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SEMISTABILITY OF FROBENIUS DIRECT IMAGES OVER CURVES

VIKRAM B. MEHTA AND CHRISTIAN PAULY

Abstract. Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$. Given a semistable vector bundle $E$ over $X$, we show that its direct image $F_*E$ under the Frobenius map $F$ of $X$ is again semistable. We deduce a numerical characterization of the stable rank-$p$ vector bundles $F_*L$, where $L$ is a line bundle over $X$.

1. Introduction

Let $X$ be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field $k$ of characteristic $p > 0$ and let $F: X \to X_1$ be the relative $k$-linear Frobenius map. It is now a well-established fact that on any curve $X$ there exist semistable vector bundles $E$ such that their pull-back under the Frobenius map $F^*E$ is not semistable [LanP], [LasP]. In order to control the degree of instability of the bundle $F^*E$, one is naturally lead (through adjunction $\text{Hom}_{O_X}(F^*E, E') = \text{Hom}_{O_{X_1}}(E, F_*E')$) to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

1.1. Theorem. Assume that $g \geq 2$. If $E$ is a semistable vector bundle over $X$ (of any degree), then $F_*E$ is also semistable.

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that $F_*L$ is stable for a line bundle $L$ ([LanP] Proposition 1.2) and that in small characteristics the bundle $F_*E$ is stable for any stable bundle $E$ of small rank [IRXY]. The main ingredient of the proof is Faltings’ cohomological criterion of semistability. We also need the fact that the generalized Verschiebung $V_r$, defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank-$r$ vector bundles over $X_1$ with fixed trivial determinant to the moduli space $\mathcal{M}_{X}(r)$ induced by pull-back under the relative Frobenius map $F$,

$$V_r: \mathcal{M}_{X_1}(r) \to \mathcal{M}_{X}(r), \quad E \mapsto F^*E$$

is dominant for large $r$. We actually show a stronger statement for large $r$.

1.2. Proposition. If $l \geq g(p - 1) + 1$ and $l$ prime, then the generalized Verschiebung $V_l$ is generically étale for any curve $X$. In particular $V_l$ is separable and dominant.

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant $\nu$ of a vector bundle $E$, defined as

$$\nu(E) := \mu_{\text{max}}(F^*E) - \mu_{\text{min}}(F^*E),$$

where $\mu_{\text{max}}$ (resp. $\mu_{\text{min}}$) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of $F^*E$.

1.3. Proposition. For any semistable rank-$r$ vector bundle $E$

$$\nu(E) \leq \min((r - 1)(2g - 2), (p - 1)(2g - 2)).$$

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We note that the inequality \( \nu(E) \leq (r - 1)(2g - 2) \) was proved in \cite{SB} Corollary 2 and in \cite{S} Theorem 3.1. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (=number of pieces) of the Harder-Narasimhan filtration of \( F^*E \) is at most \( p \) for semistable \( E \).

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant \( \nu \).

1.4. Proposition. Let \( E \) be a stable rank-\( p \) vector bundle over \( X \). Then the following statements are equivalent.

(1) There exists a line bundle \( L \) such that \( E = F_*L \).
(2) \( \nu(E) = (p - 1)(2g - 2) \).

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case \( \mu(E) = g - 1 \).

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles \( E \) with slope \( \mu(E) = g - 1 \).

Let \( E \) be a semistable vector bundle over \( X \) of rank \( r \) and let \( s \) be the integer defined by the equality

\[
\mu(E) = g - 1 + \frac{s}{r}.
\]

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map \( F : X \to X_1 \), we obtain

\[
\mu(F_*E) = g - 1 + \frac{s}{pr}.
\]

Let \( \pi : \tilde{X} \to X \) be a connected étale covering of degree \( n \) and let \( \pi_1 : \tilde{X}_1 \to X_1 \) denote its twist by the Frobenius of \( k \) (see \cite{R} section 4). The diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{F} & \tilde{X}_1 \\
\pi \downarrow & & \downarrow \pi_1 \\
X & \xrightarrow{F} & X_1
\end{array}
\]

(2.1)

is Cartesian and we have an isomorphism

\[
\pi_1^*(F_*E) \cong F_*(\pi^*E).
\]

Since semistability is preserved under pull-back by a separable morphism of curves, we see that \( \pi^*E \) is semistable. Moreover if \( F_*(\pi^*E) \) is semistable, then \( F_*E \) is also semistable.

Let \( L \) be a degree \( d \) line bundle over \( \tilde{X}_1 \). The projection formula

\[
F_*(\pi^*E \otimes F^*L) = F_*(\pi^*E) \otimes L
\]

shows that semistability of \( F_*(\pi^*E) \) is equivalent to semistability of \( F_*(\pi^*E \otimes F^*L) \).

Let \( \tilde{g} \) denote the genus of \( \tilde{X} \). By the Riemann-Hurwitz formula \( \tilde{g} - 1 = n(g - 1) \). We compute

\[
\mu(\pi^*E \otimes F^*L) = n(g - 1) + n\frac{s}{r} + pd = \tilde{g} - 1 + n\frac{s}{r} + pd,
\]

which gives

\[
\mu(F_*(\pi^*E \otimes F^*L)) = \tilde{g} - 1 + n\frac{s}{pr} + d.
\]

2.1. Lemma. For any integer \( m \) there exists a connected étale covering \( \pi : \tilde{X} \to X \) of degree \( n = p^km \) for some \( k \).
Proof. If the \( p \)-rank of \( X \) is nonzero, the statement is clear. If the \( p \)-rank is zero, we know by Corollaire 4.3.4 [R] that there exist connected étale coverings \( Y \to X \) of degree \( p^l \) for infinitely many integers \( t \) (more precisely for all \( t \) of the form \( (l-1)(g-1) \) where \( l \) is a large prime). Now we decompose \( m = p^su \) with \( p \) not dividing \( u \). We then take a covering \( Y \to X \) of degree \( p^l \) with \( t \geq s \) and a covering \( \tilde{X} \to Y \) of degree \( u \).

Now the lemma applied to the integer \( m = pr \) shows existence of a connected étale covering \( \pi : \tilde{X} \to X \) and a line bundle \( \tilde{X}^G \) with trivial determinant such that \( \pi \). □

To summarize, we have shown that for any semistable \( E \) over \( X \) there exists a covering \( \tilde{X} \to X \) and a line bundle \( L \) over \( \tilde{X} \) such that the vector bundle \( \tilde{E} := \pi^* E \otimes F^* L \) is semistable with \( \mu(\tilde{E}) = \tilde{g} - 1 \) and such that semistability of \( F_* \tilde{E} \) implies semistability of \( F_* E \).

3. Proof of Theorem 1.1

In order to prove semistability of \( F_* E \) we shall use the cohomological criterion of semistability due to Faltings [F].

3.1. Proposition ([F] Théorème 2.4). Let \( E \) be a rank-\( r \) vector bundle over \( X \) with \( \mu(E) = g - 1 \) and \( l \) an integer \( > \frac{1}{r}(g-1) \). Then \( E \) is semistable if and only if there exists a rank-\( l \) vector bundle \( G \) with trivial determinant such that
\[
h^0(X, E \otimes G) = h^1(X, E \otimes G) = 0.
\]

Moreover if the previous condition holds for one bundle \( G \), it holds for a general bundle by upper semicontinuity of the function \( G \mapsto h^0(X, E \otimes G) \).

Remark. The proof of this proposition (see [F] section 2.4) works over any algebraically closed field \( k \).

By Proposition 1.2 (proved in section 4) we know that \( V_l \) is dominant when \( l \) is a large prime number. Hence a general vector bundle \( G \in \mathcal{M}_X(l) \) is of the form \( F^* G' \) for some \( G' \in \mathcal{M}_{X_1}(l) \). Consider a semistable \( E \) with \( \mu(E) = g - 1 \). Then by Proposition 3.1 \( h^0(X, E \otimes G) = 0 \) for general \( G \in \mathcal{M}_X(l) \). Assuming \( G \) general, we can write \( G = F^* G' \) and we obtain by adjunction
\[
h^0(X, E \otimes F^* G') = h^0(X_1, F_* E \otimes G') = 0.
\]

This shows that \( F_* E \) is semistable by Proposition 3.1.

4. Proof of Proposition 1.2

According to [MS] section 2 it will be enough to prove the existence of a stable vector bundle \( E \in \mathcal{M}_{X_1}(l) \) satisfying \( F^* E \) stable and
\[
h^0(X_1, B \otimes \text{End}_0(E)) = 0,
\]
because the vector space \( H^0(X_1, B \otimes \text{End}_0(E)) \) can be identified with the kernel of the differential of \( V_l \) at the point \( E \in \mathcal{M}_X(l) \). Here \( B \) denotes the sheaf of locally exact differentials over \( X_1 \) (see [R] section 4).

Let \( l \neq p \) be a prime number and let \( \alpha \in JX_1[l] \cong JX[l] \) be a nonzero \( l \)-torsion point. We denote by
\[
\begin{align*}
\pi : \tilde{X} & \to X \\
\pi_1 : \tilde{X}_1 & \to X_1
\end{align*}
\]
the associated cyclic étale cover of \( X \) and \( X_1 \) and by \( \sigma \) a generator of the Galois group \( \text{Gal}(\tilde{X}/X) = \mathbb{Z}/l\mathbb{Z} \). We recall that the kernel of the Norm map
\[
\text{Nm} : J\tilde{X} \to JX
\]
has \( l \) connected components and we denote by 
\[ i : P := \text{Prym}(\hat{X}/X) \subset J\hat{X} \]
the associated Prym variety, i.e., the connected component containing the origin. Then we have an isogeny 
\[ \pi^* \times i : JX \times P \to J\hat{X} \]
and taking direct image under \( \pi \) induces a morphism 
\[ P \to \mathcal{M}_X(l), \quad L \mapsto \pi_* L. \]
Similarly we define the Prym variety \( P_1 \subset JX_1 \) and the morphism \( P_1 \to \mathcal{M}_{X_1}(l) \) (obtained by twisting with the Frobenius of \( k \)). Note that \( \pi_{1*} L \) is semistable for any \( L \in P_1 \) and stable for general \( L \in P_1 \) (see e.g. \([3]\)). Since \( F^*(\pi_{1*} L) \cong \pi_1(F^* L) \) — see diagram (2.3) — and since \( F^* \) induces the Verschiebung \( V_P : P_1 \to P \), which is surjective, we obtain that \( \pi_{1*} L \) and \( F^*(\pi_{1*} L) \) are stable for general \( L \in P_1 \).

Therefore Proposition 1.2 will immediately follow from the next Proposition.

4.1. **Proposition.** If \( l \geq g(p-1) + 1 \) then there exists a cyclic degree \( l \) étale cover \( \pi_1 : \hat{X}_1 \to X_1 \) with the property that 
\[ h^0(X_1, B \otimes \text{End}_0(\pi_{1*} L)) = 0 \]
for general \( L \in P_1 \).

**Proof.** By relative duality for the étale map \( \pi_1 \) we have \( (\pi_{1*} L)^* \cong \pi_{1*} L^{-1} \). Therefore 
\[ \text{End}(\pi_{1*} L) \cong \pi_{1*} L \otimes \pi_{1*} L^{-1} \cong \pi_{1*}(L^{-1} \otimes \pi_{1*} L) \]
by the projection formula. Moreover since \( \pi_1 \) is Galois étale we have a direct sum decomposition 
\[ \pi_{1*} \pi_{1*} L \cong \oplus_{i=0}^{l-1}(\sigma^i)^* L. \]
Putting these isomorphisms together we find that 
\[
H^0(X_1, B \otimes \text{End}(\pi_{1*} L)) = H^0(X_1, B \otimes \pi_{1*}(\bigoplus_{i=0}^{l-1} L^{-1} \otimes (\sigma^i)^* L)) = \\
= \bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)) = \\
\bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \pi_{1*}L_{X_1}) \oplus \bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)).
\]
Moreover \( \pi_{1*} \mathcal{O}_{\hat{X}_1} = \bigoplus_{i=0}^{l-1} \alpha^i \), which implies that 
\[
(4.1) \quad H^0(X_1, B \otimes \text{End}_0(\pi_{1*} L)) = \bigoplus_{i=0}^{l-1} H^0(X_1, B \otimes \alpha^i) \oplus \bigoplus_{i=1}^{l-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^* L)).
\]
Let us denote for \( i = 1, \ldots, l-1 \) by \( \phi_i \) the isogeny 
\[ \phi_i : P_1 \to P_1, \quad L \mapsto L^{-1} \otimes (\sigma^i)^* L. \]
Since the function \( L \mapsto h^0(X_1, B \otimes \text{End}_0(\pi_{1*} L)) \) is upper semicontinuous, it will be enough to show the existence of a cover \( \pi_1 : \hat{X}_1 \to X_1 \) satisfying
\begin{enumerate}
  \item[(1)] for \( i = 1, \ldots, l-1 \), \( h^0(X_1, B \otimes \alpha^i) = 0 \) (or equivalently, \( P \) is an ordinary abelian variety).
  \item[(2)] for \( M \) general in \( P \), \( h^0(X_1, B \otimes \pi_{1*} M) = 0 \).
\end{enumerate}
Note that these two conditions imply that the vector space \( (4.1) \) equals \( \{0\} \) for general \( L \in P_1 \), because the \( \phi_i \)'s are surjective.

We recall that \( \ker(\pi_1^* : JX_1 \to J\hat{X}_1) = \langle \alpha \rangle \cong \mathbb{Z}/l\mathbb{Z} \) and that
\[ P_1[l] = P_1 \cap \pi_{1*}(JX_1) \cong \alpha^\perp/\langle \alpha \rangle \]
where \( \alpha^\perp = \{ \beta \in JX_1[l] \mid \omega(\alpha^*, \beta) = 1 \} \) and \( \omega : JX_1[l] \times JX_1[l] \to \mu_l \) denotes the symplectic Weil form. Consider a \( \beta \in \alpha^\perp \setminus \langle \alpha \rangle \). Then \( \pi_1^* \beta \in P_1[l] \) and 
\[ \pi_{1*} \pi_1^* \beta = \bigoplus_{i=0}^{l-1} \beta \otimes \alpha^i. \]
Again by upper semicontinuity of the function $M \mapsto h^0(X_1, B \otimes \pi_1^* M)$ one observes that the conditions (1) and (2) are satisfied because of the following lemma (take $M = \pi_1^* \beta$).

4.2. Lemma. If $l \geq g(p - 1) + 1$ then there exists a pair $(\alpha, \beta) \in JX_1[l] \times JX_1[l]$ satisfying

1. $\alpha \neq 0$ and $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$,
2. for $i = 1, \ldots, l - 1$ $h^0(X_1, B \otimes \alpha^i) = 0$,
3. for $i = 0, \ldots, l - 1$ $h^0(X_1, B \otimes \beta \otimes \alpha^i) = 0$.

Proof. We adapt the proof of [R] Lemme 4.3.5. We denote by $\mathbb{F}_l$ the finite field $\mathbb{Z}/l\mathbb{Z}$. Then there exists a symplectic isomorphism $JX_1[l] \cong \mathbb{F}_l^g \times \mathbb{F}_l^g$, where the latter space is endowed with the standard symplectic form. Note that composition is written multiplicatively in $\mathbb{F}_l^g$.

Let $\Theta_B \subset JX_1$ denote the theta divisor associated to $B$. Then by [R] Lemma 4.3.5 the cardinality $A(l)$ of the finite set $\Sigma(l) := JX_1[l] \cap \Theta_B$ satisfies

$$A(l) \leq l^{2g-2} g(p - 1).$$

Suppose that there exists an isotropic 2-plane $\Pi \subset \mathbb{F}_l^g \times \mathbb{F}_l^g$ which contains $\leq l - 2$ points of $\Sigma(l)$. Then we can find a pair $(\alpha, \beta)$ satisfying the 3 properties of the Lemma as follows: any nonzero point $x \in \Pi$ determines a line (=$\mathbb{F}_l$-vector space of dimension 1). Since a line contains $l - 1$ nonzero points, we obtain at most $(l - 1)(l - 2)$ nonzero points lying on lines generated by $\Sigma(l) \cap \Pi$. Since $(l - 1)(l - 2) < l^2 - 1$ there exists a nonzero $\alpha$ in the complement of these lines. Now we note that there are $l - 1$ affine lines parallel to the line generated by $\alpha$ and the $l$ points on any of these affine lines are of the form $\beta \alpha^i$ for $i = 0, \ldots, l - 1$ for some $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. The points $\Sigma(l) \cap \Pi$ lie on at most $l - 2$ such affine lines, hence there exists at least one affine line parallel to $\langle \alpha \rangle$ avoiding $\Sigma(l)$. This gives $\beta$.

Finally let us suppose that any isotropic 2-plane contains $\geq l - 1$ points of $\Sigma(l)$. Then we will arrive at a contradiction as follows: we introduce the set

$$S = \{(x, \Pi) \mid x \in \Pi \cap \Sigma(l) \text{ and } \Pi \text{ isotropic 2-plane}\},$$

with cardinality $|S|$. Then by our assumption we have

$$|S| \geq (l - 1)N(l).$$

On the other hand, since any nonzero $x \in \mathbb{F}_l^g \times \mathbb{F}_l^g$ is contained in $\frac{l^{2g-2} - 1}{l - 1}$ isotropic 2-planes, we obtain

$$|S| \leq \frac{l^{2g-2} - 1}{l - 1} A(l).$$

Putting (4.2) and (4.3) together, we obtain

$$A(l) \geq \frac{l^{2g} - 1}{l + 1}.$$

But this contradicts the inequality $A(l) \leq l^{2g-2} g(p - 1)$ if $l \geq g(p - 1) + 1$. □

This completes the proof of Proposition 4.1. □

Remark. It has been shown [O] Theorem A.6 that $V_r$ is dominant for any rank $r$ and any curve $X$, by using a versal deformation of a direct sum a $r$ line bundles.

Remark. We note that $V_r$ is not separable when $p$ divides the rank $r$ and $X$ is non-ordinary. In that case the Zariski tangent space at a stable bundle $E \in \mathcal{M}_{X_1}(r)$ identifies with the quotient
The quotient $M$ of the inclusion $\langle \rangle$ and satisfying the inequalities $\mu$ denote we obtain that $\mu \leq (p-1)(2g-2)$.

Hence $\mu(F^*E) \leq \mu(F_\ast Q) + (p-1)(g-1)$.

Similarly we consider the subbundle $\langle \rangle$ with maximal slope, i.e., $\mu(S) = \mu_{\max}(F^*E)$ and $S$ semistable. Taking the dual and proceeding as above, we obtain that $\mu(F^*E) \geq \mu_{\max}(F^*E) - (p-1)(g-1)$.

Now we combine both inequalities and we are done.

**Remark.** We note that the inequality of Proposition 1.3 is sharp. The maximum $(p-1)(2g-2)$ is obtained for the bundles $E = F_\ast E'$ (see [JRXY] Theorem 5.3).

6. Characterization of direct images

Consider a line bundle $L$ over $X$. Then the direct image $F_\ast L$ is stable ([LanP] Proposition 1.2) and the Harder-Narasimhan filtration of $F^*F_\ast L$ is of the form (see [JRXY])

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{p-1} \subset V_p = F^*F_\ast L,$$

with $V_i/V_{i-1} \cong L \otimes \omega_X^{p-i}$.

In particular $\nu(F_\ast L) = (p-1)(2g-2)$. In this section we will show a converse statement.

More generally let $E$ be a stable rank-$rp$ vector bundle with $\mu(E) = g - 1 + \frac{d}{rp}$ for some integer $d$ and satisfying

1. the Harder-Narasimhan filtration of $F^*E$ has $l$ terms.
2. $\nu(E) = (p-1)(2g-2)$.

**Questions.** Do we have $l \leq p$? Is $E$ of the form $E = F_\ast G$ for some rank-$r$ vector bundle $G$?

We will give a positive answer in the case $r = 1$ (Proposition 6.1).

Let us denote the Harder-Narasimhan filtration by

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{l-1} \subset V_l = F^*E, \quad V_i/V_{i-1} = M_i.$$satisfying the inequalities

$$\mu_{\max}(F^*E) = \mu(M_1) > \mu(M_2) > \ldots > \mu(M_l) = \mu_{\min}(F^*E).$$

The quotient $F^*E \rightarrow M_l$ gives via adjunction a nonzero map $E \rightarrow F_\ast M_l$. Since $F_\ast M_l$ is semistable, we obtain that $\mu(E) \leq \mu(F_\ast M_l)$. This implies that $\mu(M_l) \geq -\frac{d}{rp}$. Similarly taking the dual of the inclusion $M_1 \subset F^*E$ gives a map $F^*(E^*) \rightarrow M_1^*$ and by adjunction $E^* \rightarrow F_\ast (M_1^*)$. Let us denote $\mu(M_1^*) = g - 1 + \delta$, so that $\mu(F_\ast (M_1^*)) = g - 1 + \frac{\delta}{p}$. Because of semistability of $F_\ast (M_1^*)$, we obtain $-\frac{d}{rp} = \mu(E^*) \leq \mu(F^*(M_1^*))$, hence $\delta \geq -2p(g-1) - \frac{d}{p}$. This implies that

$$H^1(X_1, \text{End}_0(E))/\langle e \rangle$$
where $e$ denotes the nonzero extension class of $\text{End}_0(E)$ by $\mathcal{O}_{X_1}$ given by $\text{End}(E)$. Then the inclusion of homotheties $\mathcal{O}_{X_1} \hookrightarrow \text{End}_0(E)$ induces an inclusion $H^1(X_1, \mathcal{O}_{X_1}) \subset H^1(X_1, \text{End}_0(E))/\langle e \rangle$ and the restriction of the differential of $V_\prime$ at the point $E$ to $H^1(X_1, \mathcal{O}_{X_1})$ coincides with the non-injective Hasse-Witt map.
\[
\mu(M_1) \leq (2p-1)(g-1) + \frac{d}{r}. \]
Combining this inequality with \( \mu(M_i) \geq g-1 + \frac{d}{r} \) and the assumption \( \mu(M_1) - \mu(M_i) = (p-1)(2g-2) \), we obtain that
\[
\mu(M_1) = (2p-1)(g-1) + \frac{d}{r}, \quad \mu(M_i) = g-1 + \frac{d}{r}.
\]

Let us denote by \( r_i \) the rank of the semistable bundle \( M_i \). We have the equality
\[
(6.1) \quad \sum_{i=1}^{l} r_i = rp.
\]
Since \( E \) is stable and \( F_*(M_i) \) is semistable and since these bundles have the same slope, we deduce that \( r_i \geq r \). Similarly we obtain that \( r_i \geq r \).

Note that it is enough to show that \( r_i = r \). Since \( E \) is stable and \( F_*M_i \) semistable and since the two bundles have the same slope and rank, they will be isomorphic.

We introduce the integers for \( i = 1, \ldots, l-1 \)
\[
\delta_i = \mu(M_{i+1}) - \mu(M_i) + 2(g-1) = \mu(M_{i+1} \otimes \omega) - \mu(M_i).
\]
Then we have the equality
\[
(6.2) \quad \sum_{i=1}^{l-1} \delta_i = \mu(M_1) - \mu(M_1) + 2(l-1)(g-1) = 2(l-p)(g-1).
\]
We note that if \( \delta_i < 0 \), then \( \text{Hom}(M_i, M_{i+1} \otimes \omega) = 0 \).

### 6.1. Proposition.

Let \( E \) be stable rank-\( p \) vector bundle with \( \mu(E) = g-1 + \frac{d}{r} \) and \( \nu(E) = (p-1)(2g-2) \). Then \( E = F_*L \) for some line bundle \( L \) of degree \( g-1 + d \).

**Proof.** Let us first show that \( l = p \). We suppose that \( l < p \). Then \( \sum_{i=1}^{l-1} \delta_i = 2(l-p)(g-1) < 0 \) so that there exists a \( k \leq l-1 \) such that \( \delta_k < 0 \). We may choose \( k \) minimal, i.e., \( \delta_i \geq 0 \) for \( i < k \). Then we have
\[
(6.3) \quad \mu(M_k) > \mu(M_i) + 2(g-1) \quad \text{for} \quad i > k.
\]
We recall that \( \mu(M_i) \leq \mu(M_{k+1}) \) for \( i > k \). The Harder-Narasimhan filtration of \( V_k \) is given by the first \( k \) terms of the Harder-Narasimhan filtration of \( F^*E \). Hence \( \mu_{\text{min}}(V_k) = \mu(M_k) \).

Consider now the canonical connection \( \nabla \) on \( F^*E \) and its first fundamental form
\[
\phi_k : V_k \hookrightarrow F^*E \xrightarrow{\nabla} F^*E \otimes \omega_X \rightarrow (F^*E/V_k) \otimes \omega_X.
\]
Since \( \mu_{\text{min}}(V_k) > \mu(M_i \otimes \omega) \) for \( i > k \) we obtain \( \phi_k = 0 \). Hence \( \nabla \) preserves \( V_k \) and since \( \nabla \) has zero \( p \)-curvature, there exists a subbundle \( E_k \subset E \) such that \( F^*E_k = V_k \).

We now evaluate \( \mu(E_k) \). By assumption \( \delta_i \geq 0 \) for \( i < k \). Hence
\[
\mu(M_i) \geq \mu(M_1) - 2(i-1)(g-1) \quad \text{for} \quad i \leq k,
\]
which implies that
\[
\deg(V_k) = \sum_{i=1}^{k} r_i \mu(M_i) \geq \text{rk}(V_k) \mu(M_1) - 2(g-1) \sum_{i=1}^{k} r_i (i-1).
\]
Hence we obtain
\[
p \mu(E_k) = \mu(V_k) \geq \mu(M_1) - 2(g-1)C,
\]
where \( C \) denotes the fraction \( \frac{\sum_{i=1}^{k} r_i (i-1)}{\text{rk}(V_k)} \). We will prove in a moment that \( C \leq \frac{\mu_1}{2} \), so that we obtain by substitution
\[
p \mu(E_k) \geq (2p-1)(g-1) + d - (g-1)(p-1) = p(g-1) + d = p \mu(E),
\]
contradicting stability of $E$. Now let us show that $C \leq \frac{p-1}{2}$ or equivalently

$$\sum_{i=1}^{k} ir_i \leq \frac{p+1}{2} \sum_{i=1}^{k} r_i.$$ 

But that is obvious if $k \leq \frac{p-1}{2}$. Now if $k > \frac{p-1}{2}$ we note that passing from $E$ to $E^*$ reverses the order of the $\delta_i$'s, so that the index $k^*$ for $E^*$ satisfies $k^* \leq \frac{p-1}{2}$. This proves that $l = p$.

Because of (6.1) we obtain $r_i = 1$ for all $i$ and therefore $E = F^*M_p$. □

7. Stability of $F^*E$?

Is stability also preserved by $F^*$?

We show the following result in that direction.

7.1. Proposition. Let $E$ be a stable vector bundle over $X$. Then $F^*E$ is simple.

Proof. Using relative duality $(F^*E)^* \cong F^*_*(E^* \otimes \omega_{X,-p})$ we obtain

$$H^0(X, \operatorname{End}(F^*E)) = H^0(X, F^*F^*F^*E \otimes E^* \otimes \omega_{X,-p}).$$

Moreover the Harder-Narasimhan filtration of $F^*F^*E$ is of the form (see [JRXY])

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{p-1} \subset V_p = F^*F^*E,$$

with $V_i/V_{i-1} \cong E \otimes \omega_{X,i}^{p-i}$. We deduce that

$$H^0(X, F^*F^*E \otimes E^* \otimes \omega_{X,-p}) = H^0(X, V_1 \otimes E^* \otimes \omega_{X,-p}) = H^0(X, \operatorname{End}(E)),$$

and we are done. □

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