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A Time Domain Method for Modeling Wave Propagation Phenomena in Viscoacoustic Media

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Summary. In many applications, realistic propagation media disperse and attenuate the acoustic waves. This behavior can be taken into account by a viscoacoustic model. Since in general, the viscoacoustic modulus is a function of frequency, incorporating this into time domain computations is practically non-feasible with classical discretization methods. However, one can use an approximation of the viscoacoustic modulus by a low-order rational function of frequency. We use here, such an approximation and show how it can be incorporated in the velocity-pressure formulation for viscoacoustic wave propagation. For the discretization in space we use a finite-element method and for the time discretization a second order centered finite difference scheme.

1 Viscoacoustic wave propagation

In an isotropic viscoacoustic medium occupying a domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, wave propagation phenomena are described by the following system of equations,

$$\begin{cases} \varrho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla p = \mathbf{f}, & \text{in } \Omega \times]0, T[, \\ p = \mu \star_t \operatorname{div} \mathbf{u}, & \text{in } \Omega \times]0, T[, \end{cases} \quad (1)$$

with some initial conditions at time $t = 0$, that we will systematically omit in the following, and where T is some finite time, up to which we compute the solution. In (1), $\mathbf{u}$ is the displacement, p the pressure, $\varrho = \varrho(\mathbf{x})$ the density and $\mu = \mu(\mathbf{x}, \omega)$ the viscoacoustic modulus, which is complex and frequency dependent. Given that dispersion and attenuation are related through the Kramer-Kronig relation [6], $\mu(\mathbf{x}, \omega)$ is uniquely determined from the quality factor $Q(\mathbf{x}, \omega)$ defined by,

$$Q(\mathbf{x}, \omega) = \frac{\operatorname{Re}(\mu(\mathbf{x}, \omega))}{\operatorname{Im}(\mu(\mathbf{x}, \omega))}. \quad (2)$$

Since $\mu(\mathbf{x}, \omega)$ is a function of frequency, the constitutive relation in the time domain (second equation of (1)), is expressed in terms of a convolution operator, denoted here by $\star_t$:

$$p(t) = \mu \star_t \operatorname{div} \mathbf{u} = \int_{-\infty}^t \mu(\mathbf{x}, t-s) \operatorname{div} \mathbf{u}(\mathbf{x}, s) ds. \quad (3)$$

This equation, however, is of no practical use for numerical calculations, as it requires saving in memory the whole history of the solution at all points of the computational domain. To overcome this inconvenience, we approximate the viscoacoustic modulus by a rational function in frequency, as it was proposed in [4, 5, 3]. To explain this approach, it is convenient to introduce the relaxation function $R(\mathbf{x}, t)$, defined by

$$\mu(\mathbf{x}, t) = \frac{\partial R(\mathbf{x}, t)}{\partial t} ; R(\mathbf{x}, t) = \left(\mu_R(\mathbf{x}) + \delta\mu(\mathbf{x}) \int_0^\infty r(\mathbf{x}, \omega') e^{-\omega' t} d\omega' \right) H(t),$$

where $\mu_R(\mathbf{x}) = \lim_{t \rightarrow +\infty} R(\mathbf{x}, t)$ is the relaxed modulus, $\mu_U(\mathbf{x}) = \mu_R(\mathbf{x}) + \delta\mu(\mathbf{x}) = \lim_{t \rightarrow 0} R(\mathbf{x}, t)$ the unrelaxed modulus, $r(\omega', \mathbf{x})$ the normalized relaxation spectrum ($\int_0^\infty r(\mathbf{x}, \omega') d\omega' = 1$) and $H(t)$ the Heaviside function. Replacing this into equation (3) gives,

$$p(\mathbf{x}, t) = \mu_U(\mathbf{x}) \operatorname{div} \mathbf{u}(\mathbf{x}, t) - \delta\mu(\mathbf{x}) \int_{-\infty}^t \int_0^{+\infty} \omega' r(\omega') e^{-\omega'(t-\tau)} \operatorname{div} \mathbf{u}(\mathbf{x}, \tau) d\omega' d\tau. \quad (4)$$

We assume now that the relaxation spectrum can be approximated by L single peaks of amplitude α_j at relaxation frequencies ω_j :

$$r(\mathbf{x}, \omega) = \sum_{j=1}^L \alpha_j(\mathbf{x}) \delta(\omega - \omega_j(\mathbf{x})) ; \sum_{j=1}^L \alpha_j(\mathbf{x}) = 1. \quad (5)$$

In this case, we get,

$$R(\mathbf{x}, t) = \left(\mu_R(\mathbf{x}) + \delta\mu(\mathbf{x}) \sum_{j=1}^L \alpha_j(\mathbf{x}) e^{-\omega_j(\mathbf{x}) t} \right) H(t) \quad (6)$$

and

$$\mu_L(\mathbf{x}, \omega) = \mu_R(\mathbf{x}) \left(1 + \sum_{j=1}^L \frac{y_j(\mathbf{x})(i\omega)}{i\omega + \omega_j(\mathbf{x})} \right). \quad (7)$$

In (7) we introduced $y_j(\mathbf{x})$ defined by,

$$y_j(\mathbf{x}) = \frac{\delta\mu(\mathbf{x})}{\mu_R(\mathbf{x})} \alpha_j(\mathbf{x}), \text{ with } \sum_{j=1}^L y_j(\mathbf{x}) = \frac{\delta\mu(\mathbf{x})}{\mu_R(\mathbf{x})}. \quad (8)$$

Remark that equation (7) can be obtained if one assumes that $\mu(\mathbf{x}, \omega)$ can be approximated by a rational function of $i\omega$,

$$\mu(\mathbf{x}, \omega) \approx \mu_L(\mathbf{x}, \omega) = \frac{P_L(\mathbf{x}, i\omega)}{Q_L(\mathbf{x}, i\omega)}, \quad (9)$$

with P_L and Q_L being polynomials of degree L in $i\omega$. Then, (7) can be re-interpreted as an expansion of (9) into partial fractions [5]. Thus approximating the viscoacoustic modulus by a rational function is equivalent to approximating the relaxation spectrum by a discrete one. For computational reasons, it is natural to seek for rational approximations of the viscoacoustic modulus, which give good results using a small number of terms. We follow the approach of Emmerich and Korn [5] which consists in taking the relaxation frequencies ω_j equidistant on a logarithmic scale and then estimate the y_j 's which minimize the difference between the approximated quality factor and the exact one. Given that μ is uniquely defined by Q , if one has a good approximation of the quality factor, this implies a good approximation for the viscoacoustic modulus. Results comparing this approximation method with the Padé approximation method [4] and the method proposed in [2] will be shown in the conference.

2 The mixed velocity-pressure formulation

Considering (4) in frequency domain and incorporating (7) leads to,

$$p(\mathbf{x}, \omega) = \mu_R(\mathbf{x}) \operatorname{div} \mathbf{u}(\mathbf{x}, \omega) + \mu_R(\mathbf{x}) \sum_{j=1}^L \frac{y_j(\mathbf{x})(i\omega)}{i\omega + \omega_j(\mathbf{x})} \operatorname{div} \mathbf{u}(\mathbf{x}, \omega). \quad (10)$$

We propose now the introduction of new variables η_j defined by,

$$(i\omega + \omega_j(\mathbf{x})) \eta_j(\mathbf{x}, \omega) = \mu_R(\mathbf{x}) y_j(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}, \omega), \quad j = 1, \dots, L,$$

where $\mathbf{v}$ is the velocity, defined as the time derivative of the displacement $\mathbf{u}$. The previous equation can be written in time domain,

$$\frac{\partial \eta_j(\mathbf{x}, t)}{\partial t} + \omega_j(\mathbf{x}) \eta_j(\mathbf{x}, t) = \mu_R(\mathbf{x}) y_j(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}, t), \quad j = 1, \dots, L.$$

Using the definition of η_j and multiplying (10) by $(i\omega)$ we get

$$(i\omega)p(\mathbf{x}, \omega) = \mu_R(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}, \omega) + \sum_{j=1}^L (i\omega) \eta_j(\mathbf{x}, \omega).$$

Our final system of equations in the time-domain is,

$$\begin{cases} \varrho \frac{\partial \mathbf{v}}{\partial t} - \nabla p = \mathbf{f}, & \text{in } \Omega \times]0, T[, \\ \frac{\partial p}{\partial t} - \sum_{j=1}^L \frac{\partial \eta_j}{\partial t} = \mu_R \operatorname{div} \mathbf{v}, & \text{in } \Omega \times]0, T[, \\ \frac{\partial \eta_j}{\partial t} + \omega_j \eta_j = \mu_R y_j \operatorname{div} \mathbf{v}, \quad \forall j & \text{in } \Omega \times]0, T[. \end{cases} \quad (11)$$

Equivalently, one can chose to eliminate the pressure and obtain a second order in time equation for the displacement by introducing adequate new variables [5]. We prefer, however, the first order velocity-pressure formulation for the following reasons:

- It can be coupled with the fictitious domain method for taking into account diffraction by objects of complicated geometry.
- A perfectly matched layer model (PML) can be written for this system. This permits us to simulate efficiently wave propagation in unbounded domains.
- This system is easier to implement in heterogeneous media, since it does not necessitate approximation of spatial derivatives of the physical parameters.

An equivalent first order velocity-pressure system was proposed in [7] where the authors used staggered finite differences schemes for the discretization. Our aim being to couple this system with the fictitious domain method, we propose instead the use of a mixed-finite element method on regular grids.

3 Discretization

A mixed formulation associated to equations (11) is given by,

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{v}, p, H) :]0, T[\longrightarrow X \times M \times (M)^L \text{ such that :} \\ \frac{d}{dt}(\varrho \mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = (\mathbf{f}, \mathbf{w}), \quad \forall \mathbf{w} \in X, \\ \frac{d}{dt}(\frac{1}{\mu_R} p, q) - \sum_{j=1}^L \frac{d}{dt}(\frac{1}{\mu_R} \eta_j, q) - b(\mathbf{v}, q) = 0, \quad \forall q \in M, \\ \frac{d}{dt}(\frac{1}{\mu_R y_j} \eta_j, q) + (\frac{\omega_j}{\mu_R y_j} \eta_j, q) - b(\mathbf{v}, q) = 0, \quad \forall j, \quad \forall q \in M, \end{array} \right. \quad (12)$$

H being the L -dimensional vector with components η_j and

$$b(\mathbf{w}, q) = \int_{\Omega} q \operatorname{div} \mathbf{w} \, d\mathbf{x}, \quad \forall (\mathbf{w}, q) \in X \times M. \quad (13)$$

The functional spaces X and M are $H(\operatorname{div}; \Omega)$ and $L^2(\Omega)$ respectively. We introduce now some finite element spaces $X_h \subset X$, $M_h \subset M$ of dimensions N_1 and N_2 respectively. The semi-discretization in space of problem (12) is,

$$\left\{ \begin{array}{l} \text{Find } (V_h, P_h, H_h) \in L^2(0, T; \mathbb{R}^{N_1}) \times L^2(0, T; \mathbb{R}^{N_2}) \times L^2(0, T; (\mathbb{R}^{N_2})^L) \text{ s.t.:} \\ M_v \frac{dV_h}{dt} + B_h P_h = F_h, \\ M_p \frac{dP_h}{dt} - \sum_{j=1}^L M_p \frac{d(H_h)_j}{dt} - B_h^T V_h = 0, \\ M_y \frac{d(H_h)_j}{dt} + M_{\omega}(H_h)_j B_h^T V_h = 0, \quad \forall j, \end{array} \right. \quad (14)$$

where B_h^T denotes the transpose of B_h . In practice we only consider regular domains in $\mathbb{R}^d$, $d = 1, 2$, that can be discretized with a uniform mesh $\mathcal{T}_h$ composed by segments or squares of size h , depending on the dimension of the problem. The finite element spaces we use are presented in [1]. We implemented for $d = 1, 2$ the lowest order elements, which are

$$\begin{cases} X_h = \{\mathbf{w}_h \text{ in } X / \forall K \in \mathcal{T}_h, \mathbf{w}_h|_K \in (Q_1)^d\}, \\ M_h = \{q_h \text{ in } M / \forall K \in \mathcal{T}_h, q_h|_K \in Q_0\}. \end{cases} \quad (15)$$

For the time discretization of (14) we use a second order centered finite differences scheme :

$$\begin{cases} \text{Find } (V_h^{n+1}, P_h^{n+\frac{3}{2}}, H_h^{n+\frac{3}{2}}) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times (\mathbb{R}^{N_2})^L \text{ such that :} \\ M_v \frac{V_h^{n+1} - V_h^n}{\Delta t} - B_h^T P_h^{n+\frac{1}{2}} = F_h^{n+1/2}, \\ M_p \frac{P_h^{n+\frac{3}{2}} - P_h^{n+\frac{1}{2}}}{\Delta t} - \sum_{j=1}^L M_p \frac{(H_h)_j^{n+\frac{3}{2}} - (H_h)_j^{n+\frac{1}{2}}}{\Delta t} + B_h^T V_h^{n+1} = 0, \\ M_y \frac{(H_h)_j^{n+\frac{3}{2}} - (H_h)_j^{n+\frac{1}{2}}}{\Delta t} + M_\omega \frac{(H_h)_j^{n+\frac{3}{2}} + (H_h)_j^{n+\frac{1}{2}}}{2} B_h V_h^{n+1} = 0, \forall j. \end{cases} \quad (16)$$

In our simulations, we assume that the problem is posed in the whole space and to solve it numerically, we couple system (16) with the perfectly matched absorbing layer model (PML).

3.1 Numerical Aspects

The numerical scheme (16) becomes explicit in time when an adequate quadrature formula is used to approximate the matrix M_v , the other mass matrices (M_p , M_y and M_ω) being diagonal, since the space M_h is composed by discontinuous functions. Stability analysis based on energy techniques shows that the scheme is stable under the following CFL condition (in homogeneous media),

$$\frac{\Delta t^2}{4} \frac{\mu_R}{\varrho} \|B_h^T B_h\| \left(1 + \sum_{j=1}^L y_j \right) \leq 1,$$

with $\|B_h^T B_h\| \geq \frac{4}{h^2}$ in 1D and $\|B_h^T B_h\| \geq \frac{8}{h^2}$ in 2D. The numerical scheme has been tested in 1D by comparing the results to the analytic solution. The 2D implementation is still in progress. Numerical results will be presented in the conference.

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