Self-inductance coefficient for toroidal thin conductors

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Abstract

We consider the inductance coefficient for a thin toroidal inductor whose thickness depends on a small parameter $\varepsilon > 0$. An explicit form of the singular part of the corresponding potential u^ε is given. This allows to construct the limit potential u (as $\varepsilon \rightarrow 0$) and an approximation of the inductance coefficient L^ε . We establish some estimates of the deviation $u^\varepsilon - u$ and of the error of approximation of the inductance. The main result shows that L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \rightarrow 0$.

Key words: Asymptotic behaviour, self inductance, eddy currents, thin domain

1 Introduction

In electrotechnical engineering, eddy current devices often involve thick conductors in which a magnetic field is induced, and circuits made of thin wires or coils, as inductors, connected to a power source generator. Mathematical modelling of such devices has then to take into account the simultaneous presence of thick conductors and thin inductors. For a two-dimensional configuration where the magnetic field has only one nonvanishing component, it was shown that the eddy current equation has the Kirchhoff circuit equation as a limit problem, as the thickness of the inductor tends to zero, see [1]. For the three-dimensional case, eddy current models require the use of a relevant quantity that is the self-inductance of the inductor, see [2], [3]. This number has to be evaluated *a priori* as a part of problem data. It is the purpose of the present

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paper to study the asymptotic behaviour of this number when the thickness of the inductor goes to zero.

Let us consider a toroidal domain of $\mathbb{R}^3$, denoted by Ω_ε , whose thickness depends on a small parameter $\varepsilon > 0$. The geometry of Ω_ε will be described in the next section. We denote by Γ_ε the boundary of Ω_ε , by n_ε the outward unit normal to Γ_ε , and by Ω'_ε the complement of its closure, that is $\Omega'_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon}$. We denote by Σ a cut in the domain Ω'_ε , that is, Σ is a smooth orientable surface such that, for any $\varepsilon > 0$, $\Omega'_\varepsilon \setminus \Sigma$ is simply connected.

Let now $\mathbf{h}^\varepsilon$ denote the time-harmonic and complex-valued magnetic field. Neglecting the displacement currents, it follows from Maxwell's equations that

$$\mathbf{curl} \mathbf{h}^\varepsilon = 0, \quad \text{div} \mathbf{h}^\varepsilon = 0 \quad \text{in } \Omega'_\varepsilon.$$

Then, by a result in [4], p. 265, $\mathbf{h}^\varepsilon$ may be written in the form

$$\mathbf{h}^\varepsilon_{|\Omega'_\varepsilon} = \nabla \varphi^\varepsilon + I^\varepsilon \nabla u^\varepsilon, \quad (1)$$

where I^ε is a complex number, $\varphi^\varepsilon \in W^1(\Omega'_\varepsilon)$ and satisfies

$$\Delta \varphi^\varepsilon = 0 \quad \text{in } \Omega'_\varepsilon,$$

and u^ε is solution of :

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, \\ \left[\frac{\partial u^\varepsilon}{\partial n} \right]_\Sigma = 0. \end{cases} \quad (2)$$

Here $W^1(\Omega'_\varepsilon)$ is the Sobolev space

$$W^1(\Omega'_\varepsilon) = \left\{ v; \rho v \in L^2(\Omega'_\varepsilon), \nabla v \in \mathbf{L}^2(\Omega'_\varepsilon) \right\},$$

equipped with the norm

$$\|v\|_{W^1(\Omega'_\varepsilon)} = \left(\|\rho v\|_{L^2(\Omega'_\varepsilon)}^2 + \|\nabla v\|_{\mathbf{L}^2(\Omega'_\varepsilon)}^2 \right)^{\frac{1}{2}}, \quad (3)$$

where $\mathbf{L}^p(\Omega'_\varepsilon)$ denotes the space $L^p(\Omega'_\varepsilon)^3$ and ρ is the weight function $\rho(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-\frac{1}{2}}$. Let us note here, see [4], pp. 649–651, that

$$|v|_{W^1(\Omega'_\varepsilon)} = \left(\int_{\Omega'_\varepsilon} |\nabla v|^2 d\mathbf{x} \right)^{\frac{1}{2}}$$

is a norm on $W^1(\Omega'_\varepsilon)$, equivalent to (3). In (2), n is the unit normal on Σ , and $[u^\varepsilon]_\Sigma$ (resp. $\left[\frac{\partial u^\varepsilon}{\partial n}\right]_\Sigma$) denotes the jump of u^ε (resp. $\frac{\partial u^\varepsilon}{\partial n}$) across Σ .

In (1), the number I^ε can be interpreted as the total current flowing in the inductor, see [3].

The inductance coefficient is then defined by the expression

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 d\mathbf{x}.$$

Our goal is to study the asymptotic behaviour of u^ε and L^ε as ε goes to zero. We first give an explicit form of the singular part of the potential u^ε which allows to construct the limit potential u (as $\varepsilon \rightarrow 0$) and an approximation of the inductance L^ε . We then prove that the deviation $\|u^\varepsilon - u\|_{W^1(\Omega'_\varepsilon)}$ and the error of approximation of L^ε are of order $O(\varepsilon^{\frac{5}{6}-\eta})$ for every $\eta > 0$. Finally we show that the inductance coefficient L^ε behaves asymptotically as $\ln \varepsilon$, when $\varepsilon \rightarrow 0$, and we thus recover the result stated (without proof) in [5], p. 137.

Let us outline the organization of this paper. In Section 2 we specify the geometry of the inductor, assuming that it is a toroidal neighborhood of a closed curve, the internal radius of the torus being proportional to a small positive number ε . Section 3 states the main result and gives the main steps in its proof. Let us note here that an extended version of this paper with detailed proofs can be consulted in [6].

2 Geometry of the domain

We consider a toroidal domain, with a small cross section. This domain may be defined as a tubular neighborhood of a closed curve. Let γ denote a closed Jordan arc of class $\mathcal{C}^3$ in $\mathbb{R}^3$, with a parametric representation defined by a function $\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^3$ satisfying

$$\mathbf{g}(0) = \mathbf{g}(1), \quad \mathbf{g}'(0) = \mathbf{g}'(1), \quad |\mathbf{g}'(s)| \geq C_0 > 0. \quad (4)$$

For each $s \in (0, 1]$ we denote by $(\mathbf{t}(s), \boldsymbol{\nu}(s), \mathbf{b}(s))$ the Serret–Frénet coordinates at the point $\mathbf{g}(s)$, *i.e.*, $\mathbf{t}(s), \boldsymbol{\nu}(s), \mathbf{b}(s)$ are respectively the unit tangent vector to γ , the principal normal and the binormal, given by

$$\mathbf{t} = \frac{\mathbf{g}'}{|\mathbf{g}'|}, \quad \boldsymbol{\nu} = \frac{\mathbf{t}'}{|\mathbf{t}'|}, \quad \mathbf{b} = \mathbf{t} \times \boldsymbol{\nu},$$

and by κ and τ respectively the curvature and the torsion of the arc γ .

Let $\widehat{\Omega} = (0, 1)^2 \times (0, 2\pi)$ and let δ denote a positive number to be chosen in a convenient way. We define, for any ε , $0 \leq \varepsilon < \delta$, the mapping $\mathbf{F}_\varepsilon : \widehat{\Omega} \rightarrow \mathbb{R}^3$ by

$$\mathbf{F}_\varepsilon(s, \xi, \theta) = \mathbf{g}(s) + r_\varepsilon(\xi)(\cos \theta \boldsymbol{\nu}(s) + \sin \theta \mathbf{b}(s)),$$

where $r_\varepsilon(\xi) = (\delta - \varepsilon)\xi + \varepsilon$. The jacobian of $\mathbf{F}_\varepsilon$ is therefore given by

$$J_\varepsilon(s, \xi, \theta) = (\delta - \varepsilon)a_\varepsilon(s, \xi, \theta)r_\varepsilon(\xi),$$

where

$$a_\varepsilon(s, \xi, \theta) = |\mathbf{g}'(s)| - r_\varepsilon(\xi)\kappa(s) \cos \theta.$$

According to (4), if δ is chosen such that

$$\delta|\kappa(s)| < |\mathbf{g}'(s)|, \quad 0 \leq s \leq 1,$$

then

$$0 < C_1 \leq a_\varepsilon \leq C_2,$$

and the mapping $\mathbf{F}_\varepsilon$ is a $\mathcal{C}^1$ -diffeomorphism from $\widehat{\Omega}$ into $\Lambda_\varepsilon^\delta = \mathbf{F}_\varepsilon(\widehat{\Omega})$.

Here and in the sequel, the quantities $C, C_1, C_2, \dots$ denote generic positive numbers that do not depend on ε .

We now set, for any $0 < \varepsilon < \delta$,

$$\Omega_\delta = \Lambda_0^\delta = \mathbf{F}_0(\widehat{\Omega}), \quad \Omega'_\delta = \mathbb{R}^3 \setminus \overline{\Omega}_\delta, \quad \Omega'_\varepsilon = \text{Int}(\overline{\Omega}'_\delta \cup \overline{\Lambda}_\varepsilon^\delta), \quad \Omega_\varepsilon = \mathbb{R}^3 \setminus \overline{\Omega}'_\varepsilon.$$

For technical reasons, we choose in the sequel $0 < \varepsilon \leq \frac{\delta}{2}$.

Given a function v on $\Lambda_\varepsilon^\delta$, we define the function $\widehat{v}$ on $\widehat{\Omega}$ by $\widehat{v} = v \circ \mathbf{F}_\varepsilon$. If $v \in L^p(\Lambda_\varepsilon^\delta)$, $1 \leq p \leq \infty$, then $\widehat{v} \in L^p(\widehat{\Omega})$ and we have

$$\int_{\Lambda_\varepsilon^\delta} v d\mathbf{x} = \int_{\widehat{\Omega}} \widehat{v} (\delta - \varepsilon) a_\varepsilon r_\varepsilon d\widehat{\mathbf{x}}.$$

Moreover, for u and v in $H^1(\Lambda_\varepsilon^\delta)$, we have

$$\begin{aligned} \int_{\Lambda_\varepsilon^\delta} \nabla u \cdot \nabla v d\mathbf{x} &= (\delta - \varepsilon) \int_{\widehat{\Omega}} \left(\frac{r_\varepsilon}{a_\varepsilon} \frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial s} + \frac{r_\varepsilon a_\varepsilon}{(\delta - \varepsilon)^2} \frac{\partial \widehat{u}}{\partial \xi} \frac{\partial \widehat{v}}{\partial \xi} \right. \\ &\quad \left. + \left(\frac{a_\varepsilon}{r_\varepsilon} + \frac{\tau^2 r_\varepsilon}{a_\varepsilon} \right) \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial \theta} \right. \\ &\quad \left. - \frac{r_\varepsilon \tau}{a_\varepsilon} \left(\frac{\partial \widehat{u}}{\partial s} \frac{\partial \widehat{v}}{\partial \theta} + \frac{\partial \widehat{u}}{\partial \theta} \frac{\partial \widehat{v}}{\partial s} \right) \right) d\widehat{\mathbf{x}}. \end{aligned}$$

We also define the set $\widehat{\Gamma} = (0, 1) \times (0, 2\pi)$ and the mapping $\mathbf{G}_\varepsilon : \widehat{\Gamma} \rightarrow \mathbb{R}^3$ by

$$\mathbf{G}_\varepsilon(s, \theta) = \mathbf{g}(s) + \varepsilon(\cos \theta \boldsymbol{\nu}(s) + \sin \theta \mathbf{b}(s)).$$

The boundary of Ω'_ε is then represented by $\Gamma_\varepsilon = \overline{\mathbf{G}_\varepsilon(\widehat{\Gamma})}$. If $w \in L^2(\Gamma_\varepsilon)$, we define $\widehat{w} \in L^2(\widehat{\Gamma})$ by $\widehat{w} = w \circ \mathbf{G}_\varepsilon$, and we have

$$\int_{\Gamma_\varepsilon} w \, d\sigma = \int_{\widehat{\Gamma}} \widehat{w} \, \varepsilon (|\mathbf{g}'| - \varepsilon \kappa \cos \theta) \, d\widehat{\sigma}.$$

Clearly, Ω_ε and its complement Ω'_ε are connected domains but they are not simply connected. To define a cut in Ω'_ε , we denote by Σ_0 the set $\mathbf{F}_0((0, 1)^2 \times \{0\})$ and $\partial\Sigma_0 = \mathbf{F}_0((0, 1) \times \{1\} \times \{0\})$. Let Σ' denote a smooth simple surface that has $\partial\Sigma_0$ as a boundary and such that the surface $\Sigma = \Sigma' \cup \Sigma_0$ is oriented and of class $\mathcal{C}^1$ (cf. [7]). We denote by Σ^+ (*resp.* Σ^-) the oriented surface with positive (*resp.* negative) orientation, and by $\mathbf{n}$ the unit normal on Σ directed from Σ^+ to Σ^- . If $w \in W^1(\mathbb{R}^3 \setminus \Sigma)$, we denote by $[w]_\Sigma$ the jump of w across Σ in the direction of $\mathbf{n}$, *i.e.*

$$[w]_\Sigma = w|_{\Sigma^+} - w|_{\Sigma^-}.$$

3 Formulation of the problem and statement of the result

We consider the boundary value problem

$$\begin{cases} \Delta u^\varepsilon = 0 & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \frac{\partial u^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [u^\varepsilon]_\Sigma = 1, \\ \left[\frac{\partial u^\varepsilon}{\partial n} \right]_\Sigma = 0, \end{cases} \quad (5)$$

where n_ε denotes the unit normal on Γ_ε pointing outward Ω'_ε and $\mathbf{n}$ is the unit normal on Σ oriented from Σ^+ toward Σ^- . The inductance coefficient is defined by

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla u^\varepsilon|^2 \, d\mathbf{x}. \quad (6)$$

We want to describe the asymptotic behaviour of u^ε and L^ε as $\varepsilon \rightarrow 0$.

We first exhibit a function that has the same singularity as might be expected for the solution of Problem (5) (as $\varepsilon \rightarrow 0$). Let us define

$$\widehat{v}(s, \xi, \theta) = \frac{\theta}{2\pi} \widehat{\varphi}(\xi), \quad (s, \xi, \theta) \in \widehat{\Omega},$$

where $\widehat{\varphi} \in \mathcal{C}^2(\mathbb{R})$ and such that

$$\widehat{\varphi}(\xi) = 1 \text{ for } 0 \leq \xi \leq \frac{1}{2}, \quad \widehat{\varphi}(\xi) = 0 \text{ for } \xi \geq \frac{3}{4}.$$

We then define $v : \mathbb{R}^3 \rightarrow \mathbb{R}$ by :

$$v(\mathbf{x}) = \begin{cases} \widehat{v}(\mathbf{F}_0^{-1}(\mathbf{x})) & \text{if } \mathbf{x} \in \Omega_\delta, \\ 0 & \text{if } \mathbf{x} \in \Omega'_\delta. \end{cases}$$

Let us also define

$$\begin{aligned} \widehat{f}(s, \xi, \theta) &= \frac{1}{2\pi a_0} \left(\frac{\kappa \sin \theta}{\delta \xi} - \frac{\tau^2 \delta \xi \kappa \sin \theta}{a_0^2} - \frac{\partial}{\partial s} \left(\frac{\tau}{a_0} \right) \right) \widehat{\varphi} \\ &\quad + \frac{\theta}{2\pi a_0 \delta^2 \xi} (2a_0 - |\mathbf{g}'|) \widehat{\varphi}' + \frac{\theta}{2\pi \delta^2} \widehat{\varphi}'', \quad (s, \xi, \theta) \in \widehat{\Omega}, \\ f(\mathbf{x}) &= \begin{cases} \widehat{f}(\mathbf{F}_0^{-1}(\mathbf{x})) & \text{if } \mathbf{x} \in \Omega_\delta, \\ 0 & \text{if } \mathbf{x} \in \Omega'_\delta, \end{cases} \\ \varphi(\mathbf{x}) &= \begin{cases} \widehat{\varphi}(\xi) & \text{if } \mathbf{x} \in \Omega_\delta, \text{ with } (s, \xi, \theta) = \mathbf{F}_0^{-1}(\mathbf{x}), \\ 0 & \text{if } \mathbf{x} \in \Omega'_\delta. \end{cases} \end{aligned}$$

By straightforward calculations, we see that function v is solution of

$$\begin{cases} \Delta v = f & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [v]_\Sigma = \varphi, \\ \left[\frac{\partial v}{\partial n} \right]_\Sigma = 0. \end{cases} \quad (7)$$

Moreover, it satisfies

$$\frac{\partial v}{\partial n_\varepsilon} = 0 \quad \text{on } \Gamma_\varepsilon.$$

Furthermore, we have for any $1 \leq p < 2$,

$$f \in L^p(\mathbb{R}^3), \quad v \in L^\infty(\mathbb{R}^3) \cap W^{1,p}(\mathbb{R}^3 \setminus \Sigma).$$

We note here that $v \notin H^1(\mathbb{R}^3 \setminus \Sigma)$. However, for any ε , $v \in H^1(\Omega'_\varepsilon \setminus \Sigma)$.

Let us now set $w^\varepsilon = u^\varepsilon - v$. We have by subtracting (7) from (5),

$$\begin{cases} -\Delta w^\varepsilon = f & \text{in } \Omega'_\varepsilon \setminus \Sigma, \\ \frac{\partial w^\varepsilon}{\partial n_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon, \\ [w^\varepsilon]_\Sigma = 1 - \varphi, \\ \left[\frac{\partial w^\varepsilon}{\partial n} \right]_\Sigma = 0. \end{cases} \quad (8)$$

We note here that Problem (8) differs from (5) by the value of the jump of the solution across Σ and by the presence of a right-hand side f . However, we notice that $1 - \varphi$ vanishes in a neighborhood of $\partial\Sigma$ and then, for Problem (8), the jump of w^ε vanishes in a neighborhood of $\partial\Sigma$.

Now, to study the asymptotic behaviour of w^ε and L^ε as $\varepsilon \rightarrow 0$ we consider the following decomposition. Let w_1 denote the solution of

$$\begin{cases} \Delta w_1 = 0 & \text{in } \mathbb{R}^3 \setminus \Sigma, \\ [w_1]_\Sigma = 1 - \varphi, \\ \left[\frac{\partial w_1}{\partial n} \right]_\Sigma = 0, \\ w_1(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (9)$$

Using [4], p. 654, and the fact that $1 - \varphi$ vanishes in a neighborhood of $\partial\Sigma$, we see that Problem (9) has a unique solution in $W^1(\mathbb{R}^3 \setminus \Sigma)$ given by

$$w_1(\mathbf{x}) = \frac{1}{4\pi} \int_\Sigma (1 - \varphi(\mathbf{y})) \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \Sigma.$$

Then we write $w^\varepsilon = w_1 + w_2^\varepsilon$, where the function w_2^ε is the unique solution, in $W^1(\Omega'_\varepsilon)$, of the exterior Neumann problem

$$\begin{cases} -\Delta w_2^\varepsilon = f & \text{in } \Omega'_\varepsilon, \\ \frac{\partial w_2^\varepsilon}{\partial n_\varepsilon} = -\frac{\partial w_1}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon, \\ w_2^\varepsilon(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (10)$$

Finally, let w_2 denote the unique solution in $W^1(\mathbb{R}^3)$ of

$$\begin{cases} -\Delta w_2 = f & \text{in } \mathbb{R}^3, \\ w_2(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & |\mathbf{x}| \rightarrow \infty. \end{cases} \quad (11)$$

As it is classical (see [8] for instance) the function w_2 is given by

$$w_2(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^3.$$

Summarizing the decomposition process of the solution to Problem (5), we have

$$u^\varepsilon = v + w_1 + w_2^\varepsilon \quad \text{in } \Omega'_\varepsilon \setminus \Sigma,$$

where v , w_1 and w_2^ε are solutions of (7), (9) and (10) respectively.

We now state our main result.

Theorem 3.1 *Let u^ε be the solution of Problem (5) and let L^ε be the inductance coefficient defined by (6). Let u be the function defined in $\mathbb{R}^3 \setminus \Sigma$ by $u = v + w_1 + w_2$, where v , w_1 and w_2 are solutions of (7), (9) and (11)*

respectively. Then, for every $\eta > 0$,

$$\begin{aligned} \|u - u^\varepsilon\|_{W^1(\Omega'_\varepsilon)} &= O(\varepsilon^{\frac{5}{6}-\eta}), \\ L^\varepsilon &= -\frac{\ell_\gamma}{2\pi} \ln \varepsilon + L' - \int_{\mathbb{R}^3} f(w_1 + w_2) d\mathbf{x} \\ &\quad + \int_\Sigma (1 - \varphi) \left(\frac{\partial w_1}{\partial n} + \frac{\partial w_2}{\partial n} + 2 \frac{\partial v}{\partial n} \right) d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}), \end{aligned}$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi} \ln \frac{\delta}{2} + \frac{1}{4\pi^2} \int_{\hat{\Omega}} \left(a_0 \xi \theta^2 (\hat{\varphi}')^2 + \frac{\delta^2 \xi \tau^2}{a_0} \hat{\varphi}^2 \right) d\hat{\mathbf{x}} + \ell_\gamma \int_{\frac{1}{2}}^1 \frac{\hat{\varphi}^2}{2\pi \xi} d\xi.$$

The next section is devoted to the proof of this result.

3.1 Proof of error estimate

Let $\tilde{w}_2^\varepsilon = w_2^\varepsilon - w_2$. Clearly $\tilde{w}_2^\varepsilon = u^\varepsilon - u$, $\tilde{w}_2^\varepsilon \in W^1(\Omega'_\varepsilon)$ and it satisfies

$$\begin{cases} \Delta \tilde{w}_2^\varepsilon = 0 & \text{in } \Omega'_\varepsilon, \\ \frac{\partial \tilde{w}_2^\varepsilon}{\partial n_\varepsilon} = -\frac{\partial w_1}{\partial n_\varepsilon} - \frac{\partial w_2}{\partial n_\varepsilon} & \text{on } \Gamma_\varepsilon, \\ \tilde{w}_2^\varepsilon(\mathbf{x}) = O(|\mathbf{x}|^{-1}), & |\mathbf{x}| \rightarrow +\infty. \end{cases} \quad (12)$$

To estimate the solution of Problem (12), we need the following result.

Lemma 3.1 *There is a constant C , independent of ε , such that :*

$$\|\psi\|_{L^2(\Gamma_\varepsilon)} \leq C \varepsilon^{\frac{1}{2}} |\ln \varepsilon|^{\frac{1}{2}} \|\psi\|_{W^1(\Omega'_\varepsilon)} \quad \text{for all } \psi \in W^1(\Omega'_\varepsilon), \quad (13)$$

$$\begin{aligned} \|\psi\|_{L^2(\Gamma_\varepsilon)} &\leq C \left(\varepsilon^{\frac{1}{2}} \|\psi\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{4}{3}-\frac{2}{p}} \|\nabla \psi\|_{L^p(\Lambda_\varepsilon^\delta)} \right) \\ &\quad \text{for all } \psi \in W^{1,p}(\Omega'_\varepsilon) \text{ with compact support, } \frac{3}{2} < p < 2. \end{aligned} \quad (14)$$

For the proof we refer to [6].

Using the variational formulation associated with (12), Cauchy–Schwarz inequality and Estimate (13), we deduce

$$\begin{aligned}
\int_{\Omega'_\varepsilon} |\nabla \tilde{w}_2^\varepsilon|^2 d\mathbf{x} &= \int_{\Gamma_\varepsilon} \left(\frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right) \tilde{w}_2^\varepsilon d\sigma \\
&\leq \left\| \frac{\partial w_1}{\partial n_\varepsilon} + \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \|\tilde{w}_2^\varepsilon\|_{L^2(\Gamma_\varepsilon)} \\
&\leq C \varepsilon^{\frac{1}{2}} |\ln \varepsilon|^{\frac{1}{2}} \left(\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} + \left\| \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \right) \|\nabla \tilde{w}_2^\varepsilon\|_{L^2(\Omega'_\varepsilon)}.
\end{aligned} \tag{15}$$

Using the integral representation of w_1 , we easily check that

$$\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^\infty(\Gamma_\varepsilon)} \leq C.$$

Therefore

$$\left\| \frac{\partial w_1}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \leq C (\text{meas } \Gamma_\varepsilon)^{\frac{1}{2}} \leq C_1 \varepsilon^{\frac{1}{2}}. \tag{16}$$

To estimate $\frac{\partial w_2}{\partial n_\varepsilon}$, we use standard regularity results for elliptic problems, see [9], p. 343, to deduce, since $f \in L^p(\mathbb{R}^3)$ for $p < 2$, that $w_2 \in W_{\text{loc}}^{2,p}(\mathbb{R}^3)$. Then we apply Estimate (14) to the function $u = \frac{\partial w_2}{\partial x_i}$, $1 \leq i \leq 3$ with $p = 2 - \eta$, $0 < \eta < \frac{1}{2}$,

$$\left\| \frac{\partial w_2}{\partial x_i} \right\|_{L^2(\Gamma_\varepsilon)} \leq C \left(\varepsilon^{\frac{1}{2}} \left\| \frac{\partial w_2}{\partial x_i} \right\|_{W^{1,p}(\Omega'_\varepsilon)} + \varepsilon^{\frac{1}{3} - \frac{\eta}{2-\eta}} \left\| \frac{\partial}{\partial x_i} \nabla w_2 \right\|_{L^p(\Lambda_\varepsilon^\delta)} \right).$$

Since both norms on the right-hand side of the above inequality are uniformly bounded and since the outward unit normal n_ε is uniformly bounded we obtain

$$\left\| \frac{\partial w_2}{\partial n_\varepsilon} \right\|_{L^2(\Gamma_\varepsilon)} \leq C \varepsilon^{\frac{1}{3} - \frac{\eta}{2-\eta}}. \tag{17}$$

Substituting (16) and (17) into (15) and using the inequality $|\ln \varepsilon| \leq C \varepsilon^{-2\eta}$, we get

$$\int_{\Omega'_\varepsilon} |\nabla \tilde{w}_2^\varepsilon|^2 d\mathbf{x} \leq C_1 \varepsilon^{\frac{5}{6} - \frac{\eta}{2-\eta} - \eta} \|\nabla \tilde{w}_2^\varepsilon\|_{L^2(\Omega'_\varepsilon)}.$$

Therefore

$$\|\nabla \tilde{w}_2^\varepsilon\|_{L^2(\Omega'_\varepsilon)} \leq C_2 \varepsilon^{\frac{5}{6} - \eta} \quad \text{for all } \eta > 0.$$

□

3.2 Proof of asymptotic expansion

Using the decomposition $u^\varepsilon = v + w^\varepsilon = v + w_1 + w_2^\varepsilon$ it follows

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla v|^2 d\mathbf{x} + \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla w^\varepsilon|^2 d\mathbf{x} + 2 \int_{\Omega'_\varepsilon \setminus \Sigma} \nabla v \cdot \nabla w^\varepsilon d\mathbf{x}.$$

Using Green's formulae, we can write L^ε in the form

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla v|^2 d\mathbf{x} - \int_{\Omega'_\varepsilon} f w^\varepsilon d\mathbf{x} + \int_{\Sigma} (1 - \varphi) \left(\frac{\partial w^\varepsilon}{\partial n} + 2 \frac{\partial v}{\partial n} \right) d\sigma. \quad (18)$$

Using the previous estimate for w_2^ε and some regularity results ($w_2 \in W^{2,p}(\Omega_\delta)$, $w_1 \in H^2(\Omega_{\frac{\delta}{2}})$), we can estimate each term in (18), to obtain for all $\eta > 0$,

$$L^\varepsilon = \int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla v|^2 d\mathbf{x} - \int_{\mathbb{R}^3} f w d\mathbf{x} + \int_{\Sigma} (1 - \varphi) \left(\frac{\partial w}{\partial n} + 2 \frac{\partial v}{\partial n} \right) d\sigma + O(\varepsilon^{\frac{5}{6}-\eta}),$$

where $w = w_1 + w_2$.

To complete the result, an explicit calculation yields

$$\int_{\Omega'_\varepsilon \setminus \Sigma} |\nabla v|^2 d\mathbf{x} = -\frac{\ell_\gamma}{2\pi} \ln \varepsilon + L' + O(\varepsilon),$$

where ℓ_γ is the length of the curve γ and

$$L' = \frac{\ell_\gamma}{2\pi} \ln \frac{\delta}{2} + \frac{1}{4\pi^2} \int_{\hat{\Omega}} \left(a_0 \xi \theta^2 (\hat{\varphi}')^2 + \frac{\delta^2 \xi \tau^2}{a_0} \hat{\varphi}^2 \right) d\hat{\mathbf{x}} + \frac{\ell_\gamma}{2\pi} \int_{\frac{1}{2}}^1 \frac{\hat{\varphi}^2}{\xi} d\xi.$$

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