \mathbb{N}} \mathcal{L}\dagger_k$. $\square$

Recall that a symbolic lamination $L \in \Lambda_{\mathcal{A}}$ is *minimal* if L is equal to the closure of any of its orbits, with respect to both, shift and inversion. This is equivalent to saying that L does not contain a proper sublamination. One can easily characterize laminary languages of such a minimal lamination:

Definition 5.5. A language $\mathcal{L}$ has the *bounded gap property* if for any word u in $\mathcal{L}$ there exists an integer $n = n(u) \in \mathbb{N}$ such that any word $w \in \mathcal{L}$ of length greater than n contains u or u^{-1} as a subword.

The following is part of symbolic dynamics folklore [Fog02]:

Proposition 5.6. *A (symbolic) lamination is minimal if and only if its laminary language has the bounded gap property.* $\square$

Note that, if in addition the lamination is non-orientable, then for n big enough any word w of the laminary language will contain both, u and u^{-1}.

6 Metrics and topology on the set of laminations

For any laminary languages $\mathcal{L}, \mathcal{L}' \in \Lambda_{\mathcal{L}}$ we define:

$$d(\mathcal{L}, \mathcal{L}') = \exp(-\max(\{n \geq 0 \mid \mathcal{L}_{2n+1} = \mathcal{L}'_{2n+1}\} \cup \{0\})).$$

This defines a distance on $\Lambda_{\mathcal{L}}$ which is easily seen to be ultra-metric, and it is clear that $\Lambda_{\mathcal{L}}$ is a compact Haussdorf totally disconnected perfect metric space: a Cantor set.

Similarly, one can define on the set $\Sigma_{\mathcal{A}}$ of biinfinite reduced words in $\mathcal{A}^{\pm 1}$ a metric, by defining for any $Z, Z' \in \Sigma_{\mathcal{A}}$ the distance

$$d(Z, Z') = \exp(-\max(\{n \geq 0 \mid Z_n = Z'_n\} \cup \{0\}))\,,$$

where for any reduced biinfinite word $Z = \ldots z_{i-1} z_i z_{i+1} \ldots$ we denote the *central subword of length* $2n+1$ by $Z_n = z_{-n} z_{-n+1} \ldots z_n$.

From these definitions and the shift-invariance of a symbolic lamination we obtain directly that a symbolic lamination $L_{\mathcal{A}}$ is contained in the ε-neighborhood in $\Sigma_{\mathcal{A}}$ of a second symbolic lamination $L'_{\mathcal{A}}$ if and only if $\mathcal{L}_{2n+1}(L_{\mathcal{A}})$ is a subset of $\mathcal{L}_{2n+1}(L'_{\mathcal{A}})$, for $\varepsilon = e^{-n}$. This metric on $\Sigma_{\mathcal{A}}$ induces a Hausdorff metric on the set $\Lambda_{\mathcal{A}}$ of symbolic laminations in $\mathcal{A}^{\pm 1}$. We obtain directly:

Proposition 6.1. *The bijection $\rho^{\mathcal{A}}_{\mathcal{L}} : \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{L}}$ given by $L_{\mathcal{A}} \mapsto \mathcal{L}(L_{\mathcal{A}})$ is an isometry with respect to the above defined metrics:*

$$d(L_{\mathcal{A}}, L'_{\mathcal{A}}) \leq e^{-n} \iff \mathcal{L}_{2n+1}(L_{\mathcal{A}}) = \mathcal{L}_{2n+1}(L'_{\mathcal{A}})$$

□

As indicated in §2, the choice of a basis $\mathcal{A}$ of the free group F_N defines a word metric on F_N and also a (ultra-)*metric at infinity* on ∂F_N, by specifying for any $X, Y \in \partial F_N$, with prefixes X_n and Y_n respectively, the distance

$$d_{\mathcal{A}}(X, Y) = \exp(-\max\{n \geq 0 \mid X_n = Y_n\}).$$

In a similar vein as above for $\Sigma_{\mathcal{A}}$, this distance can be used to define a distance on $\partial^2 F_N$, and we can define a Hausdorff metric $d_{\mathcal{A}}$ on $\Lambda^2(F_N)$. With a little care we can show that this makes the bijection $\rho^2_{\mathcal{A}} : \Lambda^2(F_N) \to \Lambda_{\mathcal{A}}$ from Proposition 4.4 an isometry. However, contrary to the case of $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{L}}$, the choice of a basis in F_N and hence of the metric on ∂F_N is not really natural, so that we prefer for $\Lambda^2(F_N)$ only to consider the topology induced by these metrics. Whenever a basis is specified, it is in any case more convenient to pass directly to $\Lambda_{\mathcal{A}}$ or to $\Lambda_{\mathcal{L}}$. It is well known (and can easily be derived from the material presented in §7 below) that different bases of F_N induce Hölder-equivalent metrics on ∂F_N and on $\partial^2 F_N$, and thus also on $\Lambda^2(F_N)$. Thus we obtain:

Proposition 6.2. *The canonical bijections*

$$\Lambda^2(F_N) \xrightarrow{\rho^2_{\mathcal{A}}} \Lambda_{\mathcal{A}} \xrightarrow{\rho^{\mathcal{A}}_{\mathcal{L}}} \Lambda_{\mathcal{L}}$$

are homeomorphisms. They also preserve the partial order structure defined on each of them by the inclusion as subsets. □

The topology on the space of laminations is explicitly encapsulated in the following:

Remark 6.3. A sequence $(L^2_k)_{k\in\mathbb{N}}$ of algebraic laminations converges to an algebraic lamination L^2 if and only if, for some (and hence any) basis $\mathcal{A}$ of F_N, the sequence of corresponding symbolic laminations $L_k = \rho^2_{\mathcal{A}}(L^2_k)$ and their presumed limit $L = \rho^2_{\mathcal{A}}(L^2)$ satisfy the following:
Convergence criterion: For any integer $n \geq 1$ there exists a constant $K(n) \geq 1$ such that for all $k \geq K(n)$ one has:

$$\mathcal{L}_n(L_k) = \mathcal{L}_n(L)\,.$$

The following lemma will be used in [CHL-III].

Lemma 6.4. *For any given algebraic lamination L^2 the set $\delta(L^2)$ of sublaminations of L^2 is a compact subset of Λ^2.*

Proof. Since Λ^2 is compact, it suffices to show that $\delta(L^2)$ is closed. Any sublamination of L^2 has as laminary language a sublanguage of the laminary language $\mathcal{L}(L^2)$ defined by L^2, and conversely. Moreover, for laminary languages the analogous statement as given by the lemma is trivially true, as follows directly from the above Convergence criterion. □

We would like to point the reader's attention to the fact that the space Λ^2 is rather large, and for some purposes perhaps too large: it contains more objects than one would naturally think of as analogues of surface laminations. Of particular interest seems to be the natural subspace of Λ^2 given by the closure $\overline{\Lambda}_{\mathrm{rat}} = \overline{\Lambda}_{\mathrm{rat}}(F_N)$ of the the space Λ_{rat} of rational laminations (compare §2). We can now restate and prove Proposition 2.2:

Proposition 6.5. *The inclusion $\overline{\Lambda}_{rat} \subset \Lambda^2(F_N)$, for $N \geq 2$, is not an equality.*

Proof. For a and b in $\mathcal{A}$ consider the symbolic lamination $L(\mathcal{L}(Z))$ generated by the biinfinite word $Z = \ldots aaa \cdot bbb \ldots$. It consists precisely of the σ-orbit of Z and of the two periodic words $\ldots aaa \cdot aaa \ldots$ and $\ldots bbb \cdot bbb \ldots$, together with all of their inverses. The laminary language $\mathcal{L}_n(Z)$ consists of the words $a^n, a^{n-1}b, a^{n-2}b^2, \ldots, ab^{n-1}, b^n$ and their inverses. However, every rational lamination L, with the property that the corresponding laminary language contains these words, must contain the rational sublamination $L(w)$ for some $w \in F(a,b)$ that contains both letters, a and b, or their inverses. But then $\mathcal{L}_n(L)$ must also contain the word bx in $\mathcal{L}_2(L)$, for some $x \in \mathcal{A} \cup \mathcal{A}^{-1} \smallsetminus \{b, b^{-1}\}$. This contradicts the above Convergence criterion from Remark 6.3, for any $L_k = L$ as above. □

On the other hand, the closure of the rational laminations seems to be a reasonable subspace of Λ^2, as shown by the following:

Proposition 6.6. *$\overline{\Lambda}_{rat}$ contains all minimal algebraic laminations.*

Proof. We prove the proposition for non-orientable minimal laminations, where F_N-orbits and $< F_N, \text{flip} >$-orbits agree, and leave the generalization for orientable laminations to the reader.

Let L^2 be a minimal algebraic lamination and $\mathcal{A}$ a basis of F. Let $L_{\mathcal{A}} = \rho^2_{\mathcal{A}}(L^2)$ be the symbolic lamination and $\mathcal{L} = \rho^{\mathcal{A}}_{\mathcal{L}}(L_{\mathcal{A}})$ the laminary language canonically associated to L^2. By minimality of L^2 the language $\mathcal{L}$ has the bounded gap property (see Proposition 5.6): For any integer n there exists a bound $K = K(n)$ such that for any words u and w of $\mathcal{L}$ where the length of u is smaller than n and the length of w is greater than K, u occurs as a subword of w.

This proves that for any word w of $\mathcal{L}$ of length greater than K we have $\mathcal{L}_n(w) = \mathcal{L}_n(L^2)$. If moreover w is cyclically reduced, we obtain:

$$\mathcal{L}_n(L(w)) \supset \mathcal{L}_n(w) = \mathcal{L}_n(L^2)$$

Now let u be any word of $\mathcal{L}$ of length n and v another word of $\mathcal{L}$ of length $3K$. Write $v = w_1 w_2 w_3$ where w_1, w_2, w_3 are all of length K: The product $w_1 w_2 w_3$ is reduced, and each w_i is a subword of v. Now u must be a subword of both, w_1 and w_3: We can write the corresponding reduced products $w_1 = w'_1 u w''_1$ and $w_3 = w'_3 u w''_3$, and we define:

$$v' = u w''_1 w_2 w'_3$$

Since v' contains w_2 as subword, its length is bigger than K, and hence the previous equality applies: $\mathcal{L}_n(v') = \mathcal{L}_n(L^2)$. Moreover, since $w'_3 u$ is a subword of the reduced word w_3, it follows that v' is cyclically reduced, and hence $\mathcal{L}_n(L(v')) \supset \mathcal{L}_n(L^2)$. Finally, since u has length n, any subword of length n of the reduced biinfinite word $\ldots v'v' \cdot v'v' \ldots$ that is not a subword of v' is necessarily a subword of $w_2 w'_3 u$, and hence of v. Hence we get $\mathcal{L}_n(L(v')) \subset \mathcal{L}_n(L^2)$ and thus

$$\mathcal{L}_n(L(v')) = \mathcal{L}_n(L^2)\,.$$

Thus, for any integer n we found a word $v' = v'(n) \in F(\mathcal{A})$ such that the rational lamination $L(v'(n))$ satisfies $\mathcal{L}_n(L(v'(n))) = \mathcal{L}_n(L^2)$. Hence the Convergence criterion of Remark 6.3 gives directly that $L(v'(n)) \overset{n\to\infty}{\longrightarrow} L^2$. $\square$

The two previous propositions imply directly Theorem 1.2.

7 Bounded cancellation

An important tool when dealing with more than one basis in a free group F_N is *Cooper's cancellation bound* [Coo87]. We denote by $|w|_{\mathcal{A}}$ the length of the element $w \in F_N$ when written as reduced word in a basis $\mathcal{A}$ of F_N.

Lemma 7.1. *Let α be an automorphism of a free group F_N and let $\mathcal{A}$ be a basis of F_N. Then there exists a constant $C \geq 0$ such that, for any elements $u, v \in F_N$ with*

$$|u|_{\mathcal{A}} + |v|_{\mathcal{A}} \;=\; |uv|_{\mathcal{A}}$$

(i.e. there is no cancellation in the product uv of the reduced words u and v) one has

$$0 \leq |\alpha(u)|_{\mathcal{A}} + |\alpha(v)|_{\mathcal{A}} - |\alpha(uv)|_{\mathcal{A}} \;\leq\; 2C$$

As any second base $\mathcal{B}$ is the preimage of $\mathcal{A}$ under some $\alpha \in \mathrm{Aut}(F_N)$, the last line of the above statement can equivalently be replaced by

$$0 \leq |u|_{\mathcal{B}} + |v|_{\mathcal{B}} - |uv|_{\mathcal{B}} \;\leq\; 2C$$

We denote by $\mathrm{BBT}(\mathcal{A}, \alpha)$ or $\mathrm{BBT}(\mathcal{A}, \mathcal{B})$ the smallest such constant C.

An elementary proof of the above lemma can be given inductively, by decomposing the given automorphism (or basis change) into elementary Nielsen transformations. In modern geometric group theory language, one can restate the lemma as a special case of the fact that any two word metrics on a group G based on two different finite generating systems give rise to a quasi-isometry which realizes the identity on G.

This lemma has been interpreted and generalized in term of maps between trees in [GJLL98]. We describe now this interpretation; a generalization is given in [CHL-II].

Let $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ be the metric realisations (with constant edge length 1) of the Cayley graphs of F_N with respect to $\mathcal{A}$ and $\mathcal{B}$. Let $i = i_{\mathcal{A},\mathcal{B}}$ the equivariant map from $T_{\mathcal{A}}$ to $T_{\mathcal{B}}$ which is the identity on vertices and which is linear (and thus locally injective) on edges. Then Cooper's cancellation lemma 7.1 can be rephrased as:

Lemma 7.2. *For any (possibly infinite) geodesic $[P, Q]$ in $T_{\mathcal{A}}$ the image $i([P, Q])$ lies in the C-neighborhood in $T_{\mathcal{B}}$ of $[i(P), i(Q)]$, for some $C > 0$ (in particular for $C = BBT(\mathcal{A}, \mathcal{B})$ as above) independent on the choice of $P, Q \in T_{\mathcal{A}}$.* □

Finally, we state the following lemma that is used in [CHL-II]:

Lemma 7.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two bases of F_N. Any element w of F_N which is cyclically reduced with respect to the basis $\mathcal{A}$ is "almost cyclically reduced with respect to $\mathcal{B}$". More specifically, if*

$$w = y_1 \cdots y_r y_{r+1} \cdots y_n y_r^{-1} \cdots y_1^{-1}$$

with $y_i \in \mathcal{B}^{\pm 1}$ is a reduced word (in particular with $y_{r+1} \neq y_r^{-1}$ and $y_n \neq y_r$), then one has $r \leq BBT(\mathcal{A}, \mathcal{B})$.

Proof. Apply Lemma 7.1 to w^2. □

8 The Out(F_N)-action on laminations and laminary languages

In §2 we briefly mentioned that there is a natural action by any automorphism of F_N as homeomorphism on the boundary ∂F_N, and thus on Λ^2. This is a well known result in geometric group theory: Indeed the very fact that the boundary of a free group can be defined without any reference to a given basis is exactly equivalent to that statement. The key fact here is that a basis change in F_N (or, equivalently, an automorphism of F_N) induces a change of the metric on F_N (see §6) in a Lipschitz equivalent way. Therefore it changes the induced metric on the boundary (viewed as the set of one-sided infinite reduced words, see §6) in a Hölder equivalent way.

A more direct combinatorial way to define the action of $\mathrm{Out}(F_N)$ on languages is given as follows: Notice first that the elementwise image $\alpha(\mathcal{L})$ of a laminary language $\mathcal{L}$ under an automorphism $\alpha \in \mathrm{Aut}(F_N)$ is in general not a laminary language.

By Lemma 7.1, for $C = \mathrm{BBT}(\mathcal{A}, \alpha)$ the language $\alpha(\mathcal{L})\dagger_C$ is laminary, and by Lemma 5.4 we have $L(\alpha(\mathcal{L})) = L(\alpha(\mathcal{L})\dagger_C)$. Thus, if we consider the outer automorphism $\widehat{\alpha} \in \mathrm{Out}(F_N)$ defined by α, we can define:

$$\widehat{\alpha}(\mathcal{L}) = \alpha(\mathcal{L})\dagger_C = \mathcal{L}(L(\alpha(\mathcal{L})))$$

It follows directly from the second equality that this does not depend on the choice of the automorphism α in the class $\widehat{\alpha}$. It also follows directly from our definitions that this action of $\widehat{\alpha}$ is in fact a homeomorphism of the space $\Lambda_{\mathcal{L}}$ of laminary languages in $\mathcal{A}^{\pm 1}$.

Similarly, for any symbolic lamination $L_{\mathcal{A}}$ we define

$$\widehat{\alpha}(L_{\mathcal{A}}) = L(\alpha(\mathcal{L}(L)))\,.$$

From these definitions we see directly that the actions of $\widehat{\alpha}$ commute with the (bijective) map $\rho^{\mathcal{A}}_{\mathcal{L}} : \Lambda_{\mathcal{A}} \to \Lambda_{\mathcal{L}}$ given in Proposition 5.3.

If β is a second automorphism of F_N and $C' = \mathrm{BBT}(\mathcal{A}, \beta)$, one gets from Lemma 7.1 that

$$\alpha(\beta(\mathcal{L})\dagger_{C'})\dagger_C = (\alpha\beta)(\mathcal{L})\dagger_{C''}\,,$$

with $C'' = |\alpha|_{\mathcal{A}}\, C' + C$ and $|\alpha|_{\mathcal{A}} = \max\{|\alpha(x)|_{\mathcal{A}} \ : \ x \in \mathcal{A}\}$. This shows that the definitions above give an action of $\mathrm{Out}(F_N)$ on $\Lambda_{\mathcal{L}}$ and on $\Lambda_{\mathcal{A}}$.

Applying Lemma 7.1 again, we get that, if (X, X') is a leaf of an algebraic lamination L^2, then any subword of $\rho_{\mathcal{A}}(\alpha(X), \alpha(X'))$ is a word in $\alpha(\mathcal{L}(X^{-1}X'))\dagger_C$. $\blacksquare$ This proves that $\rho^2_{\mathcal{A}}$ is $\mathrm{Out}(F_N)$-equivariant and thus concludes the proof of Theorem 1.1.

Each of the above two versions of the $\mathrm{Out}(F_N)$-actions has its own virtues: Surprisingly, the action on laminary languages generalizes much more directly to more general homomorphisms $\varphi : F_N \to F_M$ of free groups. It is noteworthy in this context that non-injective substitutions on biinfinite sequences are treated classically in symbolic dynamics in a similar vein as injective ones, while from a geometric group theory standpoint it is impossible to extend a non-injective map φ as above in any meaningful way to a map $\partial\varphi : \partial F_N \to \partial F_M$. The more common injective case, however, is easy to understand even from the geometric group theory standpoint:

Remark 8.1. It is well known that every finitely generated subgroup of a free group is quasi-convex. Thus an embedding $\varphi : F_M \subset F_N$ induces canonically an embedding $\partial\varphi : \partial F_M \subset \partial F_N$, see [GdlH90]. Clearly, this extends to an embedding $\partial\varphi^2 : \partial^2 F_M \subset \partial^2 F_N$, but since the image $\partial\varphi^2(\partial^2 F_M) \subset \partial^2 F_N$ is in general not F_N-invariant, an algebraic lamination $L^2 \subset \partial^2 F_M$ is mapped by $\partial\varphi^2$ to a set $\partial\varphi^2(L^2) \subset \partial^2 F_N$ that is in general *not* an algebraic lamination. By taking the closure of $\partial\varphi^2(L^2)$ with respect to the topology, the F_N-action, and the flip map, one obtains however a well defined algebraic lamination, which we denote by $\varphi_\Lambda(L^2)$, thus defining a natural map:

$$\varphi_\Lambda : \Lambda^2(F_M) \to \Lambda^2(F_N)$$

However, it has to be noted immediately that this map φ_Λ does not have to be injective: It suffices that the embedding φ maps elements $v, w \in F_M$ which are not conjugate in F_M to elements $\varphi(v), \varphi(w)$ that are conjugate in F_N: Then the associated rational laminations satisfy

$$L^2(v) \neq L^2(w) \in \Lambda^2(F_M)\,,$$

but also

$$\varphi_\Lambda(L^2(v)) = L^2(\varphi(v)) = L^2(\varphi(w)) = \varphi_\Lambda(L^2(w)) \in \Lambda^2(F_N)\,.$$

On the other hand, we note that if F_M is a free factor of F_N, then the lamination space $\Lambda^2(F_M)$ is canonically embedded into $\Lambda^2(F_N)$: it suffices to consider a basis of F_N which contains as a subset a basis of F_M.

It seems to be an interesting question of when precisely the map $\varphi_\Lambda : \Lambda^2(F_M) \to \Lambda^2(F_N)$ induced by an embedding $\varphi : F_M \subset F_N$ is injective, and in particular, if this is the case if and only if the subgroup F_M is malnormal in F_N.

We finish this paper with an answer to the question we posed in §2.

Proposition 8.2. *Let $\mathcal{A}$ be a basis of F_N, and let a be an element of $\mathcal{A}$. Then, for any $N \geq 2$, the closure of the Out(F_N)-orbit of the rational lamination $L(a)$ in Λ^2 is not the only non-empty minimal closed Out(F_N)-invariant subspace of Λ^2.*

Proof. Let a be as above, and let b be another element of $\mathcal{A}$. Consider the rational lamination $L([a, b])$. Then for any outer automorphism $\hat{\alpha}$ of F_N and any automorphism α representing it, one has

$$\hat{\alpha}(L([a, b])) = L(\alpha([a, b])).$$

As the derived subgroup is characteristic, the $\mathrm{Out}(F_N)$-orbit of $L([a,b])$ consists of some minimal rational laminations associated to cyclically reduced words of the derived subgroup. Now any cyclically reduced word of the derived subgroup contains a subword of the form xy, where x, y are distinct elements of $\mathcal{A}^{\pm 1}$ with $x \neq y^{-1}$. This proves that for any outer automorphism $\hat{\alpha}$, the laminary language $\mathcal{L}(\hat{\alpha}(L([a,b])))$ contains a reduced word of the form xy. It follows from the Convergence criterion in Remark 6.3 that $L(a)$ is not in the closure of the $\mathrm{Out}(F_N)$-orbit of $L([a,b])$. $\square$

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