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GEOMETRIC OPTIMAL CONTROL OF ELLIPTIC KEPLERIAN ORBITS

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Abstract. This article deals with the transfer of a satellite between Keplerian orbits. We study the controllability properties of the system and make a preliminary analysis of the time optimal control using the maximum principle. Second order sufficient conditions are also given. Finally, the time optimal trajectory to transfer the system from an initial low orbit with large eccentricity to a terminal geostationary orbit is obtained numerically.

1. **Introduction.** An important problem in astronautics is to transfer a satellite between elliptic orbits. Recent research projects concern orbital transfer with electro-ionic propulsion where the thrust is very low. Two techniques are mainly used. First of all, the transfer can be achieved using stabilization methods, see for instance [11, 12]. This approach provides simple feedback controllers but the transfer time is not taken into account. Secondly, orbital transfers can be performed by minimizing a cost: the time optimal control problem is important for low propulsion because the transfer towards the geostationary orbit can take several months. Moreover, minimizing the consumption with maximal thrust is equivalent to minimizing the time.

In this article, we focus on the time optimal control problem. Preliminary results are contained in [18, 14, 9] and we follow the same line. First of all, we make an analysis of the extremal solutions of the maximum principle with geometric methods. Then, using the theoretical results of [7, 21], we give second order sufficient conditions which can easily be implemented.

The organization of the paper is the following. In §2, we recall all the preliminaries to analyze the problem: choices of coordinates to represent the equation,

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maximum principle and extremal solutions. In §3 we describe the Lie bracket structure of the system. Indeed, the thrust can be decomposed into a tangential-normal or radial-orthoradial frame. Using geometric control techniques, the effect of each controller can be analyzed. Our computations allow us to derive the controllability properties of the system and preliminary results about extremal solutions. We give in §4 a nilpotent model to analyze the structure of the local time optimal control, pointing the connection with systems from sub-Riemannian geometry. This model resolves a singularity of the problem that was observed numerically in the transfer to the geostationary orbit where the satellite inversed its thrust at a pericenter. The final section addresses the question of second order conditions for which, in our analysis, tests can be implemented. In particular, we present an algorithm to compute the optimal control steering a satellite from a low eccentric orbit towards the geostationary one. A candidate being selected using the maximum principle and computed by means of a shooting method, sufficient second order optimality conditions are checked.

2. **Generalities.**

2.1. **Models and basic properties.** In this section, we recall the representations of the system as well as basic properties which are crucial for our analysis. We denote by m the mass of the satellite and by F the thrust of the engine. The equation describing the dynamics in Cartesian coordinates is

$$\ddot{q} = -q\frac{\mu}{|q|^3} + \frac{F}{m}$$

where q is the position of the satellite measured in a fixed frame I, J, K whose origin is the Earth center, and μ the gravitation constant. The free motion where $F = 0$ is the Kepler equation. The thrust is bounded, $|F| \leq F_{\max}$, and the mass variation is described by

$$\dot{m} = -\frac{|F|}{v_e} \tag{1}$$

where v_e is a positive constant. Practically, the mass has to remain greater than the mass of the satellite without fuel, $m \geq \chi_0$, and we have a simplified model called the *constant mass model* if (1) is not taken into account. Roughly, the latter is sufficient for our geometric analysis though the mass equation has to be included for numerical computations. Besides, observe that if the thrust is maximal, maximizing the final mass boils down to minimizing the transfer time. If $q \wedge \dot{q} \neq 0$, the thrust can be decomposed into the *tangential-normal* frame according to

$$F = u_t F_t + u_n F_n + u_c F_c$$

where

$$F_t = \frac{\dot{q}}{|\dot{q}|}\frac{\partial}{\partial \dot{q}},\ F_c = \frac{q \wedge \dot{q}}{|q \wedge \dot{q}|}\frac{\partial}{\partial \dot{q}},$$

and $F_n = F_c \wedge F_t$. Another important decomposition used in the sequel is the *radial-orthoradial* frame

$$F = u_r F_r + u_{\mathrm{or}} F_{\mathrm{or}} + u_c F_c,$$

with

$$F_r = \frac{q}{|q|}\frac{\partial}{\partial \dot{q}},$$

and $F_{\rm or} = F_c \wedge F_r$. In both cases, if $u_c = 0$ we have a 2D-problem, restricting our system to the osculating plane spanned by $q(0)$ and $\dot q(0)$. The following results are standard.

PROPOSITION 2.1. *Consider the Kepler equation $\ddot q = -q\mu/|q|^3$. The vectors below are first integrals:*

$$\begin{array}{rcll} c & = & q \wedge \dot q & \textit{(angular momentum)},\\ L & = & -q\frac{\mu}{|q|} + \dot q \wedge c & \textit{(Laplace integral)}. \end{array}$$

Moreover, the energy $H(q,\dot q) = \frac{1}{2}|\dot q|^2 - \frac{\mu}{|q|}$ is preserved and the following relations hold:

$$L \cdot c = 0,\ L^2 = \mu^2 + 2Hc^2.$$

PROPOSITION 2.2. *For the Kepler motion, if $c = 0$ then q and $\dot q$ are collinear and the motion is on a line. If $c \neq 0$,*

1. *if $L = 0$, then the motion is circular;*
2. *if $L \neq 0$ and $H < 0$, then the trajectory is an ellipse,*

$$|q| = \frac{c^2}{\mu + |L|\cos(\theta - \theta_0)},$$

where θ_0 is the argument of pericenter.

DEFINITION 2.1. The domain $\Sigma_e = \{(q,\dot q) \mid H < 0,\ c \neq 0\}$ is filled by elliptic orbits and is called the *elliptic* (*2D-elliptic*, in the planar case) *domain*. To each (c, L) corresponds a unique oriented ellipse.

Remark 2.1. Using (c, L) coordinates, we have a neat representation of the state space. In particular, Σ_e is a fiber bundle — the fiber being $\mathbf{S}^1$ — whose topology is clear. In these coordinates, the dynamics of the orbital transfer becomes

$$\begin{array}{rcl} \dot c & = & q \wedge \dfrac{F}{m},\\ \dot L & = & F \wedge c + \dot q \wedge (q \wedge \dfrac{F}{m}). \end{array}$$

This representation was introduced in [11] to compute controls in orbital transfer using a stabilization method. A more detailed representation is provided by the orbit elements. In the referential (I, J, K), we identify the plane (I, J) to the equatorial plane, so that each point $x = (q,\dot q)$ of the elliptic domain is represented by the geometric parameters of the osculating orbit [20]:

Ω, longitude of the ascending node;
ω, argument of the pericenter;
i, inclination of the osculating plane;
a, semi-major axis of the ellipse;
$|e|$, eccentricity;
l, cumulated longitude, or longitude (modulo 2π).

If e is the eccentricity vector collinear to L, with modulus $|e|$, we denote by $\widetilde\omega$ the angle between I, and

$$e_x = |e|\cos\widetilde\omega,\ e_y = |e|\sin\widetilde\omega.$$

The line of the nodes contained in the plane (I, J) is represented by

$$h_x = \tan\frac{i}{2}\cos\Omega,\ h_y = \tan\frac{i}{2}\sin\Omega,$$

and we get the two systems of equations hereafter [24]. Both are commonly referred to as *Gauss equations*.

Tangential-normal system. We use the coordinates $x = (a, e_x, e_y, h_x, h_y, l)$ and the thrust is decomposed into the tangential-normal frame

$$\dot{x} = F_0 + \frac{1}{m}(u_t F_t + u_n F_n + u_c F_c).$$

In these coordinates,

$$\begin{aligned}
F_0 &= \sqrt{\frac{\mu}{P}}\frac{W^2}{P}\frac{\partial}{\partial l},\\
F_t &= \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(\frac{2WP|\eta|}{(1-e^2)^2}\frac{\partial}{\partial a} + \frac{2W\eta_x}{|\eta|}\frac{\partial}{\partial e_x} + \frac{2W\eta_y}{|\eta|}\frac{\partial}{\partial e_y}\right),\\
F_n &= \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(\frac{D\eta_x - W\eta_y}{|\eta|}\frac{\partial}{\partial e_x} + \frac{W\eta_x + D\eta_y}{|\eta|}\frac{\partial}{\partial e_y}\right),\\
F_c &= \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(-Ze_y\frac{\partial}{\partial e_x} + Ze_x\frac{\partial}{\partial e_y} + \frac{C\cos l}{2}\frac{\partial}{\partial h_x} + \frac{C\sin l}{2}\frac{\partial}{\partial h_y} + Z\frac{\partial}{\partial l}\right),
\end{aligned}$$

where

$$\begin{aligned}
P &= a(1-e^2),\\
\eta &= (\eta_x, \eta_y) \;=\; (e_x + \cos l, e_y + \sin l),\\
W &= 1 + e_x\cos l + e_y \sin l,\\
D &= e_x \sin l - e_y \cos l,\\
C &= 1 + h^2,\\
Z &= h_x \sin l - h_y \cos l.
\end{aligned}$$

The variable P is the so-called semi-latus rectum of the osculating ellipse and is used in the second system.

Radial-orthoradial system. We set $x = (P, e_x, e_y, h_x, h_y, l)$ and the thrust is decomposed into the radial-orthoradial frame

$$\dot{x} = F_0 + \frac{1}{m}(u_r F_r + u_{\mathrm{or}} F_{\mathrm{or}} + u_c F_c).$$

In these coordinates, F_0 and F_c are unchanged, while

$$\begin{aligned}
F_r &= \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(W\sin l\frac{\partial}{\partial e_x} - W\cos l\frac{\partial}{\partial e_y}\right),\\
F_{\mathrm{or}} &= \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(2P\frac{\partial}{\partial P} + (W\cos l + \eta_x)\frac{\partial}{\partial e_x} + (W\sin l + \eta_y)\frac{\partial}{\partial e_y}\right).
\end{aligned}$$

A geostationary orbit corresponds to $|e| = |h| = 0$. The 2D-model is obtained by setting $u_c = 0$. A standard representation is then to parametrize the trajectories by the cumulated longitude l.

Remark 2.2. The Gauss coordinates are important in celestial mechanics and in orbital transfer with low thrust: the longitude is the fast variable, and the remaining variables are describing the slow evolution of the orbit elements. They are used in numerical simulations.

To understand the controllability properties of the 2D-problem, we can use the following approach due to Lagrange-Binet [15]. We write the system in polar coordinates

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{\mu}{r^2} + \frac{u_r}{m}, \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{u_{\mathrm{or}}}{m}, \end{aligned}$$

so that, up to a renormalization, it is equivalent to

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -\frac{1}{r^2} + \varepsilon u_r, \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \varepsilon u_{\mathrm{or}}. \end{aligned}$$

If we set $v = 1/r$ and parametrize the equations by θ, our system can be written as

$$\begin{aligned} v'' + v - (v^2 t')^2 &= -\varepsilon v^2 t'^2 (u_r + \frac{v'}{v} u_{\mathrm{or}}), \\ (v^2 t')' &= -\varepsilon v^3 t'^3 u_{\mathrm{or}}, \end{aligned}$$

where $'$ denotes the derivative with respect to θ.

This representation shows the relation with the control of a linear oscillator and is useful to apply averaging techniques (see [15]).

2.2. **Maximum principle and extremal solutions.** In this section, we consider a smooth control system in $\mathbf{R}^n$ of the form

$$\dot{x}(t) = F(x(t), u(t)),\ u(t) \in U, \tag{2}$$

where the set $\mathscr{U}$ of admissible controls is the set of locally essentially bounded mappings valued in the control domain $U \subset \mathbf{R}^m$. We note $x(t, t_0, u)$ the solution of (2) associated to an admissible control $u \in \mathscr{U}$ with initial condition x_0 at $t = 0$.

Definition 2.2. The accessibility set in time T is

$$\mathscr{A}_{x_0,T} = \{x(T, x_0, u),\ u \in \mathscr{U}\},$$

and $\mathscr{A}_{x_0} = \bigcup_{T \geq 0} \mathscr{A}_{x_0,T}$ is the accessibility set. For any fixed x_0, T, the endpoint mapping is

$$E_{x_0,T} : u \in \mathscr{U} \mapsto x(T, x_0, u).$$

We recall the maximum principle [19] in the time optimal case. Consider the minimum time control problem for $\dot{x} = F(x, u)$, $x \in \mathbf{R}^n$, $u \in U$, with boundary conditions $x(0) \in M_0$ and $x(T) \in M_1$, M_0 and M_1 regular submanifolds. Pontryagin's maximum principle asserts that if u is an optimal control on $[0, T]$ with response x, then there exists an absolutely continuous covector function p, valued in $(\mathbf{R}^n)^* \backslash \{0\}$, such that, with $H = \langle p, F(x, u)\rangle$, almost everywhere on $[0, T]$,

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \tag{3}$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \tag{4}$$

$$H(x, p, u) = \mathscr{H}(x, p), \tag{5}$$

where $\mathscr{H}(x, p) = \max_{v \in U} H(x, p, v)$. Moreover, $\mathscr{H}$ is constant and positive along the curve $z = (x, p)$, and the following *transversality conditions* are satisfied at endpoints:

$$z(0) \in M_0^\perp,\ z(T) \in M_1^\perp, \tag{6}$$

where, for a manifold M, we denote by $M^\perp$ the Lagrangian submanifold of T^*M defined by

$$M^\perp = \{(x,p) \in T^*M \mid p \perp T_xM\}.$$

DEFINITION 2.3. The function H is called the *Hamiltonian* and p is the *adjoint vector*. We call *extremal* a triple (x,p,u) solution of (3-5), and *BC-extremal* a triple satisfying moreover (6). If $\mathscr{H} = 0$ along the extremal, it is called *exceptional*. If the maximization condition (5) implies $\partial H/\partial u = 0$, the extremal is said to be *singular*.

Consider a smooth single-input affine system $\dot{x} = F_0 + uF_1$, $|u| \leq 1$. Singular extremals satisfy $H_1 = \langle p, F_1\rangle = 0$ along the solution.

DEFINITION 2.4. The extremal is called *regular* if $u(t) = \text{sign}\, H_1(x(t),p(t))$ almost everyhere, and *bang-bang* if it is moreover piecewise constant.

We now state some standard properties of singular extremals in the single-input affine case [4]. Let $H_0 = \langle p, F_0\rangle$ and $H_1 = \langle p, F_1\rangle$ be the Hamiltonian lifts of F_0 and F_1. Generic singular extremals are computed as follows. Let $z(t) = (x(t),p(t))$ be a singular extremal; differentiating twice the relation $H_1(z(t)) = 0$, we get, using Poisson brackets,

$$H_1 = \{H_0, H_1\} = 0,$$

and

$$\{H_0,\{H_0,H_1\}\} + u_s\{H_1,\{H_0,H_1\}\} = 0$$

where $\{H_0,H_1\} = dH_1(\overrightarrow{H}_0)$. If $\{H_1,\{H_0,H_1\}\} \neq 0$, the singular extremal is said to be *of order two* and the singular control is obtained as the dynamic feedback

$$u_s(z) = -\frac{\{H_0,\{H_0,H_1\}\}(z)}{\{H_1,\{H_0,H_1\}\}(z)}.$$

Introduce the Hamilton function $H_s(x,p) = H(x,p,u_s(x,p))$. Singular extremals are integral curves of the smooth Hamiltonian vector field $\overrightarrow{H}_s$ contained in $H_1 = \{H_0,H_1\} = 0$. As a result of the maximum principle, H_0 is nonnegative for time minimal solutions. Besides, using second order necessary conditions — namely, the generalized Legendre-Clebsch condition —, we get

$$\frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u}(z(t)) = \{H_1,\{H_0,H_1\}\}(z(t)) \geq 0.$$

The projection on the state-space of singular extremals gives trajectories of (2) along which the linearized system is not controllable. They correspond to singularities of the endpoint mapping $E_{x_0,T}$, where the Fréchet derivative is computed on the set of controls endowed with the L^∞-topology. Let $0 < t \leq T$, then $E'_{x_0,t}$ is singular on the restriction of the singular control to $[0,t]$ and we denote by $K(t)$ its image, also known as the *first order Pontryagin cone* [19]. Consider the generic case where $K(t)$ is of codimension one. By construction, the adjoint vector p is orthogonal to $K(t)$ and is oriented by the condition $H_0 \geq 0$. The singular trajectories are then classified using the Legendre-Clebsch condition. Assuming $\{H_1,\{H_0,H_1\}\} \neq 0$ (singular extremal of order two), we have the following definition.

DEFINITION 2.5. Under our assumptions, a singular trajectory is called *exceptional* if $H_0 = 0$. If $H_0 \neq 0$, it is called *elliptic* if $\{H_1,\{H_0,H_1\}\} < 0$, *hyperbolic* if $\{H_1,\{H_0,H_1\}\} > 0$.

It is well known that hyperbolic (*resp.* elliptic) trajectories are candidates to minimize (*resp.* maximize) time. More precisely, the time optimality status of singular trajectories has been analyzed in [7] and the results will be used in §5.

Other trajectories, meaningful to understand the structure of the boundary of the accessibility set, are the regular extremals. In the generic case, they are classified by their contact with the switching surface $\Sigma = \{H_1 = 0\}$ (see [17] for details on what follows). We fix a reference extremal $z(t) = (x(t), p(t))$ on $[0, T]$ and we introduce the switching function $\Phi(t) = H_1(z(t))$, denoted respectively Φ_+ or Φ_- depending on whether the control is $+1$ or -1. Differentiating twice we get the relations

$$\dot{\Phi}_+ = \dot{\Phi}_- = \{H_0, H_1\},$$
$$\ddot{\Phi}_\pm = \{H_0, \{H_0, H_1\}\} \pm \{H_1, \{H_0, H_1\}\},$$

and we consider two cases.

Regular case. Take a point z_0 on Σ which is assumed to be locally a smooth hypersurface. The point z_0 is called a *regular switching point* if $\{H_0, H_1\} \neq 0$ (contact of order one). At such a point, extremal arcs are transverse to Σ and locally, every extremal is bang-bang with at most one switching.

Fold case. Let z_0 belong to $\Sigma' = \{H_1 = \{H_0, H_1\} = 0\}$ which is assumed to be locally a smooth submanifold of codimension two. At such a point, extremal arcs meet the switching surface with a contact of order at least two since $\Phi_\pm = \dot{\Phi}_\pm = 0$. Assume that the contact is of order two for both arcs, that is $\ddot{\Phi}_\pm \neq 0$. This defines the *fold case* and we have three subcases.
Parabolic case. Both $\ddot{\Phi}_+$ and $\ddot{\Phi}_-$ have the same sign at z_0. Then each extremal is locally bang-bang with at most two switchings and no admissible singular can pass through z_0.
Hyperbolic case. One has $\ddot{\Phi}_+ > 0$ and $\ddot{\Phi}_- < 0$ at z_0. Then, it is necessary that $\{H_1, \{H_0, H_1\}\}(z_0) > 0$ and through z_0 passes a singular extremal of order two (exceptional or hyperbolic) with singular control $|u_s| < 1$. Each extremal is a concatenation of arcs of the form $\gamma_\pm \gamma_s \gamma_\pm$.
Elliptic case. One has $\ddot{\Phi}_+ < 0$ and $\ddot{\Phi}_- > 0$ at z_0. As before, there exists through z_0 a singular extremal of order two (exceptional or elliptic) which is admissible, but the connection with a regular extremal is impossible. Each regular extremal is locally bang-bang but the number of switchings is not uniformly bounded.

This analysis deals only with the generic case of codimension one and little more is known about the behaviour of extremal solutions, the analysis being intricate. One reason is the *saturation phenomenon*. For instance, if $\ddot{\Phi}_+ = 0$ at z_0 and if $\{H_1, \{H_0, H_1\}\}(z_0) \neq 0$, the singular control corresponding to the singular extremal of order two through z_0 is saturating. Nevertheless, a Fuller situation where a regular extremal connects the switching surface with an infinite number of switchings has been analyzed in [17]. We recall the according definition.

DEFINITION 2.6. An extremal (z, u) defined on $[0, T]$ is called a *Fuller extremal* if the switching times form a sequence $0 \leq t_1 < t_2 < \cdots \leq T$ such that $(t_n)_n \to T$ when $n \to \infty$ and if there exists $k > 1$ with the property that $t_{n+1} - t_n \simeq 1/k^n$ as $n \to \infty$ (see [4]).

3. **Lie algebraic structure in orbital transfer and consequences.**

3.1. **Preliminaries.** We can restrict our analysis to the constant mass model and work with Cartesian coordinates, the computations being intrinsic,

$$\dot{x} = F_0 + F,$$

where $x = (q, \dot{q})$, $F_0 = \dot{q}\partial/\partial q - q\mu/|q|^3\partial/\partial\dot{q}$ and F is the thrust. It can be decomposed into the tangential-normal or radial-orthoradial frames. Having chosen a frame, we can restrict the thrust to one direction, and the system $\dot{x} = F_0 + uF_1$ is a single-input affine system. We make the computation of the corresponding Lie algebra $\mathrm{Lie}(F_0, F_1)$. This will allow us to understand the action of every physical actuator. In practice, one can have technical limitations such as $u \in C(\alpha)$ where $C(\alpha)$ is a cone of axis F_t or F_{or} of angle α. Moreover, in space mechanics, the effect of forces oriented along F_r or F_t is important and well studied. In particular, the effect of a *drag* force corresponds to a force opposite to F_t whose modulus is proportional to $\rho\dot{q}^2$, where ρ is the atmosphere density.

Definition 3.1. The *orbit* of a point is the integral manifold of the involutive distribution $\mathrm{Lie}(F_0, F_1)$ passing through this point.

3.2. **Thrust oriented along $\mathbf{F_t}$.** We have a 2D-system and we denote $q = (q_1, q_2)$, $r = |q|$, and $v = \dot{q} = (v_1, v_2)$. The Lie bracket of two smooth vector fields X, Y, is the commutator $[X, Y] = XY - YX$. Singular trajectories are responses along which the linearized system is not controllable. Beyond their importance in time optimal control, they code a lot of information about the system and have to be computed first. Since they are feedback invariants [4], we use the feedback $u' = u/|v|$ and write the system

$$\dot{x} = F_0 + u'F'_t,$$

where $F'_t = v\partial/\partial v$. We get

$$\begin{aligned}
[F_0, F'_t] &= -v\frac{\partial}{\partial q} - q\frac{\mu}{r^3}\frac{\partial}{\partial v}, \\
[F'_t, [F_0, F'_t]] &= -F_0, \\
[F_0, [F_0, F'_t]] &= q\frac{2\mu}{r^3}\frac{\partial}{\partial q} + \frac{2\mu}{r^5}\left((2q_1^2 - q_2^2)v_1 + 3q_1q_2v_2\right)\frac{\partial}{\partial v_1} \\
&\quad + \frac{2\mu}{r^5}\left(3q_1q_2v_1 + (2q_2^2 - q_1^2)v_2\right)\frac{\partial}{\partial v_2},
\end{aligned}$$

and since $q \wedge \dot{q} \neq 0$, the vector fields F_0, F'_t, $[F_0, F'_t]$, $[F_0, [F_0, F'_t]]$ form a frame. From our computations, we can easily deduce the controllability result and the structure of singular extremals.

Proposition 3.1. *Consider the single-input control system $\dot{x} = F_0 + uF_t$, $|u| \leq \varepsilon$, $\varepsilon > 0$. Then, for each $(q, \dot{q})$, $q \wedge \dot{q} \neq 0$, the rank of $\mathrm{Lie}_x(\{F_0, F_t\})$ is four and the system restricted to the 2D-elliptic domain is controllable, i.e., each point of an elliptic orbit can be transfered on a prescribed elliptic orbit, with imposed final longitude.*

Proof. The rank of the Lie algebra generated by F_0 and F_t is four whenever $q \wedge \dot{q} \neq 0$. The system restricted to the 2D-elliptic domain is thus controllable because the drift is periodic [2]. □

Proposition 3.2. *Consider the single-input control system $\dot{x} = F_0 + uF_t$. Then,*

(i) *all singular trajectories are of order two, elliptic and solution of*

$$\dot{z} = \overrightarrow{H}_0 + u_s \overrightarrow{H}_t$$

on $\Sigma' = \{H_0 = \{H_0, H_t\} = 0\}$ *with* $u_s = |v|u'_s$ *and*

$$u'_s = -\frac{\{H_0, \{H_0, H'_t\}\}}{\{H'_t, \{H_0, H'_t\}\}}.$$

(ii) *For* $|u| \leq \varepsilon$, $\varepsilon > 0$, *the relations*

$$H_t = \{H_0, H_t\} = \{H_0 \pm \varepsilon H_t, \{H_0, H_t\}\} = 0$$

are incompatible.

Proof. Singular extremals are solutions of

$$H'_t = \{H_0, H'_t\} = 0,$$
$$\{H_0, \{H_0, H'_t\}\} + u'_s \{H'_t, \{H_0, H'_t\}\} = 0,$$

and since $[F'_t, [F_0, F'_t]]$ is collinear to F_0, singular extremals that are not of order two are exceptional. Now, as an exceptional singular such that $\{H'_t, \{H_0, H'_t\}\} = 0$ holds it has to verify $\{H_0, \{H_0, H'_t\}\} = 0$, which is impossible since F_0, F'_t, $[F_0, F'_t]$, $[F_0, [F_0, F'_t]]$ form a frame. Then H_0 is positive and, since $[F'_t, [F_0, F'_t]] = -F_0$, $\{H'_t, \{H_0, H'_t\}\}$ is negative and every singular trajectory is elliptic. Assertion (ii) is obvious because we have a frame. □

More complicated computations are required to analyze regular extremals. Indeed,

$$\begin{aligned} [F_0, F_t] &= -\frac{1}{|v|} F_0 - \frac{\mu q \cdot v}{r^3 |v|^2} F_t + \frac{2\mu (q \wedge v) \wedge v}{r^3 |v|^3} \frac{\partial}{\partial v}, \\ [F_t, [F_0, F_t]] &= -\frac{1}{|v|^2} F_0 \bmod \mathrm{span}(\{F_t, [F_0, F_t]\}), \end{aligned}$$

and there exists a scalar function λ such that

$$[F_0, [F_0, F_t]] = \lambda F_0 \bmod \mathrm{span}(\{F_t, [F_0, F_t]\}).$$

According to the classification of fold points, we can have elliptic, parabolic, but not hyperbolic points. Moreover, there may be cusp points whenever one of the extremals with $u = \pm 1$, not both, has a contact of order three. Further analysis is needed but an interesting result is the following (see [17] for the proof).

PROPOSITION 3.3. *There is no Fuller point and regular extremals are bang-bang.*

The following proposition follows.

PROPOSITION 3.4. *Every time optimal trajectory of the system* $\dot{x} = F_0 + uF_t$, $|u| \leq \varepsilon$, $\varepsilon > 0$, *is bang-bang.*

3.3. **Thrust oriented along $\mathbf{F_n}$.** Consider the 2D-system $\dot{x} = F_0 + uF_n$ with

$$F_n = -\frac{v_2}{|v|} \frac{\partial}{\partial v_1} + \frac{v_1}{|v|} \frac{\partial}{\partial v_2}.$$

Computations give us

$$[F_0, F_n] = -\frac{(q \wedge v) \wedge v}{|(q \wedge v) \wedge v|} \frac{\partial}{\partial q} - v \frac{\mu |q \wedge v|}{r^3 |v|^3} \frac{\partial}{\partial v},$$

and the brackets of length three are contained in $\mathrm{span}(\{F_0, F_n\})$.

PROPOSITION 3.5. *Consider the system $\dot{x} = F_0 + uF_n$, $|u| \leq \varepsilon$, $\varepsilon > 0$, restricted to the elliptic domain. Then, the orbit is of dimension three and is the intersection of the elliptic domain with the osculating plane $\{a = a(0)\}$.*

Proof. From our computations, the orbit is of dimension three and, plugging $u_t = u_c = 0$ into the tangential-normal system, we see that the semi-major axis a cannot be controlled. □

3.4. **Thrust oriented along $\mathbf{F_c}$.** Inspecting the tangential-normal system we also observe that, if $u_t = u_n = 0$, we cannot control either the semi-major axis a or the eccentricity $|e|$, the case $|e| = 0$ being singular (circular orbits). More precisely,

$$\begin{aligned}
[F_0, F_c] &= -\frac{q \wedge v}{|q \wedge v|} \frac{\partial}{\partial q}, \\
[F_0, [F_0, F_c]] &= -\frac{\mu}{r^3} F_c, \\
[F_c, [F_0, F_c]] &= \frac{(q \wedge v) \wedge q}{|q \wedge v|^2} \frac{\partial}{\partial q} + \frac{(q \wedge v) \wedge v}{|q \wedge v|^2} \frac{\partial}{\partial v}, \\
[F_0, [F_c, [F_0, F_c]]] &= 0, \\
[F_c, [F_c, [F_0, F_c]]] &= -\frac{r^2}{|q \wedge v|^2} [F_0, F_c] + \frac{q \cdot v}{|q \wedge v|^2} F_c.
\end{aligned}$$

LEMMA 3.1. (i) *The vectors F_0, F_c and $[F_0, F_c]$ are independent.*

(ii) *The vectors F_0, F_c, $[F_0, F_c]$, $[F_c, [F_0, F_c]]$ form a frame of* $\mathrm{Lie}(F_0, F_c)$ *if and only if $L(0) \neq 0$, where L is the Laplace vector.*

(iii) *If $L(0) = 0$, the Lie algebra generated by the system is finite-dimensional of dimension three.*

Proof. The first assertion is clear. Moreover, F_0, F_c, $[F_0, F_c]$ and $[F_c, [F_0, F_c]]$ are dependent if and only if F_0 and $[F_c, [F_0, F_c]]$ are linearly dependent. In this case, $q \cdot v = 0$, $\mu = |v|^2 r$. This corresponds to circular orbits where $L = 0$. Then, r and $|v|$ are constants and

$$\begin{aligned}
[F_c, [F_0, F_c]] &= \frac{1}{|v|^2} F_0, \\
[F_0, [F_0, F_c]] &= -\frac{\mu}{r^3} F_c.
\end{aligned}$$

The associated Lie algebra is therefore finite dimensional. □

In particular, the proposition hereafter holds (see [2, 10]).

PROPOSITION 3.6. *Consider the system $\dot{x} = F_0 + uF_c$, $|u| \leq \varepsilon$, $\varepsilon > 0$, restricted to the elliptic domain.*

(i) *If $L(0) \neq 0$, then the orbit is of dimension four and is the intersection of the elliptic domain with $\{a = a(0),\ |e| = |e(0)| \neq 0\}$.*

(ii) *If $L(0) = 0$, then the orbit is of dimension three and is the intersection of the elliptic domain with $\{a = a(0),\ |e| = |e(0)| = 0\}$.*

In both cases, any point of the orbit is accessible.

Another relevant consequence of our computations is the following.

PROPOSITION 3.7. *Assume $L(0) = 0$ and restrict the system to its orbit $\{a = a(0),\ |e| = |e(0)| = 0\}$. Then, all singular trajectories are responses to the zero control and are hyperbolic.*

3.5. **Radial-orthoradial decomposition.** We shall proceed as for the tangential-normal system. For simplicity, we shall only analyze the singular flow. Since we are in the 2D-case,

$$\begin{aligned} F_0 &= v\frac{\partial}{\partial q} - q\frac{\mu}{r^3}\frac{\partial}{\partial v}, \\ F_r &= \frac{q}{r}\frac{\partial}{\partial v}, \\ F_{\rm or} &= \frac{q_2}{r}\frac{\partial}{\partial v_1} - \frac{q_1}{r}\frac{\partial}{\partial v_2}. \end{aligned}$$

As before, we introduce a feedback $u' = u/r$, and $F_r' = rF_r$, $F_{\rm or}' = rF_{\rm or}$. In the radial case, brackets of length up to three are

$$[F_0, F_r'] = -q\frac{\partial}{\partial q} + v\frac{\partial}{\partial v}, \tag{7}$$

$$[F_0, [F_0, F_r']] = -(q+v)\frac{\partial}{\partial q} - q\frac{\mu}{r^3}\frac{\partial}{\partial v}, \tag{8}$$

$$[F_r', [F_0, F_r']] = 2F_r'. \tag{9}$$

LEMMA 3.2. *Consider the system $\dot{x} = F_0 + uF_r$, $|u| \leq \varepsilon$, $\varepsilon > 0$, restricted to the elliptic domain. Then, the Lie algebra generated by F_0 and F_r is of dimension three and F_0, F_r, $[F_0, F_r]$ form a frame. The orbit is the intersection of the elliptic domain with the osculating plane and $\{P = P(0)\}$, where P is the semi-latus rectum. Any point of the orbit is accessible.*

Proof. The dimension of the orbit is clear using (7-9). Obviously, from the definition of the radial-orthoradial system, the semi-latus rectum P cannot be controlled if $u_{\rm or} = 0$. In such a case, $r^2\dot{\theta}$ is a constant and $P = (r^2\dot{\theta})/\mu$. □

In the orthoradial case, we get

$$\begin{aligned} [F_0, F_{\rm or}'] &= -q_2\frac{\partial}{\partial q_1} + q_1\frac{\partial}{\partial q_2} + v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2}, \\ [F_0, [F_0, F_{\rm or}']] &= 2\left(-v_2\frac{\partial}{\partial q_1} + v_1\frac{\partial}{\partial q_2} - q_2\frac{\mu}{r^3}\frac{\partial}{\partial v_1} + q_1\frac{\mu}{r^3}\frac{\partial}{\partial v_2}\right), \\ [F_{\rm or}', [F_0, F_{\rm or}']] &= -2F_r', \end{aligned}$$

and

$$\begin{aligned} D_0 &= F_{\rm or}' \wedge [F_0, F_{\rm or}'] \wedge [F_0, [F_0, F_{\rm or}']] \wedge F_0 \;=\; 2(v \wedge q)(|v|^2 + \frac{\mu}{r}), \\ D_1 &= F_{\rm or}' \wedge [F_0, F_{\rm or}'] \wedge [F_{\rm or}, [F_0, F_{\rm or}']] \wedge F_0 \\ &= -2r^2 q \cdot v. \end{aligned}$$

LEMMA 3.3. (i) *The vectors F_0, $F_{\rm or}'$, $[F_0, F_{\rm or}']$, $[F_0, [F_0, F_{\rm or}']]$ form a frame of the Lie algebra* $\mathrm{Lie}(F_0, F_{\rm or}')$.

(ii) *The vectors F_0, $F_{\rm or}'$, $[F_0, F_{\rm or}']$ and $[F_{\rm or}', [F_0, F_{\rm or}']]$ are linearly independent if and only if $q \cdot v \neq 0$, that is outside pericenters and apocenters.*

PROPOSITION 3.8. *Consider the system $\dot{x} = F_0 + uF_{\rm or}$, $|u| \leq \varepsilon$, $\varepsilon > 0$, restricted to the 2D-elliptic domain. Then, any point of the domain is accessible.*

Important informations about time optimality are coded by exceptional singular extremals which are deduced from our computations.

PROPOSITION 3.9. *The exceptional control is such that* $u_e = ru'_e$ *with*

$$D_0 + u'_e D_1 = 0, \tag{10}$$

and all exceptional singulars are of order two outside pericenters and apocenters. Moreover, along a singular exceptional trajectory, the drift F_0 *is not contained in the vector subspace generated by* $F'_{\rm or}$ *and* $[F_0, F'_{\rm or}]$*, while the first order Pontryagin cone is spanned by* F_0*,* $F'_{\rm or}$ *and* $[F_0, F'_{\rm or}]$*.*

Proof. Along a singular exceptional extremal, the adjoint vector p has to be orthogonal to F_0, $F'_{\rm or}$, $[F_0, F'_{\rm or}]$ and $[F_0, [F_0, F'_{\rm or}]] + u'_e[F'_{\rm or}, [F_0, F'_{\rm or}]]$, where u'_e is the exceptional control. Since the first three vector fields are independent, they generate the first order Pontryagin cone and F_0 does not belong to $\mathrm{span}(F'_{\rm or}, [F_0, F'_{\rm or}])$. Because p cannot vanish, the relation $D_0 + u'_e D_1 = 0$ holds, and outside pericenters and apocenters singular exceptionals are of order two for F_0, $F'_{\rm or}$, $[F_0, F'_{\rm or}]$, $[F'_{\rm or}, [F_0, F'_{\rm or}]]$ form a frame. □

3.6. **Consequences on controllability and complexity of the time optimal control.** As previously shown, the system is controllable in the elliptic domain with a thrust oriented either along F_t or $F_{\rm or}$, plus the control direction F_c. As a result, if the final orbit is circular, the problem can be decomposed into two phases: during the first one, F_t or $F_{\rm or}$ is used to modify the geometric parameters of the ellipse in the osculating plane. Then, during the second one, F_c is used to correct the inclination and the nodal line direction. Accordingly, an important question is to analyze the 2D-time optimal control problem, the thrust being oriented along the tangential or orthoradial direction. The complexity depends on this choice. Indeed, in the single-input tangential case, all singular trajectories are elliptic and thus locally time maximizing. The time optimal are bang-bang and bounds on the number of switchings can be computed using the concept of conjugate locus introduced by [23] (see also [22]). On top of that, the time optimal control for small time can be derived thanks to a nilpotent approximation of the Lie brackets. In the single-input orthoradial case, the situation is intricate, even for small times. Actually, elliptic, exceptional and hyperbolic singular extremals are allowed, exceptional and hyperbolic trajectories being small time optimal. They can be strictly admissible if $|u_s| < \varepsilon$, not admissible if $|u_s| > \varepsilon$, and saturating when $|u_s| = \varepsilon$. Nilpotent approximations are not sufficient to analyze such situations as observed by [16].

4. **Time optimal control in orbital transfer and SR-systems with drift.** In this section, the system is written

$$\begin{aligned} \dot{x} &= F_0 + \frac{1}{m}\sum_{i=1}^{3} u_i F_i, \\ \dot{m} &= -\beta|u|, \end{aligned}$$

where β is a positive constant, where the F_i's form an orthonormal frame, and where $|u|$ is bounded by $F_{\max}$. Though the right member is only continuous in the control, the maximum principle still applies and the associated Hamiltonian is

$$H = H_0 + \frac{1}{m}\sum_{i=1}^{3} u_i H_i - \beta p_m |u|. \tag{11}$$

In (11), the H_i's are the Hamiltonian lifts $\langle p, F_i\rangle$, p being the dual to x, p_m the dual to m. We shall assume in the sequel that the final mass is free and that the constraint $m \geq \chi_0$ is not active. We first recall the following result from [10].

PROPOSITION 4.1. *Along an optimal solution,*

(i) *the mass dual variable p_m is nonpositive and increasing;*
(ii) *whenever $\Phi = (H_1, H_2, H_3)$ is not zero, the optimal control is given by*

$$u = F_{\max}\frac{\Phi}{|\Phi|}.$$

Proof. Since the Hamiltonian is maximized over the Euclidean ball of radius $F_{\max}$, clearly, $\sum_{i=1}^{3} u_i H_i$ has to be nonnegative. Now, the adjoint equation on p_m is

$$\dot{p}_m = \frac{1}{m^2}\sum_{i=1}^{3} u_i H_i,$$

so that the dual to m is increasing towards $p_m(T)$, T final time, which is zero by transversality; p_m is nonpositive and assertion (ii) readily follows. □

Let (x, p, u) be an extremal. In accordance with our classification of regular extremals based upon the order of the contact with the switching surface $\{\Phi = 0\}$, the extremal is said to be of order zero if u is smooth and given by $u = F_{\max}\Phi/|\Phi|$, singular if $\Phi \equiv 0$. We restrict ourselves to the constant mass case and introduce the concept of sub-Riemannian system with drift.

4.1. **SR-system with drift.**

DEFINITION 4.1. We call SR-problem with drift the time optimal problem for a system of the form

$$\dot{x} = F_0 + \sum_{i=1}^{m} u_i F_i,$$

with $x \in \mathbf{R}^n$, $F_0, \ldots, F_m$ smooth vector fields, the control $u \in \mathbf{R}^m$ being bounded by $\sum_{i=1}^{m} |u_i|^2 \leq 1$.

Let the H_i's be the usual Hamiltonian lifts $\langle p, F_i\rangle$, $i = 0, \ldots, m$, and let Σ be the switching surface $\{H_i = 0,\ i = 1, \ldots, m\}$. The maximization of the Hamiltonian $H = H_0 + \sum_{i=1}^{m} u_i H_i$ outside Σ implies that

$$u_i = \frac{H_i}{\sqrt{\sum_{i=1}^{m} H_i^2}},\ i = 1, \ldots, m. \tag{12}$$

Plugging (12) into H defines the Hamiltonian function

$$H_r = H_0 + \left(\sum_{i=1}^{m} H_i^2\right)^{\frac{1}{2}}. \tag{13}$$

The corresponding solutions are the order zero extremals. From the maximum principle, optimal extremals are contained in the level set $\{H_r \geq 0\}$. Those in $\{H_r = 0\}$ are exceptional. The following result is standard.

PROPOSITION 4.2. *The extremals of order zero are smooth responses to smooth controls on the boundary of $|u| \leq 1$. They are singularities of the endpoint mapping $E_{x_0,T} : u \mapsto x(T, x_0, u)$ for the L^∞-topology when u is restricted to the unit sphere $\mathbf{S}^{m-1}$.*

In order to construct all extremals, we must analyze the behaviour of those of order zero near the switching surface. On one hand, observe that we can connect two such arcs at a point located on Σ if we respect the Weierstraß-Erdmann conditions

$$p(t+) = p(t-),\ H_r(t+) = H_r(t-),$$

where t is the time of contact with the switching surface. Those conditions, obtained in classical calculus by means of specific variations, are contained in the maximum principle. On the other hand, singular extremals satisfy $H_i = 0$, $i = 1, \dots, m$, and are contained in Σ. They are singularities of the endpoint mapping if u is interior to the control domain, $|u| < 1$. Let then $z = (x, p)$ be an extremal. Evaluated along z, the H_i's are absolutely continuous functions whose time derivatives are expressed using the Poisson bracket

$$\dot{H}_i = \{H_0, H_i\} + \sum_{j \neq i} u_j \{H_j, H_i\}. \tag{14}$$

We denote by $\mathscr{D}$ the distribution spanned by $F_1, \dots, F_m$. The following is straightforward from (14).

PROPOSITION 4.3. *We can connect any extremal of order zero converging to $z_0 = (x_0, p_0)$ in Σ with another order zero extremal departing from z_0. If $[\mathscr{D}, \mathscr{D}](x_0) \subset \mathscr{D}(x_0)$, the coordinates H_i are $\mathscr{C}^1$, $i = 1, \dots, m$.*

Our aim is to give an account of the singularity encountered when making junctions between order zero smooth extremals. It is based on the preliminary work of [10] (see [8] for more details). We limit our analysis to the 2D-case, $m = 2$, the generalization being straightforward. The system is $\dot{x} = F_0 + u_1 F_1 + u_2 F_2$ and $H = H_0 + u_1 H_1 + u_2 H_2$. The extremal controls of order zero are

$$u_i = \frac{H_i}{\sqrt{H_1^2 + H_2^2}},\ i = 1, 2,$$

and (14) takes the form

$$\begin{aligned} \dot{H}_1 &= \{H_0, H_1\} - u_2 \{H_1, H_2\}, \\ \dot{H}_2 &= \{H_0, H_2\} + u_1 \{H_1, H_2\}. \end{aligned}$$

As for SR-systems [4], we make a polar blowing up

$$H_1 = \rho \cos\varphi,\ H_2 = \rho \sin\varphi,$$

in order that the system becomes

$$\dot{\rho} = \cos\varphi \{H_0, H_1\} + \sin\varphi \{H_0, H_2\}, \tag{15}$$

$$\dot{\varphi} = \frac{1}{\rho} \left(\{H_1, H_2\} - \sin\varphi \{H_0, H_1\} + \cos\varphi \{H_0, H_2\} \right). \tag{16}$$

A *nilpotent approximation* of (15-16) consists in choosing vector fields F_0, F_1, F_2 such that brackets of length greater than three are zero. Now, differentiating we get

$$\frac{d}{dt}\{H_1, H_2\} = \{H_0, \{H_1, H_2\}\} + u_1 \{H_1, \{H_1, H_2\}\} + u_2 \{H_2, \{H_1, H_2\}\}.$$

Similarly, the time derivatives of $\{H_0, H_1\}$, $\{H_0, H_2\}$ only involve length three brackets and are also zero in our approximation. Hence, for a given extremal, we set

$$\{H_0, H_1\} = a_1,\ \{H_0, H_2\} = a_2,\ \{H_1, H_2\} = b, \tag{17}$$

where a_1, a_2 and b are constants. As a consequence, (15-16) can be integrated using the time reparametrization $ds = dt/\rho$ and trajectories crossing $\Sigma = \{H_1 = H_2 = 0\}$ with a defined slope are obtained by solving $\dot\varphi = 0$. Let us assume we are in the regular case of §2.2: if $z_0 \in \Sigma$ is a point of order one where a_1 and a_2 are not both zero, up to a rotation we have $a_2 = 0$ and $a_1 \neq 0$. The equation $\dot\varphi = 0$ reduces to $a_1 \sin\varphi + b = 0$. It has two roots $\varphi_0 < \varphi_1$ on $[0, 2\pi[$ if and only if $|b/a_1| < 1$. Moreover, $\varphi_0 = 0$ and $\varphi_1 = \pi$ if and only if $b = 0$. The latter is satisfied when $\mathscr{D}$ is involutive. Besides, if $\varphi_0 \neq \varphi_1$, then $\cos\varphi$ changes sign and one extremal reaches Σ while the other leaves it. We have just proved the following.

PROPOSITION 4.4. *In the generic nilpotent model, the extremals project onto*

$$\dot\rho = a_1 \cos\varphi + a_2 \sin\varphi, \tag{18}$$

$$\dot\varphi = \frac{1}{\rho}(b - a_1 \sin\varphi + a_2 \cos\varphi), \tag{19}$$

where $H_1 = \rho\cos\varphi$, $H_2 = \rho\sin\varphi$, and a_1, a_2, b are constant parameters defined by (17). In the involutive case $b = 0$ and, when crossing Σ at a point of order one, the control rotates instantaneously of an angle π. The resulting singularity is called a Π-singularity*.*

Consider now such a system $\dot x = F_0 + u_1 F_1 + u_2 F_2$ in dimension four, that is $x \in \mathbf{R}^4$. Assume moreover that $\mathscr{D} = \mathrm{span}(\{F_1, F_2\})$ is involutive, $[\mathscr{D}, \mathscr{D}] \subset \mathscr{D}$, and that the system has the following regularity: for any $x \in \mathbf{R}^4$, the rank of F_1, F_2, $[F_0, F_1]$, $[F_0, F_2]$ is four. As a result, there is a vector mapping λ such that

$$F_0 = \lambda_1 [F_0, F_1] + \lambda_2 [F_0, F_2] \bmod \mathscr{D}.$$

PROPOSITION 4.5. *In the regular case, the only discontinuities of an optimal control are* Π*-singularities where the control rotates instantaneously of an angle π. In the non-exceptional case, the extremals cross the switching surface with a given orientation.*

Proof. The result is a byproduct of proposition 4.4: since $H_1 = H_2 = \{H_0, H_1\} = \{H_0, H_2\} = 0$ imply $p = 0$, the only singularities are Π-singularities. With previous notations, the extremal is solution of (18), the slope at the contact being defined by $-a_1 \sin\varphi + a_2 \cos\varphi = 0$. Moreover, when crossing Σ, $H_1 = H_2 = 0$ and $H = H_0$. Thus,

$$H_0 = \lambda_1 \{H_0, H_1\} + \lambda_2 \{H_0, H_2\}$$

and $H \geq 0$ imposes $\langle p, \lambda_1 [F_0, F_1] + \lambda_2 [F_0, F_2] \rangle \geq 0$. Hence, the orientation of any trajectory but the exceptional one is fixed when crossing Σ with slope φ. □

COROLLARY 4.1. *In the regular case, all the optimal trajectories are bang-bang and the number of switchings is uniformly bounded on each compact subset of $\mathbf{R}^4$.*

To analyze the optimal control problem, we choose a representation of the nilpotent model. Denoting $x = (x_1, \ldots, x_4)$, we set

$$F_0 = (1 + x_1)\frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4},$$

and $F_1 = \partial/\partial x_1$, $F_2 = \partial/\partial x_2$. Then, $[F_0, F_1] = -\partial/\partial x_3$, $[F_0, F_2] = -\partial/\partial x_4$ and all Lie brackets with length greater than three are zero. We have

$$F_0 = -(1 + x_1)[F_0, F_1] - x_2 [F_0, F_2] = -[F_0, F_1]$$

whenever $x = 0$. If $p = (p_1, \ldots, p_4)$ is the adjoint vector, the condition $H_0 \geq 0$ orientates p according to $p_3 \geq 0$ and $p_3 = 0$ in the exceptional case. Introducing the planes $E_1 = \{(x_1, x_3)\}$ and $E_2 = \{(x_2, x_4)\}$, the system is decoupled,

$$\begin{array}{rclcrcl} \dot{x}_3 & = & 1 + x_1, & & \dot{x}_4 & = & x_2, \\ \dot{x}_1 & = & u_1, & & \dot{x}_2 & = & u_2, \end{array}$$

and the optimal synthesis around zero can easily be computed in each plane. In E_1, time minimal (*resp.* maximal) trajectories are of the form $\gamma_+\gamma_-$ (*resp.* $\gamma_-\gamma_+$) and $u_1 = \text{sign}(H_1)$, $u_2 = 0$. Conversely, in E_2, optimal policies can be either of the form $\gamma_+\gamma_-$ or $\gamma_-\gamma_+$ with $u_1 = 0$, $u_2 = \text{sign}(H_2)$, and $u = 0$ corresponds to an exceptional direction which is locally controllable. In particular,

PROPOSITION 4.6. *There are optimal trajectories with a* Π*-singularity.*

Remark 4.1. The analysis using the dimension 4 representation shows that they fill a subset of codimension one. This singularity can be handled numerically by adjusting the steplength during the integration of the system.

4.2. **Application to the orbital transfer.** We can apply our analysis to the coplanar orbital transfer. Assuming the mass constant, the system is

$$m\ddot{q} = K(q) + u_1 F_1(q, \dot{q}) + u_2 F_2(q, \dot{q}),$$

where K is the Kepler vector field and the thrust lies in the osculating plane (*e.g.* $F_1 = F_r$, $F_2 = F_{\text{or}}$). In order to avoid collisions, we must have $r = |q| \geq r_T$, r_T being the Earth radius.

PROPOSITION 4.7. *Consider the 2D-orbital transfer problem. Then, for each pair of points* x_0*,* x_1*, in the elliptic domain, there exists a trajectory transferring* x_0 *to* x_1*. If* r_0 *is the distance to collision of this trajectory, there exists a time minimal trajectory such that* $r \geq r_0$*. Each optimal arc not meeting the boundary* $r = r_0$ *is bang-bang with maximal thrust and is a concatenation of order zero arcs, the switchings being* Π*-singularities.*

Proof. According to our Lie bracket computations, the orbit of each point of the system restricted to the elliptic domain Σ_e is the domain itself. Since the free motion is periodic, the system is controllable. Take two points x_0, x_1 in Σ_e and let $x = (q, \dot{q})$ be a trajectory joining them in time T. Then, let $r_0 > 0$ be the minimum of $|q|$ on the compact $[0, T]$. If we add the constraint $|q| \geq r_0$, we observe that x is uniformly bounded on any real compact subinterval since $K(q) \to 0$ when $|q| \to \infty$ and since the control is bounded by $F_{\max}$ (hence $\dot{q}$ is bounded, as well as q). As a result, the control domain being also convex, x_1 is accessible in minimum time according to Filippov theorem [19]. Each optimal solution not meeting the boundary is extremal and the result proceeds from the analysis of §3. □

Remark 4.2. The trajectory is feasible if and only if $r_0 \geq r_T$. An optimal trajectory may be made of boundary arcs where $|q| = r_0$, and of arcs not contained in the elliptic domain.

5. **Second order optimality conditions in orbital transfer.** The purpose of the last section is to present second order conditions which can be implemented in the orbit transfer case. They are based on the concept of conjugate point and use results from [7, 21]. We start with the standard case.

5.1. **Second order conditions in the regular case.** Consider the minimum time control of the system

$$\dot{x} = F(x,u),\ x(0) = x_0,$$

where x belongs to a smooth manifold M identified with $\mathbf{R}^n$. The right hand side $F : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is smooth and u takes values in $\mathbf{R}^m$. Since the control domain is unbounded, every optimal control u on $[0,T]$ is a singularity of the endpoint mapping $E_{x_0,t}$ for t in $]0,T]$, and the resulting trajectory is the projection of an extremal (x,p,u) solution of

$$\dot{x} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial x},$$

and

$$\frac{\partial H}{\partial u} = 0,$$

where $H = \langle p, F(x,u)\rangle$ is the standard Hamiltonian, constant and nonnegative along the extremal. Without losing any generality, we can assume that the trajectory is one to one on $[0,T]$. We make the strong Legendre assumption,

(A1) The quadratic form $\partial^2 H/\partial u^2$ is negative definite along the reference extremal.

Therefore, using the implicit function theorem, the extremal control can be locally defined as a smooth function of $z = (x,p)$, solution of $\partial H/\partial u = 0$. Plugging u into H as a dynamic feedback controller defines a true Hamiltonian function

$$H_r(x,p) = H(x,p,u(x,p)),$$

and the reference extremal is a smooth solution of

$$\dot{z} = \overrightarrow{H}_r(z). \tag{20}$$

Let $(x(t,x_0,p_0), p(t,x_0,p_0))$ denote the solution of (20) for the initial condition (x_0,p_0).

LEMMA 5.1. *One has* $u(x,\lambda p) = u(x,p)$ *and*

$$\begin{aligned} x(t,x_0,\lambda p_0) &= x(t,x_0,p_0), \\ p(t,x_0,\lambda p_0) &= \lambda p(t,x_0,p_0). \end{aligned}$$

DEFINITION 5.1. Let $z = (x,p)$ be the reference extremal defined on $[0,T]$. The variational equation

$$\delta\dot{z} = \overrightarrow{H}'_r(z(t))\delta z \tag{21}$$

is called the *Jacobi equation*. A *Jacobi field* is a non trivial solution J of (21). It is said to be *vertical* at time t if $\delta x(t) = d\Pi(z(t))(J(t)) = 0$ where $\Pi : (x,p) \mapsto x$ is the standard projection.

The geometric result hereafter is crucial.

PROPOSITION 5.1. *Let* L_0 *be the fiber* $T^*_{x_0}M$ *and* $L_t = \exp t\overrightarrow{H}_r(L_0)$ *be its image by the one parameter subgroup generated by* $\overrightarrow{H}_r$. *Then* L_t *is a Lagrangian submanifold whose tangent space at* $z(t)$ *is generated by the Jacobi fields* J *vertical at* $t = 0$, *and the rank of the restriction of* Π *to* L_t *is at most* $n-1$ *at* $z(t)$.

Proof. The fiber L_0 is a Lagrangian submanifold, so is L_t as its image by a symplectomorphism. By definition, the Jacobi fields with $\delta x(0) = 0$ will form the tangent space. By Lemma 5.1, making a variation $p + \lambda p$ in the fiber at x_0, we get a Jacobi field with $\delta x(t) = 0$ so that the rank of $d\Pi(z(t))$ cannot be more than $n-1$. □

In order to derive second order optimality conditions, we make the following additional generic assumptions on the reference extremal.

(A2) The singularity of the endpoint mapping $E_{x_0,t}$ at u is of codimension one, for all $0 < t \leq T$.

(A3) We are not in the exceptional case, that is, $H_r \neq 0$ along the extremal.

DEFINITION 5.2. We define the *exponential mapping* by

$$\exp_{x_0}(t, p_0) = x(t, x_0, p_0).$$

It is defined for small enough nonnegative t. We can assume that p_0 belongs to $\mathbf{S}^{n-1}$.

DEFINITION 5.3. Let $z = (x, p)$ be the reference extremal defined on $[0, T]$. Under our assumptions, the time $0 < t_c \leq T$ is called *conjugate* if the mapping $\exp_{x_0}(t_c, .)$ is not an immersion at p_0. The associated point $x(t_c)$ is said to be *conjugate* to x_0. We denote by t_{1c} the first conjugate time.

The following result is fundamental (see [21]).

THEOREM 5.1. *Under our assumptions, the extremities being fixed, the reference trajectory is locally time optimal (for the* L^∞*-topology on the set of controls) up to the first conjugate time.*

The exponential mapping at time t is an immersion if and only if the rank of the derivative of $\exp_{x_0}(t, .)$ with respect to p_0 is $n - 1$.

Test 1. Consider the vector space of dimension $n-1$ generated by the Jacobi fields $J_i = (\delta x_i, \delta p_i)$, $i = 1, \ldots, n-1$, which are vertical at $t = 0$: the $\delta x_i(0)$ are zero and the $\delta p_i(0)$ are normalized by the condition

$$\langle p_0, \delta p_i(0) \rangle = 0.$$

A conjugate time corresponds to a Jacobi field J in this subspace which is vertical at 0 and $t_c > 0$:

$$\mathrm{rank}(\delta x_1(t_c), \ldots, \delta x_{n-1}(t_c)) < n - 1. \tag{22}$$

If we augment the previous family of vectors by the dynamics, an equivalent test is to search for t_c such that

$$\delta x_1(t_c) \wedge \ldots \wedge \delta x_{n-1}(t_c) \wedge \dot{x}(t_c) = 0. \tag{23}$$

Remark 5.1. In the exceptional case, $H_r = 0$ and $\dot{x}$ belongs to the vector space spanned by $\delta x_1, \ldots, \delta x_{n-1}$ and (23) is identically zero. Moreover, even the test (22) will not provide the first point where the trajectory loses its optimality (see [7, 13]). The exceptional case will be discussed at the end of the section for single-input affine systems.

Consider the reference extremal z for times t smaller than the first conjugate time, $t < t_{1c}$, starting from $z(0) = (x_0, \star)$. Then, the reference trajectory x can be imbedded in the central field with origin x_0, consisting of extremal trajectories starting from x_0, which will cover a $\mathscr{C}^0$-neighbourhood of x. By standard arguments, x is optimal with respect to all trajectories with same extremities contained in this neighbourhood. Besides, the *shooting mapping*

$$S(t, p_0) = \exp_{x_0}(t, p_0) - x_1$$

where x_1 is the target is smooth and of full rank. This is a necessary condition to compute numerically the trajectory by means of Newton-like algorithms.

Consider now the more general problem where the final target is a submanifold M_1. By virtue of the maximum principle, the reference extremal has to satisfy the transversality condition $z(T) \in M_1^\perp$.

DEFINITION 5.4. Let $z = (x, p)$ be the reference extremal defined on $[0, T]$, $z(T)$ in $M_1^\perp$. We say that $t_f > 0$ is a *focal time* if there exists a Jacobi field $J = (\delta x, \delta p)$ such that $\delta x(0) = 0$ and $J(t_f)$ is tangent to $M_1^\perp$,

$$J(t_f) \in T_{z(t_f)} M_1^\perp.$$

Test 1'. The computation is the same as for conjugate points but we integrate backwards in time. Consider the vector space of dimension $n-1$ generated by the Jacobi fields $J_i = (\delta x_i, \delta p_i)$, $i = 1, \ldots, n-1$, such that $J_i(0) \in T_{z(t_f)} M_1^\perp$ and $\delta p_i(0)$ are normalized by the condition

$$\langle p(t_f), \delta p_i(0) \rangle = 0.$$

The time t_f is a focal time whenever

$$\operatorname{rank}(\delta x_1(-t_f), \ldots, \delta x_{n-1}(-t_f)) < n - 1.$$

A direct application of the previous results is to consider the time optimal control of an SR-system with drift (see §4)

$$\dot{x} = F_0 + \sum_{i=1}^{m} u_i F_i, \ x \in \mathbf{R}^n, \ \sum_{i=1}^{m} |u_i|^2 \leq 1.$$

We proceed as follows. To introduce H_r we restrict our analysis to the extremals of order zero defined in §4. Thus, H_r is the smooth Hamiltonian function (13) defined outside the switching surface $\Sigma = \{H_i = 0, \ i = 1, \ldots, m\}$. Let z be a reference extremal of order zero, it is a singularity of the endpoint mapping where $u \in S^{m-1}$ and our algorithm applies. In particular, (A2) means that on each subinterval $[0, t]$ of $[0, T]$ the singularity is of codimension one. This condition is checked in the algorithm because if the codimension is more than one at t, $\exp_{x_0}(t, .)$ cannot be an immersion. Hence, this approach can be implemented without any preliminary computations in the SR-case with drift. It is also worth noting that if the extremal flow only has Π-singularities, the singularity resolution of §3 still allows us to implement the method.

5.2. **Application to the orbital transfer.** The system is written in the 3D radial-orthoradial frame, mass variation included,

$$\begin{aligned} \dot{x} &= F_0 + \frac{1}{m}(u_r F_r + u_{\mathrm{or}} F_{\mathrm{or}} + u_c F_c), \\ \dot{m} &= -\beta |u|. \end{aligned}$$

Since the terminal mass is free, we have $p_m = 0$ by transversality. Practically, the final longitude is free as well, so that $p_l = 0$ at the final time. For geometric purposes, we will assume l fixed at terminal time. Now, we integrate backwards the variational equation with initial conditions $\delta x(0) = 0$, $\delta p_m(0) = 0$, up to a first focal point such that $\delta x = 0$, $\delta m = 0$. Observe that since $\delta \dot{m} = 0$, then $\delta m \equiv 0$ and a focal point is also a conjugate point. As a matter of fact, if $\delta m \equiv 0$, the variational equation satisfied by p is the same as in the constant mass case with the mass explicited as a function of time. Finally, the algorithm to test second

TABLE 1. Physical constants.

Variable	Value
μ	5165.8620912 $\mathrm{Mm}^3\cdot\mathrm{h}^{-2}$
β	$1.42e-2$ $\mathrm{Mm}^{-1}\cdot\mathrm{h}$
m^0	1500 kg
$F_{\max}$	3 N

order conditions for the 3D and mass-varying system consists in computing the five Jacobi fields J_i, $i = 1, \ldots, 5$, for the time dependent system

$$\begin{aligned} \dot{x} &= F_0 + \frac{1}{m(t)}(u_r F_r + u_{\mathrm{or}} F_{\mathrm{or}} + u_c F_c), \\ \dot{p} &= -p\left(\frac{\partial F_0}{\partial x} + \frac{1}{m(t)}\left(u_r \frac{\partial F_r}{\partial x} + u_{\mathrm{or}} \frac{\partial F_{\mathrm{or}}}{\partial x} + u_c \frac{\partial F_c}{\partial x}\right)\right), \end{aligned}$$

with initial conditions $\delta x_i(0) = 0$ and $\delta p_i(0)$ normalized, and $m(t) = m_0 - \beta F_{\max} t$. The physical constants and the boundary conditions are summarized in Table 1 and Table 2, respectively.

The boundary conditions are chosen as follows. The physical problem is to transfer the system from the initial orbit to the geostationary orbit whose parameters are in Table 2. From the controllability result, there exists a trajectory of the system satifying these conditions with fixed longitude (taken as an angle). We compute such an extremal curve, steering the satellite to the geostationary orbit. It is then prolongated, in order to compute (if they exist) conjugate times.

TABLE 2. Boundary conditions.

Variable	Initial cond.	Final cond.
P	11.625 Mm	42.165 Mm
e_x	0.75	0
e_y	0	0
h_x	0.0612	0
h_y	0	0
l	π rad	103 rad

In order to generate the boundary value problem as well as the Jacobi equation, first and second order derivatives of the Hamiltonian H_r associated with smooth extremals of order zero are computed by automatic differentiation [1]. Extremals are approximated using a shooting technique. Though we are quite close to a Π-singularity located around the pericenter, the numerical integration is easy. Regarding conjugate times, the numerical procedure is to detect a change of sign in $\delta x_1(t) \wedge \ldots \wedge \delta x_5(t) \wedge \dot{x}(t)$. The result is checked by evaluating the rank of $\delta x_1(t), \ldots, \delta x_5(t)$ using a singular value decomposition (see fig. 1).

Given a reference extremal trajectory starting from x_0, the first point where the extremal ceases to be (globally) optimal is the *cut point*. The set of such points for every extremal is the *cut locus*, $C(x_0)$. An ultimate goal in optimal control is to compute the cut loci. Two relevant analogies for the analysis of the orbit transfer problem are, on the one hand an analytic Riemannian problem on $\mathbf{S}^2$, the flat torus $\mathbf{T}^2$ on the other hand. On $\mathbf{S}^2$, the cut locus is a tree, and extremities of the branches are conjugate points. Other cut points are points where several

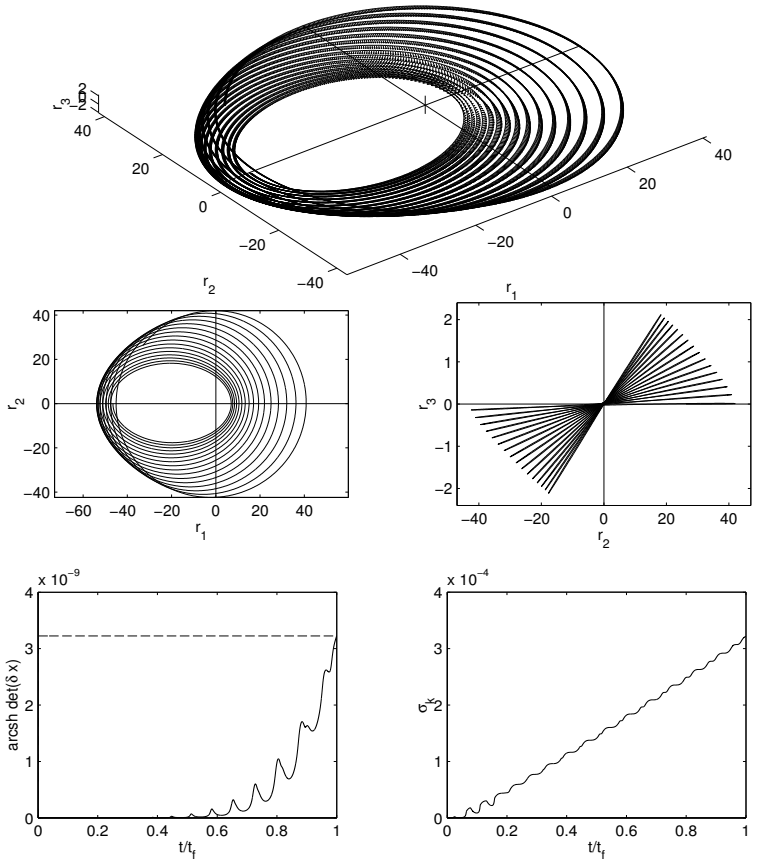


FIGURE 1. A 3 Newton transfer. The minimum time is about 12 days. There are approximatively 15 revolutions around the Earth. On the top, the optimal trajectory (with projections in the equatorial plane and a perpendicular plane to illustrate how the inclination is corrected). Bottom left, the determinant, bottom right, the smallest singular value of the Jacobi fields associated to the extremal. The positivity after $t = 0$ ensures local optimality of the trajectory.

minimizing geodesics meet. Moreover, conjugate point extremities of the cut are cusps of the conjugate locus. On the torus, the problem is flat and there exists no conjugate point. However, imbedding the torus into $[0, 1]^2$, every geodesic starting from the center of the square is minimizing until it meets a side of the square where two minimizing geodesics (four, in the case of a corner) meet. On top of that, any point can be connected to the origin by infinitely many non minimizing geodesics with increasing rotation counts on the torus. In orbital transfer, we can expect similar phenomenons, mixing both cases. Indeed, one can observe on numerical simulations that

– there are conjugate points;

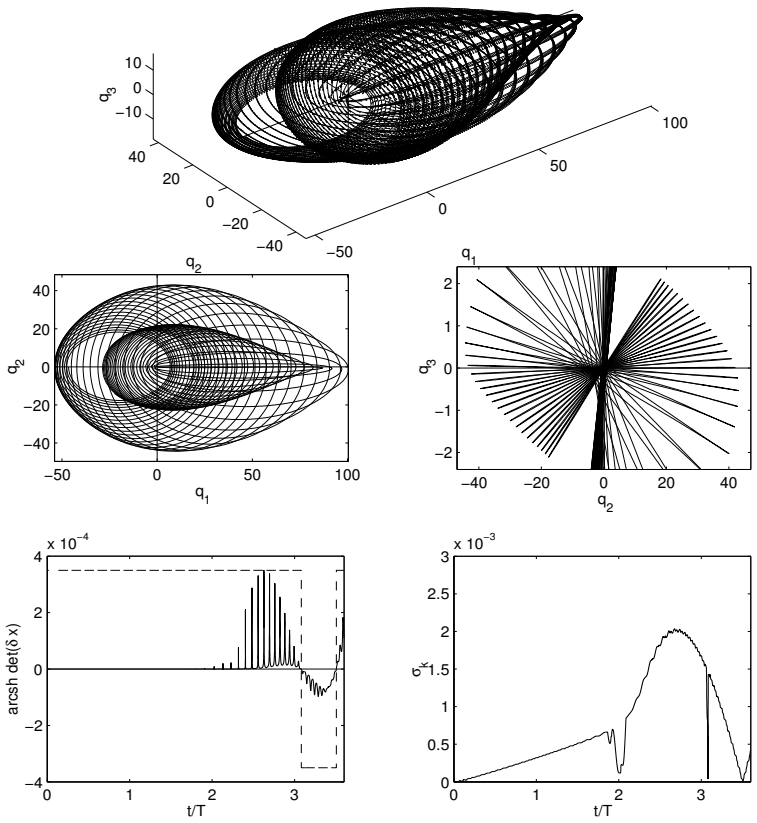


FIGURE 2. The same extremal as in fig. 1 is extended until, roughly, 3.5 times the minimum time. There, two conjugate times are detected. The optimality is lost about three times the minimum time.

– when fixing the final longitude, there are minimizing curves, but many extremals with a greater cumulated longitude also satisfy the other boundary conditions.

The second observation is the consequence of the topology of the elliptic domain which is fibered by elliptic orbits of Kepler equation.

5.3. **Second order conditions in the singular case for single-input affine systems.** In order to investigate the second order conditions in orbit transfer with thrust oriented in a single direction, we consider a single-input affine system

$$\dot{x} = F_0 + uF_1,$$

where F_0, F_1 are smooth vector fields of $\mathbf{R}^n$ and u is valued in $\mathbf{R}$. Optimal trajectories are singular, but we cannot apply the previous algorithms to check second order conditions because the strong Legendre condition is not satisfied any more. Our aim is to apply the theoretical framework of [7] so as to get sufficient conditions,

together with algorithms from [13, 6]. We first introduce some generic conditions along the reference extremal.

Let x be the reference singular trajectory on $[0,T]$, and let u be the associated control. First of all, it is convenient to apply a feedback transformation to normalize the control to $u \equiv 0$. We make the following assumptions.

(A1) The reference trajectory is smooth and injective.

(A2) For every $t \in [0,T]$, $\mathrm{span}\{\mathrm{ad}^k F_0.F_1(x(t)) \mid k = 0,\dots,n-2\}$ has codimension one.

As a result, this vector subspace is the Pontryagin cone $K(t)$ for positive t. The adjoint $p(t)$ is unique up to a constant and oriented with the convention $H \geq 0$ of the maximum principle.

(A3) The vector field $\mathrm{ad}^2 F_1.F_0$ does not belong to $\mathrm{span}\{\mathrm{ad}^k F_0.F_1 \mid k = 0,\dots,n-2\}$ along the reference trajectory.

According to §2, theses conditions imply that the reference singular extremal z is of order two and solution of

$$\dot{z} = \overrightarrow{H}_s(z) \tag{24}$$

on $\{H_1 = \{H_0, H_1\} = 0\}$ with

$$\begin{aligned} H_s &= H_0 + u_s H_1, \\ u_s &= -\frac{\{H_0,\{H_0,H_1\}\}}{\{H_1,\{H_0,H_1\}\}}. \end{aligned}$$

(A4) If $n = 2$, F_0 and F_1 are independent along the reference trajectory. If $n \geq 3$, F_0 does not belong to $\mathrm{span}\{\mathrm{ad}^k F_0.F_1(x(t)) \mid k = 0,\dots,n-3\}$ along the reference trajectory.

We recall the following result from [7].

THEOREM 5.2. *Under our assumptions, let x be the reference singular trajectory defined on $[0,T]$. In the hyperbolic and exceptional (*resp. *elliptic) case, the trajectory is locally time minimizing (*resp. *maximizing) with respect to all trajectories with same extremities and contained in a $\mathscr{C}^0$-neighbourhood of x, up to the first conjugate time t_{1c}.*

We describe now the algorithms to compute the conjugate points in these cases. Contrary to [7] where they are presented using a preliminary integral transformation, we shall lay here the emphasis on an intrinsic description coming from [13]. Nevertheless, since it is crucial in the understanding of the method, we still begin by recalling the Goh transform.

Since, by assumption, F_1 is transverse to the trajectory, we can identify F_1 with $\partial/\partial x_n$ in a tubular neighbourhood of x and the system is decomposed into

$$\begin{aligned} \dot{\widetilde{x}} &= F(\widetilde{x}, x_n), \\ \dot{x}_n &= g(\widetilde{x}, x_n) + u, \end{aligned}$$

where $\widetilde{x} = (x_1,\dots,x_{n-1})$.

DEFINITION 5.5. The *integral* (or Goh) *transformation* consists in choosing $v = x_n$ as the new control, considering thus the reduced system $\dot{\widetilde{x}} = F(\widetilde{x}, x_n)$ with associated Hamiltonian

$$\widetilde{H} = \langle \widetilde{p}, F(\widetilde{x}, v)\rangle,\ v \in \mathbf{R}.$$

The connection between the two systems is the following (see [7]).

LEMMA 5.2. *The triple (x, p, u) is an extremal if and only if $(\widetilde{x}, \widetilde{p}, x_n)$ is an extremal of the reduced system. Moreover, there holds, along $(x(t), p(t))$,*

$$\begin{aligned} \frac{d}{dt}\frac{\partial H}{\partial u} &= -\frac{\partial \widetilde{H}}{\partial x_n}, \\ \frac{\partial}{\partial u}\frac{d^2}{dt^2}\frac{\partial H}{\partial u} &= -\frac{\partial^2 \widetilde{H}}{\partial x_n^2}. \end{aligned}$$

As a consequence, the strict Legendre-Clebsch condition is equivalent to the strong Legendre condition on the reduced system.

In the elliptic and hyperbolic cases, we present two algorithms: one based on the Goh transform, the other being intrinsic.

Test 1. Having performed the integral transformation, we are in the regular case of §5.1 and we consider the reduced system

$$\begin{aligned} \dot{\widetilde{x}} &= \frac{\partial \widetilde{H}}{\partial \widetilde{p}}, \\ \dot{\widetilde{p}} &= -\frac{\partial \widetilde{H}}{\partial \widetilde{x}}. \end{aligned}$$

Let $\widetilde{J}_1, \ldots, \widetilde{J}_{n-2}$ be the $n-2$ Jacobi fields, $\widetilde{J}_i = (\delta\widetilde{x}_i, \delta\widetilde{p}_i)$, vertical at $t = 0$ and with $\delta\widetilde{p}(0)$ normalized as before. Then, t_c is a conjugate time if

$$\operatorname{rank}(\delta\widetilde{x}_1(t_c), \ldots, \delta\widetilde{x}_{n-2}(t_c)) < n-2.$$

Test 2. It is intrinsic and does not use the Goh transform. Actually, we consider the Jacobi fields associated with the variational equation of (24) together with the constraints $H_1 = \{H_0, H_1\} = 0$ linearized at $z(0)$,

$$dH_1 = d\{H_0, H_1\} = 0. \tag{25}$$

The set of $\delta z(0) = (\delta x(0), \delta p(0))$ where $\delta p(0)$ is still normalized by $\langle p(0), \delta p(0)\rangle = 0$, where, moreover, we add the condition $\delta x(0) \in \mathbf{R}F_1(x(0))$, and where (25) holds, form a linear subspace of dimension $n-2$. If $J_1, \ldots, J_{n-2}$ are the Jacobi fields for these initial conditions, t_c is a conjugate time if

$$\operatorname{rank}(\delta x_1(t_c), \ldots, \delta x_{n-2}(t_c), F_1(x(t_c))) < n-1.$$

Under our assumptions, this is equivalent to

$$\delta x_1(t_c) \wedge \ldots \wedge \delta x_{n-2}(t_c) \wedge F_1(x(t_c)) \wedge F_0(x(t_c)) = 0.$$

Remark 5.2. Observe that we replace the verticality condition $\delta x = 0$ by a verticality condition for the reduced system, $\delta x \in \mathbf{R}F_1$.

In the exceptional case, the result is not straightforward, even with the theoretical result of [7]. The test is presented without the integral transformation in order to be implemented numerically.

Test 3. Since extremals are restricted to the level set $H_0 = 0$, we consider the $n-3$ Jacobi fields that solve the variational equation of (24) with the augmented linearized constraints

$$dH_0 = dH_1 = d\{H_0, H_1\} = 0. \tag{26}$$

The corresponding set of $\delta z(0) = (\delta x(0), \delta p(0))$ where $\delta p(0)$ is normalized as before, $\delta x(0)$ is in $\mathbf{R}F_1(x(0))$, and where (26) holds, is of dimension $n-3$. Accordingly, t_c is a conjugate time whenever

$$\operatorname{rank}(\delta x_1(t_c), \ldots, \delta x_{n-3}(t_c), F_1(x(t_c)), F_0(x(t_c))) < n-1. \tag{27}$$

Eventually, under our assumptions, this is equivalent to

$$\delta x_1(t_c) \wedge \ldots \wedge \delta x_{n-3}(t_c) \wedge F_1(x(t_c)) \wedge F_0(x(t_c)) \wedge \mathrm{ad}^2 F_1.F_0(x(t_c)) = 0. \tag{28}$$

Remark 5.3. This is intricate but, geometrically, a conjugate point in this case corresponds to the existence of a Jacobi field which is tangent to the level $\{H_0 = 0\}$ with the terminal focal condition $\delta x \in \operatorname{span}(\{F_1, F_0\})$.

We end the last section with the application of such a computation to the orbit transfer problem. More precisely, we consider the single-input case of §3.5 with the thrust oriented along F_{or}. In the exceptional case, the control is the feedback control (10) and since $n = 4$, according to (28) we only have one Jacobi field to compute. The physical values for the computation are those of table 1. Since we have a 2D-constant mass model, we do not use β or h, though, and we change l_0 not to start from the pericenter or the apogee (see fig. 3).

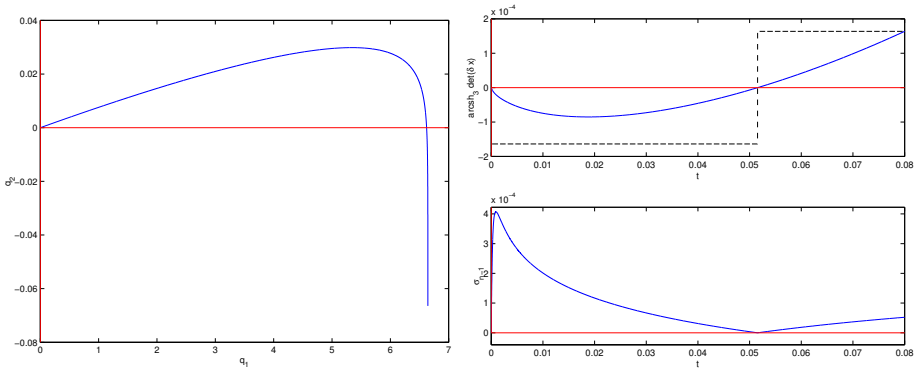


FIGURE 3. Exceptional trajectory. The initial cumulated longitude is $l_0 = -0.01$ and we tend to a collision (see the orbit on the left). A conjugate point is detected ($t_{\mathrm{cc}} \simeq 5.1e-2$) by checking the associated determinant (28) as well as the rank in (27) (see the two subplots on the right).

6. **Conclusion.** The contribution of this article is twofold. First of all, we make a geometric analysis of the controllability properties of the system, studying the role of each controller in the tangent-normal frame and in the radial-orthoradial frame. This is a preliminary step to study the time optimal control problem, in which we have technical limitations on the control. Besides, our analysis allows to construct control laws, using for instance a path planning method, completing existing control methods based on stabilization (see [11, 12]). Secondly, we analyze the time optimal problem. We give a geometric model of a singularity observed in the problem (see [14]), called π-singularity, and our resolution allows to handle this problem numerically, and proves optimality. Another contribution is to give second-order optimality conditions, which complete previous results (see [9, 14]).

Combined with numerical simulations, this allows to compute the optimal solution to transfer in minimal time our system to a geostationary orbit. Moreover, our work is a first step in the analysis of optimal trajectories for every terminal condition, computing the conjugate locus and the cut locus. Another possible future work is the investigation of the general optimal control problem with low propulsion, in which the cost is a compromise between minimizing time and maximizing the final mass.

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