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To cite this version:
Jean-Michel Coron, Emmanuel Trélat. Feedback stabilization along a path of steady-states for 1-D semilinear heat and wave equations. 2005, 6 p. hal-00086458

HAL Id: hal-00086458
https://hal.archives-ouvertes.fr/hal-00086458
Submitted on 18 Jul 2006

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Feedback stabilization along a path of steady-states for 1-D semilinear heat and wave equations

Jean-Michel Coron and Emmanuel Trélat

Abstract—We report the problem of feedback stabilization along a path of steady-states, and of exact boundary controllability of semilinear one-dimensional heat and wave equations, investigated in [5], [6]. The main result is that it is possible to move from any steady-state to any other one by means of a boundary control, provided that they are in the same connected component of the set of steady-states. The proof is based on an effective feedback stabilization procedure which is efficiently implementable.

I. INTRODUCTION

A. Semilinear one-dimensional heat equations

In this subsection, we report on results proved in [5]. Let $L > 0$ fixed and let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $C^2$. Consider the boundary control system

$$\begin{cases}
\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(y), \\
y(t, 0) = 0, \ y(t, L) = u(t),
\end{cases}$$

(1)

where the state is $y(t, \cdot) : [0, L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$.

Concerning the global controllability problem, one of the main results [8] asserts that if $f$ is globally lipschitzian then this control system is approximately globally controllable (see also [14] for exact controllability). When $f$ is superlinear, the situation is still widely open, in particular because of possible blowing up. Indeed, if $y f(y) > 0$ as $y \neq 0$, then blow-up phenomena may occur for the Cauchy problem

$$\begin{cases}
\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(y), \\
y(t, 0) = 0, \ y(t, L) = 0, \\
y(0, x) = y_0(x).
\end{cases}$$

(2)

For instance, if $f(y) = y^3$, then, for numerous initial data, there exists $T \in (0, +\infty)$ such that the unique solution to the previous Cauchy problem is well defined on $[0, T) \times [0, L]$ and satisfies

$$\lim_{t \to T} \|y(t, \cdot)\|_{L^\infty(0, L)} = +\infty,$$

see for instance [1], [11], [3], [15], [17], [20], [23] and references therein.

A natural question is then the following. Is it possible, by acting on the boundary of $[0, L]$, to avoid the blow-up phenomenon? Actually the answer is negative in general (see [10]): for some nonlinear functions $f$ satisfying

$$|f(y)| \sim |y| \log^p(1 + |y|) \quad \text{as} \quad |y| \to +\infty,$$

with $p > 2$, and for any time $T > 0$, there exist initial data which lead to blow-up before time $T$, whatever the control function $u$ is. Notice however that if

$$|f(y)| = o \left(|y| \log^{3/2}(1 + |y|)\right) \quad \text{as} \quad |y| \to +\infty,$$

then the blow-up (which could occur in the absence of control) can be avoided by means of boundary control (see [10]).

In the first case where the blow-up phenomenon cannot be compensated by means of boundary control, we propose an alternative solution. The result is that it is possible to move from any given steady-state to any other one belonging to the same connected component of the set of steady-states. More precisely, we define the notion of steady-state.

Definition 1.1: A function $y \in C^2([0, L])$ is a steady-state of the control system (1) if

$$\frac{\partial^2 y}{\partial x^2} + f(y) = 0, \ y(0) = 0.$$

We denote by $S$ the set of steady-states, endowed with the $C^2$ topology.

Introduce the Banach space

$$Y_T = \left\{ y(t, x), (t, x) \in (0, T) \times (0, L) \mid \right.\left. y \in L^2((0, T), W^{2, 2}(0, L)) \right\}$$

and endowed with the norm

$$\|y\|_{Y_T} = \|y\|_{L^2(0, T, W^{2, 2}(0, L))} + \|\frac{\partial y}{\partial t}\|_{L^2((0, T) \times (0, L))}.$$

Notice that $Y_T$ is continuously imbedded in $L^\infty((0, T) \times (0, L))$.

The following result is proved in [5].

Theorem 1.2: Let $y_0$ and $y_1$ be two steady-states belonging to a same connected component of $S$. For every neighborhood $V$ of $y_1$ in $H^1$-topology, there exists a positive real number $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a control function $u \in H^1(0, 1/\varepsilon)$ such that the solution $y(t, x)$ of

$$\begin{cases}
\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + f(y), \\
y(t, 0) = 0, \ y(t, L) = u(t), \\
y(0, x) = y_0(x),
\end{cases}$$

(4)
satisfies $y(1/\varepsilon, \cdot) \in V$.

Corollary 1.3: Under the assumptions of Theorem 1.2, there exist a time $T > 0$ and a control function $u \in L^2(0,T)$ such that the solution $y(t,x)$ in $Y_T$ of the Cauchy-Dirichlet problem (4) satisfies $y(T,\cdot) = y_1(\cdot)$.

Remark 1: In [5], we give an explicit construction of the control $u$ in a feedback-type form, and of a Lyapunov functional. We stress that the procedure is effective and consists actually in solving a stabilization problem in finite dimension. Indeed in order to construct $u$ we need to compute only a finite number of quantities related to an Hilbertian expansion of the solution. The procedure has been implemented numerically and has proved to be efficient.

Remark 2: For any $T > 0$ and $u \in L^2(0,T)$ there is at most one solution of (4) in the Banach space $Y_T$.

Remark 3: The corollary provides a (partial) global exact controllability result. The time needed in our proof is large, but on the other hand there are indeed cases where the time $T$ of controllability cannot be taken arbitrarily small. For instance in the case where $T > 0$ large enough, given initial data $(a_0, a_1)$ and final data $(b_0, b_1)$ in a suitable Hilbert space, is it possible to construct a control $u$ steering the control system (5) from the initial state $(a_0, a_1)$ to the target $(b_0, b_1)$ within time $T$? Moreover, is it possible to achieve this by an explicit and efficient numerical implementation?

If $f$ is linear, the situation is well-known (see for instance [19], [22]). In the general semilinear case, the main results as to the global controllability problem, using a variant of the Hilbert Uniqueness Method and a fixed point argument, assert that if $f$ is asymptotically linear (see [24]), and more generally if $f$ is globally Lipschitzian (see [25]), then the control system (5) is globally controllable in time $T > 2$, in the space $H^1_{(0)}(0,L) \times L^2(0,L)$, with controls in $L^2(0,T)$. The situation extends to slightly superlinear functions, or functions sharing a good sign growth condition, see [2], [18], [24], [26]. Here, $H^1_{(0)}(0,L)$ denotes the Banach space

$$H^1_{(0)}(0,L) := \{ y \in H^1(0,L) \mid y(0) = 0 \}.$$

When $f$ is highly superlinear the situation is far more intricate, in particular because of possible blowing up, as previously for the heat equation. It is proved in [26] that if $f$ satisfies

$$\liminf_{s \to -\infty} \frac{-f(s)}{s \ln^p s} > 0,$$

for $p > 2$, then the system (5) is not exactly controllable in any time $T > 0$. More precisely, for every $T > 0$, there exist initial data $(a_0, a_1) \in H^1_{(0)}(0,L) \times L^2(0,L)$ for which the solution of (5) so that $y(0,\cdot) = a_0(\cdot)$ and $y_t(0,\cdot) = a_1(\cdot)$ blows up in time $t < T$, for all control $u \in C([0,T])$. Hence there is no hope to get a general result on global controllability.

Definition 1.4: A function $y \in C^2([0,L])$ is a steady-state of the control system (5) if

$$\frac{d^2 y}{dx^2}(x) + f(y(x)) = 0, \quad y(0) = 0.$$

We denote by $S$ the set of steady-states, endowed with the $C^2$ topology.

Let us also introduce the Banach space

$$Y_T := C^0([0,T], H^1(0,L) \cap C^1([0,T], L^2(0,L))).$$

Theorem 1.5: Let $y_0$ and $y_1$ be two steady-states belonging to a same connected component of $S$. For every $\delta > 0$, there exists $\varepsilon_1 > 0$ so that, for every $\varepsilon \in (0, \varepsilon_1]$, there exists

$$\frac{d^2 y}{dx^2}(x) + f(y(x)) = 0, \quad y(0) = 0.$$
of course this path is not in general solution of system \((8), \) but if \(\varepsilon > 0\) is small enough then the \(C^1\)-path \((\tilde{y}^\varepsilon, \tilde{u}^\varepsilon)\)
\[
[0, 1/\varepsilon] \to \mathbb{R}^n \times \mathbb{R}^m
\]
\[
t \mapsto (\tilde{y}^\varepsilon(t), \tilde{u}^\varepsilon(t)) = (\tilde{y}(\varepsilon t), \tilde{u}(\varepsilon t))
\]
is “almost” a solution of system \((8).\) Indeed
\[
\|\tilde{y}^\varepsilon - g(\tilde{y}^\varepsilon, \tilde{u}^\varepsilon)\| = O(\varepsilon) \text{ as } \varepsilon \to 0^+.
\]
**Second step.** This quasi-static trajectory is not in general stable, and thus has to be stabilized. To this aim, introduce the following change of variable:

\[
z(t) = y(t) - y^\varepsilon(t),
\]
\[
v(t) = u(t) - u^\varepsilon(t),
\]
where \(t \in [0, 1/\varepsilon].\) In the new variables \(z, v,\) the control system writes, at least if \(\|z(t)\| + \|v(t)\|\) is small enough,
\[
\dot{z}(t) = A(\varepsilon t)z(t) + B(\varepsilon t)v(t) + O(\|z(t)\|^2 + \|v(t)\|^2 + \varepsilon),
\]
where \(t \in [0, 1/\varepsilon],\) and where
\[
A(\tau) = \frac{\partial g}{\partial y}(\tilde{y}(\tau), \tilde{u}(\tau)),
\]
and
\[
B(\tau) = \frac{\partial g}{\partial u}(\tilde{y}(\tau), \tilde{u}(\tau)),
\]
with \(\tau = \varepsilon t \in [0, 1].\) Therefore we have to stabilize near the origin a *slowly-varying in time* linear control system; we refer to [16] for this classical theory.

**Third step.** Under mild controllability assumptions, namely
\[
\forall \tau \in [0, 1] \quad \text{rank} \ (B(\tau), A(\tau)B(\tau), \ldots, A(\tau)^{n-1}B(\tau)) = n
\]
(Kalman condition) it is actually possible to stabilize the system by *pole shifting* and to construct a quadratic Lyapunov function. Notice that this does not work in general if the system is not slowly-varying. So if \(\varepsilon\) is small enough then using this Lyapunov function we infer that \(y(1/\varepsilon)\) belongs to some prescribed neighborhood of the target \(y_1.\) At this stage, a stabilization result is achieved.

**Fourth step.** If the system \((8)\) is *locally controllable* near the point \(y_1,\) we conclude that it is possible to steer the system in finite time from the point \(y(1/\varepsilon)\) to the desired target \(y_1.\) Usually such a local controllability result is achieved by using an implicit function argument, after proving that the linearized system is controllable.

**Remark 8:** The use of quasi-static deformation for the controllability of a nonlinear partial differential control system has already been used in [4]. But note that in [4] the quasi-static trajectory \((\tilde{y}^\varepsilon, \tilde{u}^\varepsilon)\) was stable so it was not necessary to perform steps 2 and 3.
B. Application to the heat equation

Let $y_0$ and $y_1$ in the same connected component of $S$. We construct in $S$ a $C^1$ path $(\bar{y}(\tau, \cdot), \bar{u}(\tau))$, $0 \leq \tau \leq 1$, joining $y_0$ to $y_1$. For each $i = 0, 1$ set

$$\alpha_i = y_i'(0).$$

Then $y_i(\cdot) = y^\alpha(\cdot), i = 0, 1$, where the maximal solution of

$$\frac{d^2 y}{dx^2} + f(y) = 0, \quad y(0) = 0, y'(0) = \alpha,$$

is denoted by $y^\alpha(\cdot)$. Now set

$$\bar{y}(\tau, x) = y^{1-\tau}\alpha_0 + \tau \alpha_1(x) \quad \text{and} \quad \bar{u}(\tau) = \bar{y}(\tau, L),$$

where $\tau \in [0, 1]$ and $x \in [0, L]$. By construction we have

$$\bar{y}(0, \cdot) = y_0(\cdot), \quad \bar{y}(1, \cdot) = y_1(\cdot) \quad \text{and} \quad \bar{u}(0) = \bar{u}(1) = 0,$$

and thus $(\bar{y}(\tau, \cdot), \bar{u}(\tau))$ is a $C^1$ path in $S$ connecting $y_0$ to $y_1$.

We then reduce the problem as follows.

Let $\varepsilon > 0$. For every $t \in [0, 1/\varepsilon]$ and every $x \in [0, L]$, set

$$z(t, x) = y(t, x) - \bar{y}(\varepsilon t, x),$$

$$w(t) = w(t) - \bar{u}(\varepsilon t).$$

Then $z$ satisfies the initial-boundary problem

$$\begin{cases}
  z_t = z_{xx} + f'(\bar{y})z + z^2 \int_0^1 (1-s)f''(\bar{y} + sz)ds - \varepsilon z, \\
  z(t, 0) = 0, \quad z(t, L) = v(t), \\
  z(0, x) = 0.
\end{cases}$$

(9)

To reduce the problem to a Dirichlet-type problem, set

$$w(t, x) = z(t, x) - \frac{x}{L} v(t),$$

(10)

and suppose that the control $v$ is derivable. This leads to

$$\begin{cases}
  w_t = w_{xx} + \frac{x}{L} f'(\bar{y})w + w^2 \int_0^1 (1-s)f''(\bar{y} + sz)ds - \varepsilon w, \\
  w(t, 0) = w(t, L) = 0, \\
  w(0, x) = -\frac{x}{L} v(0),
\end{cases}$$

(12)

where

$$r(\varepsilon, x, t) = -\varepsilon \bar{y} + \frac{x}{L} v + \left(w + \frac{x}{L} v\right)^2 \int_0^1 (1-s)f''(\bar{y} + s(w + \frac{x}{L} v))ds.$$

The aim is then to prove that there exist $\varepsilon$ small enough and a pair $(v, w)$ solution of (12) such that $w(1/\varepsilon, \cdot)$ belongs to some arbitrary neighborhood of 0 in $H^1_0$-topology. To achieve this proof consists in constructing an appropriate control function and a Lyapunov functional which stabilizes system (12) to 0 (see [5]).

The proof requires a precise spectral analysis of the problem. We introduce the one-parameter family of linear operators

$$A(\tau) = \Delta + f'(\bar{y}(\tau, \cdot))Id, \quad \tau \in [0, 1],$$

(14)

defined on $H^2(0, L) \cap H^1_0(0, L)$. Let $(e_j(\tau, \cdot))_{j \geq 1}$ be an Hilbertian basis of $L^2(0, L)$ of eigenfunctions of $A(\tau)$, such that for each $j \geq 1$ and each $\tau \in [0, 1]$,

$$e_j(\tau, \cdot) \in H^1_0(0, L) \cap C^2([0, L]),$$

and let $(\lambda_j(\tau))_{j \geq 1}$ denote the corresponding eigenvalues. From the minimax principle, these eigenfunctions and eigenvalues are $C^1$ functions of $\tau$. Moreover for each $\tau \in [0, 1]$,

$$-\infty < \cdots < \lambda_n(\tau) < \cdots < \lambda_1(\tau),$$

and

$$\lambda_n(\tau) \underset{n \rightarrow +\infty}{\rightarrow} -\infty.$$

From the continuity of the eigenvalues on $[0, 1]$, we can define $n$ as the maximal number of eigenvalues taking at least a nonnegative value as $\tau \in [0, 1]$, i.e. there exists $\eta > 0$ such that

$$\forall t \in [0, 1/\varepsilon], \quad \forall k > n, \quad \lambda_k(\varepsilon t) < -\eta < 0.$$  (15)

The integer $n$ can be arbitrarily large. For example if $f(y) = y^3$ and if $y'(0) \rightarrow +\infty$ then $n \rightarrow +\infty$.

We also set, for any $\tau \in [0, 1]$ and $x \in [0, L]$,

$$a(\tau, x) = \frac{x}{L} f'(\bar{y}(\tau, x)) \quad \text{and} \quad b(x) = -\frac{x}{L}.$$

(16)

In these notations system (12) leads to

$$w_t(t, \cdot) = A(\varepsilon t)w(t, \cdot) + a(\varepsilon t, \cdot)v(t) + b(\cdot)v'(t) + r(\varepsilon, t, \cdot).$$

(17)

Any solution $w(t, \cdot) \in H^1_0(0, L)$ of (16) can be expanded as series in the eigenfunctions $e_j(\varepsilon t, \cdot)$, convergent in $H^1_0(0, L)$,

$$w(t, \cdot) = \sum_{j=1}^{\infty} \hat{w}_j(t)e_j(\varepsilon t, \cdot).$$

In particular, we get, for $i = 1 \ldots n$,

$$w_i(t) = \lambda_i(\varepsilon t)w_i(t) + a_i(\varepsilon t)v(t) + b_i(\varepsilon t)v'(t) + r_1(\varepsilon, t),$$

(18)

and consider $v(t)$ as a state and $\alpha(t)$ as a control. Then the former finite dimensional system may be rewritten as

$$\begin{cases}
  \dot{v} = \alpha, \\
  \dot{w}_1 = \lambda_1 w_1 + a_1 v + b_1 \alpha + r_1, \\
  \vdots \\
  \dot{w}_n = \lambda_n w_n + a_n v + b_n \alpha + r_n.
\end{cases}$$

(19)
If we introduce the matrix notations
\[
X_1(t) = \begin{pmatrix} v(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad R_1(\varepsilon, t) = \begin{pmatrix} r_1^1(\varepsilon, t) \\ \vdots \\ r_n^1(\varepsilon, t) \end{pmatrix},
\]
\[
A_1(\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1(\tau) & \lambda_1(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n(\tau) & 0 & \cdots & \lambda_n(\tau) \end{pmatrix},
\]
\[
B_1(\tau) = \begin{pmatrix} 1 \\ b_1(\tau) \\ \vdots \\ b_n(\tau) \end{pmatrix},
\]
then equations (20) yield the finite dimensional linear control system
\[
X_1'(t) = A_1(\varepsilon t)X_1(t) + B_1(\varepsilon t)\alpha(t) + R_1(\varepsilon, t). \quad (21)
\]

We then move, by means of an appropriate feedback control, the \( n \) first eigenvalues of the operator \( A \), without moving the others, in order to make all eigenvalues negative.

It is easy to check that, for each \( \tau \in [0, 1] \) the pair \((A_1(\tau), B_1(\tau))\) satisfies the Kalman condition, i.e.
\[
\text{rank} (B_1(\tau), A_1(\tau)B_1(\tau), \ldots, A_1(\tau)^{n-1}B_1(\tau)) = n. \quad (22)
\]
Thus, for each \( \tau \in [0, 1] \) there exist scalars \( k_0(\tau), \ldots, k_n(\tau) \) such that, if we denote
\[
K_1(\tau) = (k_0(\tau), \ldots, k_n(\tau)),
\]
then the matrix \( A_1(\tau) + B_1(\tau)K_1(\tau) \) admits \(-1\) as an eigenvalue with order \( n + 1 \).

The remainder of the proof consists in proving, using Lyapunov functions, that the feedback control function \( \alpha(t) = K_1tX_1(t) \) stabilizes the complete infinite dimensional system along the path of steady-states (see [5] for details).

C. Application to the wave equation

The procedure is similar.

Let \( \varepsilon > 0 \), and let \( y \) denote the solution of (7) in \( Y_{1/\varepsilon} \), associated to a control \( u \in H^2(0, 1/\varepsilon) \). We set, for all \( t \in [0, 1/\varepsilon] \) and \( x \in [0, L] \),
\[
z(t, x) := y(t, x) - \tilde{y}(\varepsilon t, x),
\]
\[
u_1(t) := u(t) - \bar{u}(\varepsilon t).
\]
Then,
\[
z_{tt} = z_{xx} + f'\tilde{y}z + \varepsilon^2 \int_0^1 (1-s)f''\tilde{y}sz\,ds - \varepsilon^2 r_\tau y,
\]
\[
z(t, 0) = 0, \quad z_x(t, L) = u_1(t),
\]
\[
z(0, x) = 0, \quad z_t(0, x) = -\varepsilon \tilde{y}(0, x).
\]

Notice that, if the nonlinearity \( f \) and the residual term \( r \) were equal to zero, then, as explained previously, setting \( u_1(t) = -\alpha z_t(t, L) \), the energy function
\[
t \mapsto \int_0^L (z_t(t, x))^2 + z_x(t, x)^2\,dt
\]
would be exponentially decreasing. This suggests to seek the control function \( u_1(t) \) in the form
\[
u_1(t) = -\alpha z_t(t, L) + v(t),
\]
where \( \alpha > 0 \) has to be chosen in a convenient way. Set
\[
w(t, x) := z(t, x) - \frac{x(x-L)}{L} v(t).
\]
This leads to the system
\[
\begin{cases}
w_{tt} = w_{xx} + f'(\tilde{y})w - \frac{x(x-L)}{L} v''
&+ \left( \frac{x(x-L)}{L} f'\tilde{y}\frac{2}{L} \right) v + r(\varepsilon, t, x), \\
w(t, 0) = 0, \quad w_x(t, L) = -\alpha w_t(t, L), \\
w(0, x) = -\frac{x(x-L)}{L} v(0),
\end{cases}
\]
where \( r(\varepsilon, t, x) \) is a remainder term. The aim is to prove that, given a neighborhood \( V \) of \((0, 0, 0, 0) \) in \( \mathbb{R} \times \mathbb{R} \times H^1_{0}) (0, L) \times L^2(0, L) \), for \( \varepsilon > 0 \) small enough, there exists a pair \((v, w)\) solution of (26), satisfying \( v(0) = v'(0) = 0 \), such that
\[
(v(1/\varepsilon), v'(1/\varepsilon), w(1/\varepsilon), \cdot), w_t(1/\varepsilon, \cdot)) \in V.
\]
As previously, a precise spectral analysis of the system is due. Set
\[
H := \left\{ \begin{pmatrix} w_1^1 \\ w^2_1 \end{pmatrix} \in H^1 \times L^2((0, L), \mathbb{C}) \mid w^1(0) = 0 \right\}. \quad (27)
\]
Introduce the one-parameter family of linear operators
\[
\bar{A}(\tau) := \begin{pmatrix} 0 & 1 \\ A(\tau) & 0 \end{pmatrix}, \quad (28)
\]
where \( A(\tau) := \triangle + f'(\tilde{y}(\tau, \cdot))\mathbb{I} \), \( \tau \in [0, 1] \), on the domain
\[
D(\bar{A}(\tau)) := \left\{ \begin{pmatrix} w_1^1 \\ w_1^2 \end{pmatrix} \in H \mid w^1 \in H^2((0, L), \mathbb{C}), \\
w^2 \in H^1((0, L), \mathbb{C}), \quad w^2(0) = 0, \quad w_x^1(L) = -\alpha w^2(L) \right\}. \quad (29)
\]
The following nontrivial lemma is crucial (see [6] for details and for a proof).

Lemma 2.1: For every \( \tau \in [0, 1] \), the operator \( \bar{A}(\tau) \) has a compact resolvent in \( H \), and thus its spectrum consists of isolated eigenvalues \( \lambda_k(\tau) \) \( k \in I \), where \( I = \mathbb{Z} \) or \( I = \mathbb{Z} \setminus \{0\} \). There exists a Riesz basis \( \{\varepsilon_k(\tau, \cdot)\}_k \in I \) of \( H \), having a dual Riesz basis \( \{f_k(\tau, \cdot)\}_k \in I \), such that:
- \( e_k(\tau, \cdot) \in D(\bar{A}(\tau)) \), and \( \|e_k(\tau, \cdot)\|_H = 1 \), for every \( k \in I \) and every \( \tau \in [0, 1] \);
for every integer \( k \in I \), the functions \( \tau \mapsto e_k(\tau, \cdot) \) and \( \tau \mapsto f_k(\tau, \cdot) \) are of class \( C^1 \) on \([0, 1]\);

- each eigenvalue \( \lambda_k(\tau) \) is geometrically simple;
- there exists an integer \( n_0 \geq 0 \) so that, for every integer \( k \) satisfying \( |k| > n_0 \), each eigenvalue \( \lambda_k(\tau) \) is algebraically simple, and satisfies
  \[
  \lambda_k(\tau) = \frac{1}{2L} \ln \frac{\alpha - 1}{\alpha + 1} + \frac{k\pi}{L} + O\left( \frac{1}{|k|} \right),
  \]
  \( (30) \)
  as \( |k| \to +\infty \), uniformly for \( \tau \in [0, 1] \);
- if \( |k| > n_0 \), then \( e_k(\tau, \cdot) \) is an eigenfunction of \( \bar{A}(\tau) \), associated to the (algebraically simple) eigenvalue \( \lambda_k(\tau) \), and \( f_k(\tau, \cdot) \) is an eigenfunction of \( \bar{A}(\tau)^* \), associated to the (algebraically simple) eigenvalue \( \lambda_k(\tau) \);

- for every integer \( k > n_0 \) and every \( \tau \in [0, 1] \),
  \[
  \lambda_k(\tau) = \overline{\lambda_k(\tau)},
  \]
  \[
  e_k(\tau, \cdot) = e^{-k}(\tau, \cdot), \quad f_k(\tau, \cdot) = f^{-k}(\tau, \cdot);
  \]
- for every integer \( k \) so that \( |k| \leq n_0 \), there holds
  \[
  \bar{A}(\tau)e_k(\tau, \cdot) \in \text{Span}\{e_p(\tau, \cdot) | |p| \leq n_0\},
  \]
  and
  \[
  \bar{A}(\tau)^* f_k(\tau, \cdot) \in \text{Span}\{f_p(\tau, \cdot) | |p| \leq n_0\}.
  \]

Remark 9: To prove this lemma, we first prove the existence of a Riesz basis of \( H \), consisting of generalized eigenfunctions \( (\tilde{e}_k(\tau, \cdot))_{k \in \mathbb{Z}} \) of \( \bar{A}(\tau) \), associated to the eigenvalues \( (\lambda_k(\tau))_{k \in \mathbb{Z}} \). However, if \( |k| \leq n_0 \), then the corresponding eigenvalue \( \lambda_k(\tau) \) is not necessarily algebraically simple. The proof then consists in modifying the generalized eigenfunctions \( \tilde{e}_k(\tau, \cdot) \), for \( |k| \leq n_0 \), so as to obtain new vectors \( e_k(\tau, \cdot) \), \( |k| \leq n_0 \), that are \( C^1 \) functions of \( \tau \), but are not necessarily generalized eigenfunctions of \( A(\tau) \).

Let us show how to isolate the finite dimensional unstable part of the system. Let \( \alpha > 1 \) so that
\[
\frac{1}{2L} \ln \frac{\alpha - 1}{\alpha + 1} < -1.
\]
Using (30), only a finite number of eigenvalues may have a nonnegative real part as \( \tau \in [0, 1] \). More precisely, there exists an integer \( n \) so that
\[
\forall \tau \in [0, 1], \quad \forall k \in \mathbb{Z}, \quad (|k| > n) \Rightarrow (\Re(\lambda_k(\tau)) < -1).
\]
(31)

Every solution can then be expanded as series in the Riesz basis \( (e_j(\tau, \cdot))_{j \in I} \), convergent in \( H \). As previously, we then move, by means of an appropriate feedback control, the \( 2n+1 \) eigenvalues \( \lambda_0(\tau), \ldots, \lambda_{2n}(\tau) \), whose real part may be nonnegative, without moving the others, so that all eigenvalues then have a negative real part. We obtain a differential system in \( \mathbb{R}^{2n+1} \) controlled by \( u, v' \), \( v'' \). As for the heat equation, we have then to check that Kalman's condition holds, which allows one to derive a feedback control stabilizing the finite dimensional system. The rest of the proof consists in showing that this control actually stabilizes the complete system (see [6]).