# Mathematical challenges in shape optimization

Antoine Henrot, Jan Sokolowski

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## 1 Introduction

The present paper covers some theoretical investigations performed in France, in the framework of the CNRS programme GDR *Shape Optimization* which does exist during last 12 years. The programme included also some activities in Poland, in the Banach Center as well as a workshop in Poznan. We do not restrict the presentation to the French community in the research field, the list of references includes all recent monographs on the shape optimization. We refer the reader to the contributions by Gregoire Allaire, Samuel Amstutz, Dorin Bucur, Marc Dambrine, Jean-Antoine Désidéri Frédéric de Gournay, François Jouve, Mohamed Masmoudi, Jean-Rodolphe Roche, Gregory Vial, Jean-Paul Zolesio, for other aspects of the research in the domain in France. In the other paper in the issue, by M.P. Bendsoe et al., the Danish community in the domain is presented.

The outline of the paper is the following. First we present some main fields of the activity in shape optimization. To present some precise results, from mathematical point of view, we include two sections. The first is devoted to the eigenvalues, the second to the drag minimization. Many theoretical questions related to these problems are still open.

### 1.1 Applications of shape optimization

We list below the main fields of applications of the shape optimization. After that, some theoretical problems, which are still unsolved, are formulated. We do not present the topology optimization by the method of topological derivatives since it is already the subject of few contributions in the present issue of the journal. We restrict ourselves to the developpments in France.

### Aerospace engineering

This is the first field of applications of shape optimization from the historical point of view. The problems posed for mathematical and numerical solutions concerns e.g., the minimization of the drag and an improvement of the lift. In France Dassault and EADS are the companies which are strongly involved in the research in the domain and influence the mathematical community. Many examples of applied problems can be found in

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the book by Mohammadi-Pironneau [45]. Modern computers are used for modelling of complex equations of mathematical physics. The recent works in the domain includes the mathematical modelling of invisible plane for diminution of Surface Equivalent Radar, the noise of the plane during of its take-off and shape optimization of space antennas. The research in the field is performed in France among others by J.A. Desideri [43], A. Habbal [27], A. Henrot, M. Masmoudi [55], B. Rousselet [54].

### Automotive industry

This is also a field of applications for the shape optimization. The classical optimum design problem of weight minimization is one among the problems of interest. Some particular problems are solved in the so-called *industrial* PhD dissertations, which are performed in the companies like PSA, Renault, Valeo and others. It is a way for implementation and verification of mathematical methods when used for industrial problems.

### Structural mechanics

One of the most popular mathematical methods is the computations of optimal composites applied in shape and topology optimization. The method is based on homogenization and is currently used in all domains of applications of shape optimization, including car industry, aerospace engineering, civil engineering. The method is easy to apply and with relatively good numerical performances. However it requires still further mathematical studies concerning explicit formulae for homegenized coefficients in three dimensional elasticity.

Many examples can be found on the web page:
http://www.cmap.polytechnique.fr/~optopo
of the Groupe d'optimisation de Forme du CMAP (G. Allaire, F. Jouve and others).

Another example in the domain is modelling and optimization of dynamics of structures, this direction of research is supported by Thomson. B. Rousselet is now engaged in this research, in our mathematical community.

### Biology, geology, human sciences

One of the most important domains of applications becomes in the 21st century the classical field of biology and medical sciences. It seems to us that it is the inverse modelling which is vitally important here. Let us explain this by an example. We would like to find the model of an axon including the equations for the electrical field. To check if the proposed model is closed to the reality, we could optimize the shape of the axon and compare the result with the form observed in the nature. If our optimal shape ressembles the real shape it would mean that the proposed model and the shape functional optimized are related to the real word. Otherwise, one should change the functional and possibly the model to obtain better agreement.

Another applications include the tomography and the tumor (cancer) identification. These are the problems which can be classified as geometrical inverse problems, which are briefly described in the following section.

In geology, many problems concerns unknown mechanical parameters and geometrical caracterictics which can be determined based on available observations and data. These are as well geometrical inverse problems. Solutions could be helpful for determination of evolution in time, including fracture mechanics phenomena, of interfaces or boundaries in such e.g. underground structures like mines or cavities. For some model problems, an unexpected relation to the classical geometrical problems can be discovered. It is the case, in particular, for the so-called Cheeger problem, which is considered recently by P. Hild, I. Ionescu and T. Lachand-Robert see [35] for mathematical formulation of inverse problems for Bingham fluids in application to landslides modelling.

### Geometrical inverse problems

Inverse problems which are applied in the so-called nondestructive control are usually of special mathematical structure. The geometrical domain contains an unknown part of the boundary to be detected. The unknown part of the boundary should be determined on the basis of a boundary value problem for which the boundary values are overdetermined on the known part of the boundary. In particular, for the tomography and inverse scattering, the unknown part of the boundary is either its interior part, or its exterior part, respectively. There are many results for such problems, obtained in particular by H. Ammari [3] or [4], B. Canuto [14], S. Chaabane [15], M. Choulli [18], A. El Badia [23], H. Haddar [19], T. Ha Duong, M. Jaoua see [36], B. Rousselet, H. Sahli.

One of the possibilities in geometrical inverse problems is crack identification, with many applications. As usually, in the theory of inverse problems, the mathematical questions concern the uniqueness of a solution, its stability with respect to data perturbations, and finally the numerical solution which combine numerical approximation with optimization techniques.

### Control, stabilization and smart materials

Control theory for PDE's is strictly connected with shape optimization. It is due to the fact, that controls are spatially distributed in e.g. elastic body. We could list some of problems recently treated by some members of GDR.

- optimization of the structure of control system including optimal positions of actuators and sensors. (cf for example P. Destuynder [22], E. Degryse and S. Mottelet, P. Hébrard and A. Henrot [31]). Mathematical models are of different nature, including e.g., the damped wave equation, fluid and structure interaction, and the merit function in the form e.g., of the rate of energy decrease.

- exact controlability with respect to geometrical domain (D. Chenais and E. Zuazua [17]).

- stabilisation of shells (J. Cagnol, C. Lebiedzik, I. Lasiecka, J.P. Zolesio see [12], [13])

- control of plasma in Tokamak (J. Blum).

The smart materials used in engineering require for mathematical modelling the applications of asymptotic analysis and of shape optimization. We have no place here to describe in details such applications for noise reduction or for vibrations reduction in automotive industry.

### 1.2 Theoretical problems

The importance of mathematics in the analysis and understanding of shape optimization problems comes from the fact that such problems are in general ill-posed. It means that we cannot expect any existence of a solution to such a problem, under only natural constraints, from one side. From the other side, if a solution does exist, in general it is not stable with respect to imperfections, a partial reason is that shape functionals are not convex.

Therefore, when a particular mathematical model is established for a given problem, e.g. the elasticity boundary value problem with the specific functional to be minimized we can furnish an analysis which covers:

- the existence of an optimal shape by e.g. restriction of the family of admissible shapes to become *compact* for the shape functional under studies.
- the gradient analysis of optimization problem, or the necessary optimality system which gives some possibilities for determination of optimal shapes.
- the sensitivity analysis of the optimization problem which says how the optimal solution, if any, depends on the data.

However, from mathematical point of view, the regularity of the optimal shape is the most difficult problem, recognized by specialists in the free boundary modelling. We refer the reader to recent works by T. Briançon and M. Pierre, A. Chambolle and C. Larsen [16] for some results in this direction.

We also point out the recent progress in the famous Newton problem of aerodynamics due to M. Comte, T. Lachand-Robert, M. Peletier, and others (see e.g. [20], [40] and references therein).

In two following sections we describe in details results available for two classes of shape optimization problems. The first one is studied in many papers and there are still open problems - it is the problem of shape optimization of controlling the eigenvalues. Simple formulation leads to difficult questions. The second one is also difficult, and describes modelling of the compressible fluids, by means of Navier-Stokes equations. The main difficulty for shape optimization is associated with the possible nonexistence, and nonuniqueness of solutions to nonlinear partial differential equations.

## 2 Eigenvalue problems

### 2.1 Introduction

Problems linking the shape of a domain to the sequence of its eigenvalues, or some of them, are among the most fascinating of mathematical analysis and differential geometry. In particular, problems of minimization of eigenvalues, or combination of eigenvalues, brought about many deep works since the early part of the twentieth century. Actually, this question appeared in the famous book of Lord Rayleigh "The theory of sound" ( for example in the edition of 1894). Thanks to some explicit computations and "physical evidence", Lord Rayleigh conjectured that the disk should minimize the first Dirichlet eigenvalue $\lambda_1$ of the Laplacian among every open sets of given measure.

It was indeed in the 1920's that Faber [24] and Krahn [38] proved simultaneously the Rayleigh's conjecture using a rearrangement technique. In [39], Krahn also proves that the union of two identical disks minimizes the second Dirichlet eigenvalue. This result was rediscovered later by P. Szegö, as quoted by G. Pólya in [52].

In this section, we discuss known results and open problems about the minimization of the $k$-th eigenvalue of the Laplacian with Dirichlet boundary conditions. More precisely, let $\Omega$ be a bounded open set in $\mathbb{R}^N$. The Laplacian on $\Omega$ with Dirichlet boundary conditions is a self-adjoint operator with compact inverse, so there exists a sequence of positive eigenvalues (going to $+\infty$) and a sequence of corresponding eigenfunctions that we will denote respectively $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots$ and $u_1, u_2, u_3, \ldots$. In other words, we have:

$$\begin{cases} -\Delta u_k = \lambda_k(\Omega) u_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

In the sequel, we are interested in minimization problems like

$$\min\{\lambda_k(\Omega),\ \Omega \text{ open subset of } \mathbb{R}^N,\ |\Omega| = A\}$$

(where $|\Omega|$ denotes the measure of $\Omega$ and $A$ is a given constant). Let us remark that, according to the behaviour of the eigenvalues with respect to homothety, looking for the minimizer of $\lambda_k(\Omega)$ with a volume constraint is equivalent to look for a minimizer of the product $|\Omega|^{2/N}\lambda_k(\Omega)$.

For an extensive bibliography and for more details and results, especially with other constraints or other boundary conditions or various combinations of eigenvalues, we refer to the recent review papers [5], [9], [32].

## 2.2 Known results

### 2.2.1 The first eigenvalue

For the first eigenvalue, the basic result is (as conjectured by Lord Rayleigh):

**Theorem 1 (Rayleigh-Faber-Krahn)** *Let $\Omega$ be any bounded open set in $\mathbb{R}^N$, let us denote by $\lambda_1(\Omega)$ its first eigenvalue for the Laplace operator with Dirichlet boundary conditions. Let $B$ be the ball of the same volume as $\Omega$, then*

$$\lambda_1(B) = \min\{\lambda_1(\Omega),\ \Omega \textit{ open subset of } \mathbb{R}^N,\ |\Omega| = |B|\}.$$

The classical proof makes use of the *Schwarz spherical decreasing rearrangement.* Since such a rearrangement preserves any $L^p$ norm and decreases the Dirichlet integral:

$$\int_B u^*(x)^2\,dx = \int_\Omega u(x)^2\,dx \qquad \int_B |\nabla u^*(x)|^2\,dx \leq \int_\Omega |\nabla u(x)|^2\,dx \tag{2}$$

the result follows using the variational characterization of the first eigenvalue (it minimizes the so-called Rayleigh quotient).

### 2.2.2 The second eigenvalue

For the second eigenvalue, the minimizer is not one ball, but two!

**Theorem 2 (Krahn-Szegö)** *The minimum of $\lambda_2(\Omega)$ among bounded open sets of $\mathbb{R}^N$ with given volume is achieved by the union of two identical balls.*

**Proof :** Let $\Omega$ be any bounded open set, and $u_2$ its second eigenfunction. Let us denote by $\Omega_+ = \{x \in \Omega, u_2(x) > 0\}$ and $\Omega_- = \{x \in \Omega, u_2(x) < 0\}$ its nodal domains. Since $u_2$ satisfies

$$\begin{cases} -\Delta u_2 = \lambda_2 u_2 & \text{in } \Omega_+ \\ u_2 = 0 & \text{on } \partial\Omega_+ \end{cases}$$

$\lambda_2(\Omega)$ is an eigenvalue for $\Omega_+$. But, since $u_2$ is positive in $\Omega_+$, it is the first eigenvalue (and similarly for $\Omega_-$):

$$\lambda_1(\Omega_+) = \lambda_1(\Omega_-) = \lambda_2(\Omega) \,. \tag{3}$$

We now introduce $\Omega_+^*$ and $\Omega_-^*$ the balls of same volume as $\Omega_+$ and $\Omega_-$ respectively. According to the Rayleigh-Faber-Krahn inequality

$$\lambda_1(\Omega_+^*) \leq \lambda_1(\Omega_+), \qquad \lambda_1(\Omega_-^*) \leq \lambda_1(\Omega_-)\,. \tag{4}$$

Let us introduce a new open set $\tilde\Omega$ defined as

$$\tilde\Omega = \Omega_+^* \cup \Omega_-^* \quad \text{disjoint union.}$$

Since $\tilde\Omega$ is disconnected, we obtain its eigenvalues by gathering and reordering the eigenvalues of $\Omega_+^*$ and $\Omega_-^*$. Therefore,

$$\lambda_2(\tilde\Omega) \leq \max(\lambda_1(\Omega_+^*), \lambda_1(\Omega_-^*))\,.$$

According to (3), (4) we have

$$\lambda_2(\tilde\Omega) \leq \max(\lambda_1(\Omega_+), \lambda_1(\Omega_-)) = \lambda_2(\Omega)\,.$$

This shows that the minimum of $\lambda_2$ is to be obtained among the union of balls. But, if the two balls would have different radii, we would decrease the second eigenvalue by shrinking the largest one and dilating the smaller one (without changing the total volume). Therefore, the minimum is achieved by the union of two identical balls. □

Being disappointed that the minimizer be not a connected set, we could be interested in solving the minimization problem for $\lambda_2$ among **connected** sets. Unfortunately, it is clear that a connectedness constraint does not really change the situation: the domain obtained by joining the union of the two previous balls by a thin pipe of width $\varepsilon$ has obviously its eigenvalues which converge to those of the two balls, therefore the infimum is not achieved in the class of connected sets. For a study of this minimization problem among **convex** sets, we refer to [33] (where is proved, in particular, that the minimum is not achieved by *a stadium*, i.e., an ovaloidal domain, which was the natural candidate).

### 2.2.3 The third eigenvalue

The proofs we recall in the previous sections are direct ones. The minimization problem becomes much more complicated for the other eigenvalues! One of the only known result is the following, cf [10] and [58]:

**Theorem 3 (Bucur-Henrot and Wolff-Keller)** *There exists a set $\Omega_3^*$ which minimizes $\lambda_3$ among the (quasi)-open sets of given volume. Moreover $\Omega_3^*$ is connected in dimension $N$ =2 or 3.*

The question of identifying the optimal domain $\Omega_3^*$ remains open. The conjecture is the following:

**Open problem** Prove that the optimal domain for $\lambda_3$ is a ball in dimension $N$ =2 or 3, a union of three identical balls in dimension $N \geq 4$.

Wolff and Keller have proved in [58] that the disk is a local minimizer for $\lambda_3$. There are two key-points in the existence proof of the above theorem. The first one is a more general result of Buttazzo-Dal Maso, see [11]:

**Theorem 4 (Buttazzo-Dal Maso)** *Let $D$ be a fixed ball in $\mathbb{R}^N$. For every fixed integer $k \geq 1$ and $c$ fixed real number $0 < c < |D|$ the problem*

$$\min\{\lambda_k(\Omega); \quad \Omega \subset D, \quad |\Omega| = c\} \tag{5}$$

*has a solution.*
*More generally, the existence result remains valid for any function $\Phi(\lambda_1, \ldots, \lambda_k)$ of the eigenvalues,* **non decreasing** *in each of its arguments.*

This theorem does not solve the general problem of existence of a minimizer for $\lambda_k(\Omega)$ since it assumes to work with "confined" sets (that is to say, sets included in a box $D$). In order to remove this assumption in [10], we used a "concentration-compactness" argument together with the Wolff-Keller's result proving that the minimizer of $\lambda_3$ (if it exists) should be connected in dimension 2 and 3 (this is the second key-point).

## 2.3 Open problems and some partial results

For the fourth eigenvalue, it is conjectured that the minimum is attained by the union of two balls whose radii are in the ratio $\sqrt{j_{0,1}/j_{1,1}}$ in dimension 2, where $j_{0,1}$ and $j_{1,1}$ are respectively the two first zeros of the Bessel functions $J_0$ et $J_1$, but it is not proved! Even existence is not yet known in this case. The proof we did with D. Bucur can be adapted for $\lambda_4$ if we were able to prove that the minimizing domain for $\lambda_3$ is a *bounded set*!

**Open problem** Prove that the optimal domain for $\lambda_4$ is the union of two balls whose radii are in the ratio $\sqrt{j_{0,1}/j_{1,1}}$ in dimension 2.

Looking at the previous results and conjectures, P. Szegö asked the following question:
*Is it true that the minimizer of any eigenvalue of the Laplace-Dirichlet operator is a ball or a union of balls?*

The answer to this question is NO. For example, Wolff and Keller remarked that the thirteenth (!) eigenvalue of a square is lower than the thirteenth eigenvalue of any union of disks of same area. Actually, it is not necessary to go to the 13th eigenvalue. Numerical experiments, cf [48] and Figure 1, show that for the 5-th eigenvalue the minimizer is no longer a ball or a union of balls. The same numerical experiments lead to the following

Figure 1: A polygonal approximation of a domain which is a possible minimizer for $\lambda_5$.

conjecture:

**Open problem** Let $\Omega_k^*$ be an open set minimizing $\lambda_k$, $k \geq 2$ among open sets of given area. Prove that $\lambda_k(\Omega^*)$ is a double eigenvalue and, more precisely, that

$$\lambda_{k-1}(\Omega_k^*) = \lambda_k(\Omega_k^*).$$

A partial result in this direction is given by the following Lemma.

**Lemma 4.1** *Let $\Omega$ be a bounded open set of class $C^{1,1}$. We assume that $\Omega$ has a multiple eigenvalue of order $m$:*

$$\lambda_{k+1}(\Omega) = \lambda_{k+2}(\Omega) = \ldots = \lambda_{k+m}(\Omega) \quad k \geq 1.$$

*Then, we can always find a deformation field $V \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$, preserving the volume and the convexity and such that, if we set*

$$\Omega_t = (Id + tV)(\Omega)$$

*we have, for $t > 0$ small enough*

$$\lambda_{k+1}(\Omega_t) < \lambda_{k+1}(\Omega) = \lambda_{k+m}(\Omega) < \lambda_{k+m}(\Omega_t)\,.$$

Indeed, the previous result has the following consequence about minimization of eigenvalues: if $\Omega_k^*$ is a domain minimizing the $k$-th eigenvalue and if $\lambda_k(\Omega^*)$ is not simple, necessarily we have

$$\lambda_{k-1}(\Omega_k^*) = \lambda_k(\Omega_k^*) \tag{6}$$

otherwise, the only possible case would be

$$\lambda_{k-1}(\Omega_k^*) < \lambda_k(\Omega_k^*) = \lambda_{k+1}(\Omega_k^*)$$

but Lemma 4.1 would then imply that $\lambda_k(\Omega_k^*)$ can be decreased by peturbation of $\Omega_k^*$ which is impossible.

**Proof of the Lemma :** We use the classical tool of derivative with respect to the domain (or Hadamard formulae), see e.g. [57], [56], [34]. Let us deform the domain $\Omega$ according to a deformation field $V$ as described in the statement of the Lemma. In the case of a multiple eigenvalue, this eigenvalue is no longer Frechet differentiable, but nevertheless it admits directional derivatives, i.e. the differential quotients

$$\frac{\lambda_{k+p}(\Omega_t) - \lambda_{k+p}(\Omega)}{t}, \quad \text{for } p = 1, \ldots, m$$

have a limit when $t$ goes to 0. Moreover, these limits are the eigenvalues of the $m \times m$ matrix

$$\mathcal{M} = \left( - \int_{\partial\Omega} \frac{\partial u_i}{\partial n} \frac{\partial u_j}{\partial n} V.n \, d\sigma \right)_{k+1 \le i,j \le k+m} \tag{7}$$

where $\frac{\partial u_i}{\partial n}$ denotes the normal derivative of the $i$-th eigenfunction $u_i$ and $V.n$ is the normal displacement of the boundary induced by the deformation field $V$. For a proof of the above-mentioned result, we refer to [30] or [53].

Let us now choose two points $A$ and $B$ located on strictly convex parts of $\partial\Omega$. Let us consider a deformation field $V$ such that $V.n = 1$ in a small neighborhood of $A$ (on the boundary of $\Omega^*$) of size $\varepsilon$, $V.n = -1$ in a small neighborhood of $B$ (with same measure) and $V$ regularized outside in a neighborhood of size $2\varepsilon$ in such a way that $|\Omega_t| = |\Omega|$ (it is always possible since the derivative of the volume is given by $d\text{Vol} = \int_{\partial\Omega} V.n \, d\sigma$ which vanishes with an appropriate choice of the regularization).

According to the above-mentioned results about the directional derivatives, the Lemma will be proved if we can find two points $A, B$ such that the symmetric matrix $\mathcal{M}$ has both positive and negative eigenvalues. Now, when $\varepsilon$ goes to 0, it is clear that the matrix $\mathcal{M}$ behaves like the $m \times m$ matrix

$$\mathcal{M}_{A,B} = \left( - \frac{\partial u_i}{\partial n}(A) \frac{\partial u_j}{\partial n}(A) + \frac{\partial u_i}{\partial n}(B) \frac{\partial u_j}{\partial n}(B) \right)_{k+1 \le i,j \le k+m} . \tag{8}$$

Let us denote by $\phi_A$ (resp. $\phi_B$) the vector of components $\frac{\partial u_i}{\partial n}(A)$, (resp. $\frac{\partial u_i}{\partial n}(B)$), $i = k+1, \ldots, k+m$. A straightforward computation gives, for any vector $X \in \mathbb{R}^m$:

$$X^T \mathcal{M}_{A,B} X = (X.\phi_B)^2 - (X.\phi_A)^2 .$$

Therefore, the signature of the quadratic form defined by $\mathcal{M}_{A,B}$ is $(1, 1)$ as soon as the vectors $\phi_A$ and $\phi_B$ are non colinears. Now, assuming these two vectors to be colinear for every choice of points $A$, $B$ would give the existence of a constant $c$ such that, on a strictly convex part $\gamma$ of $\partial\Omega$:

$$\frac{\partial u_{k+1}}{\partial n} = c \frac{\partial u_{k+2}}{\partial n} .$$

But, $u_{k+1} - c\, u_{k+2}$ would satisfy

$$\begin{cases} -\Delta(u_{k+1} - c\, u_{k+2}) = \lambda_{k+1}(u_{k+1} - c\, u_{k+2}) & \text{in } \Omega \\ u_{k+1} - c\, u_{k+2} = 0 & \text{on } \partial\Omega \cap \gamma \\ \frac{\partial(u_{k+1} - c\, u_{k+2})}{\partial n} = 0 & \text{on } \partial\Omega \cap \gamma . \end{cases}$$

Now, by Hölmgren uniqueness theorem, the previous p.d.e. system is solvable only by $u_{k+1} - c\, u_{k+2} = 0$ (first in a neighborhood of $\gamma$ and then in the whole domain by analyticity) which gives the desired contradiction. $\square$

# 3 Drag minimization for compressible isothermal Navier-Stokes equations

One of the most challenging problems in shape optimization is the design of aircrafts. As an example, which shows the complexity of the mathematical problem, we present the existence result in two dimensions proved in [49] for stationary compressible isothermal Navier-Stokes equations. The same existence result can be established in three dimensions [50]. We point out that the main issue for mathematical analysis of compressible Navier-Stokes equations is the existence of solutions. We refer the reader to the monographs by P.L. Lions [42] and by E. Feireisl [26] for the state of art in the mathematical modelling of compressible fluids.

## 3.1 Mathematical model - weak solutions

Suppose that compressible Newtonian fluid occupies the bounded region $\Omega \subset \mathbb{R}^2$. We will assume that $\Omega = B \setminus S$, where $B$ is a sufficiently large hold all containing inside a compact obstacle $S$. We could take, e.g., for $B$ a ball of radius $R$, $B = \{x||x| < R\}$. We do not impose restrictions on the topology of the flow region. The cases of $S$ with a finite number of connected components or $S = \emptyset$ are taken into consideration.

The fluid density $\rho : \Omega \mapsto \mathbb{R}^+$ and the velocity field $\mathbf{u} : \Omega \mapsto \mathbb{R}^2$ are governed by the Navier-Stokes equations

$$
\begin{aligned}
&-\nu\Delta\mathbf{u} - \xi\nabla\mathrm{div}\,\mathbf{u} + \rho\mathbf{u}\nabla\mathbf{u} + \nabla\rho = \rho\mathbf{f}\ ,\\
&\mathrm{div}\ (\rho\mathbf{u}) = 0\ ,
\end{aligned}
$$

where $\nu$, $\xi$ are positive viscous coefficients and $\mathbf{f} : \Omega \mapsto \mathbb{R}^2$ is a given vector field. If the viscous stress tensor is defined by the equality

$$
\Sigma = \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + (\xi - \nu)\mathrm{div}\,\mathbf{u}\,\mathbf{I}\ ,
$$

then the governing equations can be written in the equivalent divergence form

$$
\mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho - \rho\mathbf{f} = \mathrm{div}\,\Sigma \quad \text{in} \quad \Omega\ , \tag{9a}
$$

$$
\mathrm{div}\,(\rho\mathbf{u}) = 0 \quad \text{in} \quad \Omega\ . \tag{9b}
$$

Equations (9) should be supplemented with the boundary conditions. In view of possible applications e.g., to the shape optimisation problem of a wing it is supposed that the velocity field satisfies the non-homogeneous boundary conditions

$$
\mathbf{u} = 0 \quad \text{on} \quad \partial S\ , \qquad \mathbf{u} = \mathbf{U}^\infty \quad \text{on} \quad \Gamma\ , \tag{10a}
$$

and the density distribution is prescribed on the entrance set

$$
\rho = \rho^\infty \quad \text{on} \quad \Gamma^+ = \{x \in \partial B :\ \mathbf{U}^\infty \cdot \mathbf{n}(x) < 0\}\ . \tag{10b}
$$

Here $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. It is assumed that $\mathbf{U}^\infty \in \mathbb{R}^2$ is a given vector, and $\rho^\infty \in L_\infty(\Gamma^+)$ is a given non-negative function.

Boundary condition (10a) can be written in the form of the equality $\mathbf{u} = \mathbf{u}^\infty$ on $\partial\Omega$, where $\mathbf{u}^\infty(x)$ is a smooth function defined for any $x \in \mathbb{R}^2$, which vanishes in the vicinity of $S$ and coincides with $\mathbf{U}^\infty$ in an open neighbourhood of $\partial B$.

For $\mathbf{u}^\infty = 0$ problem (9)-(10) becomes the classical boundary value problem with no slip condition on the boundary of the flow region

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \; . \tag{11a}$$

In this particular case there are no boundary conditions for the density and the total mass $\mathcal{M}$ of the gas must be prescribed

$$\int_\Omega \rho dx = \mathcal{M} \; . \tag{11b}$$

The other physical quantities which characterise the flow, include kinetic energy $\mathcal{E}$, rate of energy dissipation $\mathcal{D}$ and drag $J$, defined by

$$\mathcal{E} = \frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2 dx \; , \quad \mathcal{D} = \int_\Omega (\nu|\nabla\mathbf{u}|^2 + \xi|\operatorname{div}\mathbf{u}|^2)dx \; , \quad J = -\mathbf{U}^\infty \cdot \int_{\partial S} (\mathbf{\Sigma} - \rho\mathbf{I})\cdot\mathbf{n}dS \; . \tag{12}$$

The drag $J$ accounts for the reaction of the surrounding fluid on the obstacle $S$. For our purposes, the formula for the drag can be written in the equivalent form, see [49],

$$J(\rho, \mathbf{u}, \Omega) = \int_\Omega (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I}) : \nabla\mathbf{u}^\infty dx + \int_\Omega (\mathbf{U}^\infty - \mathbf{u}^\infty)\cdot\mathbf{f}\rho dx \; . \tag{13}$$

We will consider the physically reasonable solutions to problems (9)-(10) and (9)-(11) for which the density is non-negative and the quantities (12) are bounded from above.

On the other hand, the peculiarity of problem (9) is that the equations do not allow us to control any $L_r$ norm of the density $\rho$ even for $r = 1$. Moreover, we can not eliminate the possibility of concentration of finite mass of gas in very small domains. The simplest way to bypass this difficulty is to suppose that the mass of gas is a Borel measure $\mu_\rho$ in $\Omega$. This means that the mass contained in any measurable set $E$ is simply $\mu_\rho(E)$. In the paper the standard notation is used for the function spaces. The space $H^{1,p}(\Omega)$ is the Sobolev space of functions integrable along with the first order generalized derivatives in $L_p(\Omega)$ equipped with its natural norm. For $p = 2$ we use the notation $H^{1,2}(\Omega)$ rather than $H^1(\Omega)$, and for real $m > 0$ we denote the Sobolev space of order $m$ by $H^{m,2}(\Omega)$.

**Definition 1** *For given $\mathbf{U}^\infty \in \mathbb{R}^2$ and $\mathbf{f} \in C(\Omega)^2$ a generalized solution to problem* (9)-(10) *is the pair $(\mu_\rho, \mathbf{u})$, where $\mu_\rho$ is a Borel measure in $\Omega$ and $\mathbf{u} - \mathbf{u}^\infty \in H^{1,2}_0(\Omega)$, which satisfies the following conditions:*

*(a) The measure $\mu_\rho$ does not charge null capacity sets i.e., $\mu_\rho(E) = 0$ for any Borel set with cap $E = 0$ and*

$$\int_\Omega d\mu_\rho(x) = \mu_\rho(\Omega) = \mathcal{M} < \infty \; . \tag{14}$$

*It implies, in particular, that for any continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ the composed function $f(\mathbf{u})$, more precisely its quasicontinuous representative, is measurable with respect to $\mu_\rho$.*

*(b) The scalar function $|\mathbf{u}|^2$ is integrable with respect to measure $\mu_\rho$ i.e.,*

$$\mathcal{E} = \frac{1}{2} \int\limits_{\Omega} |\mathbf{u}|^2 d\mu_\rho(x) < \infty \ .$$

*This means that the kinetic energy $\mathcal{E}$ of the flow is finite. It follows from this condition that the functions $u_i$ and $u_i u_j$, where $u_i$, $i = 1, 2$, are the components of the velocity field $\mathbf{u} = (u_1, u_2)$, are integrable with respect to $\mu_\rho$.*

*(c) The energy dissipation satisfies the inequality*

$$\mathcal{D} \leq \int\limits_{\Omega} \Big( \mathbf{\Sigma} : \nabla \mathbf{u}^\infty + \frac{\xi}{2} |div\, \mathbf{u}|^2 + \frac{1}{2\xi} \Big) dx - \int\limits_{\Omega} (\mathbf{u} \otimes \mathbf{u} + \mathbf{I}) : \nabla \mathbf{u}^\infty d\mu_\rho + \int\limits_{\Omega} \mathbf{f} \cdot (\mathbf{u} - \mathbf{u}^\infty) d\mu_\rho - \int\limits_{\Gamma+} \rho_\infty \log(1 + \rho_\infty) \mathbf{U}^\infty \cdot \mathbf{n} ds \ . \tag{15}$$

*(d) The integral identities*

$$\int\limits_{\Omega} (\mathbf{u} \otimes \mathbf{u} + \mathbf{I}) : \nabla \varphi d\mu_\rho + \int\limits_{\Omega} \mathbf{f} \cdot \varphi d\mu_\rho = \int\limits_{\Omega} \mathbf{\Sigma} : \nabla \varphi dx \ , \tag{16a}$$

$$\int\limits_{\Omega} \mathbf{u} \cdot \nabla \psi d\mu_\rho + \int\limits_{\Gamma+} \psi \rho^\infty \mathbf{U}^\infty \cdot \mathbf{n} d\Gamma = 0 \tag{16b}$$

*hold for all vector fields $\varphi \in C_0^1(\Omega)^2$ and all functions $\psi \in C^1(\Omega)$ vanishing on $\partial B \setminus \Gamma^+$. Here, $C_0^k(\Omega) \subset C^k(\Omega)$ stands for the linear subspace of compactly supported functions.*

In the same way we can define generalized solutions to problem (9),(11).

**Definition 2** *For given $\mathcal{M}$ and $\mathbf{f} \in C(\Omega)^2$ a generalized solution to problem (9),(11) is a pair $(\mu_\rho, \mathbf{u})$, where $\mu_\rho$ is a Borel measure in $\Omega$ and $\mathbf{u} \in H_0^{1,2}(\Omega)$. The generalized solution satisfies conditions (a)-(b) of Definition 1 and the bound on the rate of dissipation of energy*

$$\mathcal{D} \leq \int\limits_{\Omega} \Big( \frac{\xi}{2} |div\, \mathbf{u}|^2 + \frac{1}{2\xi} \Big) dx + \int\limits_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mu_\rho \ . \tag{17}$$

*Furthermore, the integral identities*

$$\int\limits_{\Omega} (\mathbf{u} \otimes \mathbf{u} + \mathbf{I}) : \nabla \varphi d\mu_\rho + \int\limits_{\Omega} \mathbf{f} \cdot \varphi d\mu_\rho = \int\limits_{\Omega} \mathbf{\Sigma} : \nabla \varphi dx \ , \tag{18a}$$

$$\int\limits_{\Omega} \mathbf{u} \cdot \nabla \psi d\mu_\rho = 0 \tag{18b}$$

*hold for all vector fields $\varphi \in C_0^1(\Omega)^2$ and all functions $\psi \in C_0^1(\Omega)$.*

Conditions (a)-(b) in Definition 1 imply that for generalized solutions the drag functional can be defined as follows

$$J(\rho, \mathbf{u}, \Omega) = \int\limits_{\Omega} \mathbf{\Sigma} : \nabla \mathbf{u}^\infty dx - \int\limits_{\Omega} (\mathbf{u} \otimes \mathbf{u} + \mathbf{I}) : \nabla \mathbf{u}^\infty d\mu_\rho + \int\limits_{\Omega} (\mathbf{U}^\infty - \mathbf{u}^\infty) \cdot \mathbf{f} d\mu_\rho \ . \tag{19}$$

### 3.2 Shape optimization problem

The cost functional for shape optimisation problems is the drag $J(\Omega, \mathbf{u}, \mu_\rho)$ defined by formula (19). In applications, the drag is usually minimised within the class of admissible shapes. To our best knowledge there are no results on the shape optimisation problem in the framework of generalized solutions, the simpler case of evolution equations is considered in [25]. The drag depends on the solution $(\mu_\rho, \mathbf{u})$ to problem (9)-(10), however such a solution is not in general unique. Furthermore, the drag depends on an admissible shape of the obstacle $S$. The dependence of the drag on the admissible shapes is twofold, first, it depends directly on $\Omega$ since the integrals in (19) are defined over $\Omega$, and it depends on the generalized solutions defined in $\Omega$. The restrictions on the shapes of admissible obstacles $S$ are defined in such a way that the set of admissible shapes and of the associated generalized solutions is compact. The precise conditions for admissible shapes are established below. In the present paper we do not provide the necessary optimality conditions for the problem of drag minimisation, we present only the compactness of the set of solutions over the set of admissible shapes. We establish as well the relation between the drag defined by (19) compared to the particular case of incompressible flow in absence of volume forces under assumption of sufficiently small data [49]. In order to formulate the main results we introduce some notations which will be used throughout the paper.

We introduce the set of admissible shapes, we refer the reader to [49] for the details.

**Definition 3** *For every positive $T$ and $C_\Omega$ denote by $\mathfrak{S}(T, C_\Omega)$ the class of domains $\Omega = B \setminus S$ satisfying the following conditions.*

*($\alpha$) The domain $B$ is $C^2$ and there exists a compact set $B_0 \Subset B$ such that $S \subset B_0$.*

*($\beta$) The so-called both side cone condition holds which means that for every $x \in \partial\Omega$ the set $\partial\Omega \bigcap B(x, T)$ is a graph of a Lipschitz function, and the Lipschitz constant does not exceed $C_\Omega$.*

*($\gamma$) The distance function $d(x)$ belongs to the space $H^{2,\infty}_{loc}(\Omega)$ and satisfies the inequalities*

$$\frac{C_\Omega}{d(x)}\mathbf{I} \geq D^2 d(x) \geq -\frac{C_\Omega}{d(x)a(d(x))}\mathbf{I} \quad a.e.\ in\ \ \Omega\ , \tag{20}$$

*where the symmetric matrix $D^2 d(x)$ stands for the Hessian of $d$.*

The following lemma shows that the family $\mathfrak{S}(T, C_\Omega)$ supplemented with the Hausdorff metric is a compact set.

**Lemma 4** *(i) For positive constants $T, C_\Omega$ the family of obstacles $S$ such that $\Omega = B \setminus S \in \mathfrak{S}(T, C_\Omega)$ is a compact with respect to the Hausdorff metric.*

*(ii) If $S \Subset B$ is either a convex set having an interior point or a piecewise $C^2$-smooth curvilinear polygon with the interior angles strictly between $0$ and $\pi$, then $\Omega = B \setminus S$ belongs to the class $\mathfrak{S}(T, C_\Omega)$ with some constants $T, C_\Omega$. Such a class includes e.g., the typical admissible shapes of wings in applied gas dynamics.*

**Definition 5** *In the sequel we denote by $c$ a generic constant which depends on the quantities $\|\mathbf{u}^\infty\|_{C^1(\Omega)}$, $\|\mathbf{f}\|_{L^\infty(\Omega)}$, $T$, $C_\Omega$ and $R_\Omega$. We denote by $c_\alpha$ constants depending on the same quantities and, in addition, on the parameter $\alpha$ i.e.,*

$$c = c(\|\mathbf{u}^\infty\|_{C^1(\Omega)}, \|\mathbf{f}\|_{L^\infty(\Omega)}, T, C_T, diam\,\Omega)$$

*and*

$$c_\alpha = c(\alpha, \|\mathbf{u}^\infty\|_{C^1(\Omega)}, \|\mathbf{f}\|_{L^\infty(\Omega)}, T, C_T, diam\,\Omega)\ .$$

We associate the measure $d\mu_e = (2 + |\mathbf{u}|^2)d\mu_\rho$ with the generalized solution $(\mu_\rho, \mathbf{u})$, this means that for any bounded Borel function $g : \Omega \mapsto \mathbb{R}$

$$\int\limits_\Omega g(x)d\mu_e = \int\limits_\Omega g(x)(2 + |\mathbf{u}(x)|^2)d\mu_\rho\ . \tag{21}$$

The boundedness of $\mu_e(\Omega)$ is equivalent to the boundedness of the total mass and of the kinetic energy of the gas.

The first theorem shows that the set of solutions to problem (9) with the uniformly bounded cost function is a compact.

**Theorem 6** *Fix* $\mathbf{f} \in C(\mathbb{R}^2)$. *Let the sequence of domains* $\Omega_n = B\setminus S_n$ *belong to the class* $\mathfrak{S}(T, C_\Omega)$ *with some positive* $T$, $C_\Omega$ *and let* $(\mu_{\rho,n}, \mathbf{u}_n)$ *be generalized solutions to problem* (9)-(10) *in* $\Omega_n$ *such that*

$$\sup_n \mathcal{M}_n < \infty, \quad \sup_n J(\mu_{\rho,n}, \mathbf{u}_n, \Omega_n) < \infty\ .$$

*Suppose that* $\mu_{\rho,n}$ *and* $\mathbf{u}_n$ *denote the measures and functions extended by* 0 *over the obstacles* $S_n \Subset B$, *respectively. Then there exists a subsequence, still denoted by* $(\Omega_n, \mu_{\rho,n}, \mathbf{u}_n)$, *a domain* $\Omega = B\setminus S \in \mathfrak{S}(T, C_\Omega)$, *measures* $\mu_\rho$, $\mu_e$, *and a velocity field* $\mathbf{u} \in H^{1,2}(B)$, *such that the subsequence of domains* $\Omega_n$ *converges in Hausdorff metric to the domain* $\Omega = B \setminus S$,

$$\mu_{\rho,n} \to \mu_\rho, \mu_{e,n} \to \mu_e \quad \text{*-weakly in } C_0^*(B), \quad \mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } H^{1,2}(B)\ .$$

*Moreover* $\mu_e(S) = 0$ *and*

$$\mathcal{M}_n \to \mathcal{M} = \mu_\rho(\Omega), \quad \mu_{e,n}(\Omega_n) \to \mu_e(\Omega)\ .$$

*According to our definition, the pair* $(\mu_\rho, \mathbf{u})$ *is a generalized solution to problem* (9)-(10) *in* $\Omega$ *and*

$$-\infty < J(\mu_\rho, \mathbf{u}, \Omega) = \lim_{n\to\infty} J(\mu_{\rho,n}, \mathbf{u}_n, \Omega_n)\ .$$

For problem (9),(11) the cost function is equal to zero and the value of total mass is prescribed. Thus Theorem 6 implies the following result on the compactness of the set of solutions to the boundary value problem with no slip condition.

**Theorem 7** *Fix* $\mathbf{f} \in C(\mathbb{R}^2)^2$ *and* $\mathcal{M} \in \mathbb{R}^+$. *Let the sequence of domains* $\Omega_n = B \setminus S_n$ *belong to the class* $\mathfrak{S}(T, C_\Omega)$ *with some positive* $T$, $C_\Omega$ *and* $(\mu_{\rho,n}, \mathbf{u}_n)$ *are generalized solutions to problem* (9),(11) *in* $\Omega_n$. *Suppose that* $\mu_{\rho,n}$ *and* $\mathbf{u}_n$ *denote the measures and functions extended by* 0 *over the obstacles* $S_n \Subset B$. *Then there exists a subsequence still denoted by* $(\Omega_n, \mu_{\rho,n}, \mathbf{u}_n)$, *a domain* $\Omega = B\setminus S \in \mathfrak{S}(T, C_\Omega)$, *measures* $\mu_\rho$, $\mu_e$, *and a velocity field* $\mathbf{u} \in H_0^{1,2}(B)$, *such that the subsequence of domains* $\Omega_n$ *converges in Hausdorff metric to the domain* $\Omega = B \setminus S$,

$$\mu_{\rho,n} \to \mu_\rho, \mu_{e,n} \to \mu_e \quad \text{*-weakly in } C_0^*(B), \quad \mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } H_0^{1,2}(B)\ .$$

*Moreover* $\mu_e(S) = 0$ *and*

$$\mathcal{M}_n \to \mathcal{M} = \mu_\rho(\Omega), \quad \mu_{e,n}(\Omega_n) \to \mu_e(\Omega) \ .$$

*The pair* $(\mu_\rho, \mathbf{u})$ *is a generalized solution to problem* (9),(11) *in* $\Omega$*.* The proofs of these results are given in [49].

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