Non subanalyticity of sub-Riemannian Martinet spheres

Emmanuel Trélat

C. R. Acad. Sci. Paris, t. ???, Série ?, p. ???–???, 2001
Géométrie analytique/*Analytic geometry*

Non subanalyticity of sub-Riemannian Martinet spheres

Emmanuel TRÉLAT [a]

[a] **Laboratoire de Topologie, Université de Bourgogne, 9 av. A. Savary, BP 47870, 21078 Dijon**
Courriel : trelat@topolog.u-bourgogne.fr

(Reçu le ???, accepté le ???)

Abstract. Consider the sub-Riemannian Martinet structure (M, Δ, g) where $M = \mathbb{R}^3$, $\Delta = \text{Ker}(dz - \frac{y^2}{2}dx)$ and g is the general gradated metric of order 0 : $g = (1+\alpha y)^2 dx^2 + (1+\beta x+\gamma y)^2 dy^2$. We prove that if $\alpha \neq 0$ then the sub-Riemannian spheres $S(0,r)$ with small radii are not subanalytic. © Académie des Sciences/Elsevier, Paris

Non sous-analyticité des sphères sous-Riemanniennes de Martinet

Résumé. *Considérons la structure sous-Riemannienne de Martinet* (M, Δ, g) *où* $M = \mathbb{R}^3$, $\Delta = \text{Ker}(dz - \frac{y^2}{2}dx)$ *et* g *est la métrique générale graduée d'ordre* 0 : $g = (1+\alpha y)^2 dx^2 + (1+\beta x+\gamma y)^2 dy^2$*. On montre que si* $\alpha \neq 0$*, les sphères sous-Riemanniennes* $S(0,r)$ *de petit rayon ne sont pas sous-analytiques.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Une *structure sous-Riemannienne* (M, Δ, g) est constituée d'une variété M de dimension finie, d'une distribution Δ sur M, et d'une métrique g sur Δ. Elle est dite analytique si tous les objets sont analytiques. Une courbe γ sur M est dite *horizontale* si elle est tangente en tout point à la distribution Δ. Soient x_0, x_1 des points de M, et $T > 0$. Le *problème sous-Riemannien* est de déterminer une courbe horizontale joignant x_0 à x_1 en temps T et minimisant la longueur (au sens de la métrique g). La *distance sous-Riemannienne* $d_{SR}(x_0, x_1)$ entre x_0 et x_1 est la borne inférieure des longueurs des chemins horizontaux joignant x_0 à x_1 en temps T. La *sphère sous-Riemannienne* $S(x_0, r)$ centrée en x_0, de rayon r, est l'ensemble des points x de M tels que $d_{SR}(x_0, x) = r$.

Reformulons le problème sous-Riemannien dans le cadre de la théorie du contrôle optimal. Notre point de vue étant local, on peut supposer que $M = \mathbb{R}^n$ et $x_0 = 0$. Supposons que la distribution Δ est engendrée par m champs de vecteurs $X_1, \dots, X_m$. Sans perte de généralité, on peut supposer que $T = 1$. Le problème sous-Riemannien est alors équivalent au problème de *contrôle optimal* pour le système (voir par exemple [5]) :

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)),\ x(0) = 0$$

Note présentée par ???

où la fonction $u = (u_1, \dots, u_m)$, appelée le *contrôle*, doit minimiser la *longueur*

$$l(u) = \int_0^1 \sqrt{g(\dot{x}(t), \dot{x}(t))}dt$$

L'application $E \ : u \in L^2([0,1], \mathbb{R}^{\mathrm{m}}) \longmapsto x_u(1) \in \mathbb{R}^{\mathrm{n}}$, qui à un contrôle u associe l'extrémité de la solution correspondante x_u, est appelée *application entrée/sortie*. C'est une application lisse.

Si un contrôle u minimise la longueur l, alors il existe un *multiplicateur de Lagrange* $(\psi, \psi^0) \in \mathbb{R}^{\mathrm{n}} \times \mathbb{R}$ défini à scalaire près tel que

$$\psi.dE(u) = -\psi^0 dl(u)$$

Si $\psi^0 \neq 0$, le contrôle (ou la trajectoire associée) est dit *normal* ; sinon il est dit *anormal*. Un contrôle anormal est une singularité de l'application entrée/sortie.

Dans [1] il est prouvé que s'il n'existe pas de trajectoire anormale minimisante non triviale partant de 0 alors la distance sous-Riemannienne à 0, $d_{SR}(0,.)$, est sous-analytique en dehors de 0, et donc les sphères sous-Riemanniennes $S(0,r)$ de petit rayon sont sous-analytiques. Cette situation est générique pour des distributions de rang ≥ 3, voir [3]. Cependant les anormales minimisantes apparaissent génériquement pour des distributions de rang 2, et on conjecture que dans ce cas les sphères sous-Riemanniennes ne sont jamais sous-analytiques. Remarquons que dans [3] les auteurs prouvent que l'existence d'une anormale minimisante est responsable de la non sous-analyticité du germe en 0 de $d_{SR}(0,.)$. Ici la conjecture concerne les sphères sous-Riemanniennes.

Cette conjecture a été prouvée dans [2] dans le cas dit de *Martinet plat*, mais il n'est pas générique. Ici on montre que cette conjecture est vraie pour une métrique générale graduée d'ordre 0. Le cas Martinet est le cas sous-Riemannien le plus simple pour lequel une anormale minimisante apparaît génériquement. La situation est la suivante. Considérons dans $\mathbb{R}^3$ la distribution $\Delta = \mathrm{Ker}(dz - \frac{y^2}{2}dx)$; elle est engendrée par les deux champs de vecteurs

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

Les variables x, y sont de poids 1, et z est de poids 3 (voir [5]). Soit g une métrique analytique sur Δ ; d'après [2] une forme normale générale graduée d'ordre 0 est :

$$g = (1 + \alpha y)^2 dx^2 + (1 + \beta x + \gamma y)^2 dy^2$$

où $\alpha, \beta, \gamma \in \mathbb{R}$. C'est une perturbation du cas plat où $\alpha = \beta = \gamma = 0$. L'unique direction anormale correspond à l'axe x ; elle est minimisante [4].

THÉORÈME 1. – *Si $\alpha \neq 0$ alors les sphères sous-Riemanniennes $S(0,r)$ de petit rayon ne sont pas sous-analytiques.*

---

1. Introduction

1.1. Sub-Riemannian geometry

A *sub-Riemannian structure* (M, Δ, g) is a triple, where M is a finite dimensional manifold, Δ is a distribution on M, and g is a metric on Δ. It is said analytic if all objects are analytic. A curve γ on M is called *horizontal* if it is tangent at any point to the distribution Δ. Let x_0, x_1 in M, and $T > 0$. The *sub-Riemannian problem* is to determine an horizontal curve joining x_0 and x_1 in time T and minimizing the length (in the sense of the metric g). The *sub-Riemannian distance* $d_{SR}(x_0, x_1)$ between x_0 and x_1 is the infimum of lengths of horizontal paths joining x_0 and x_1 in time T. The *sub-Riemannian sphere* $S(x_0, r)$ centered at x_0, with radius r, is the set of points x in M such that $d_{SR}(x_0, x) = r$.

1.2. Necessary optimality conditions

In order to determine optimality conditions for an horizontal curve, we shall reformulate the sub-Riemannian problem in optimal control theory. Our point of view is local, hence we can suppose that $M = \mathbb{R}^n$ et $x_0 = 0$. Assume that the distribution Δ is spanned by m vector fields $X_1, \ldots, X_m$. Without loss of generality, we may suppose that $T = 1$. Then the sub-Riemannian problem is equivalent to the *optimal control problem* for the system (see for instance [5]) :

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)), \ \ x(0) = 0$$

where the function $u = (u_1, \ldots, u_m)$, called the *control*, has to minimize the *length*

$$l(u) = \int_0^1 \sqrt{g(\dot{x}(t), \dot{x}(t))} dt$$

The mapping $E \ : u \in L^2([0,1], \mathbb{R}^m) \longmapsto x_u(1) \in \mathbb{R}^n$, which to a control u associates the extremity of the corresponding solution x_u, is called *end-point mapping.* It is smooth.

If a control u minimizes the length l, then there exists a *Lagrange multiplier* $(\psi, \psi^0) \in \mathbb{R}^n \times \mathbb{R}$ defined up to a scalar such that

$$\psi.dE(u) = -\psi^0 dl(u)$$

Moreover from the *Pontryagin's Maximum Principle* (see [9]) the trajectory x corresponding to this minimizing control u is the projection of an *extremal*, i.e. a solution of the Hamiltonian system :

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \ \dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \ \frac{\partial H}{\partial u}(x, p, u) = 0$$

where $H(x, p, u) = < p, \sum_{i=1}^{m} u_i X_i(x) > + p^0 g(\dot{x}(t), \dot{x}(t))$ is the Hamiltonian function, p is an absolutely continuous function on $[0,1]$ called *adjoint vector*, and p^0 is a constant. Moreover we have, up to a scalar : $(p(1), p^0) = (\psi, \psi^0)$. If $p^0 \neq 0$, the extremal is said to be *normal*, and we can normalize to $p^0 = -\frac{1}{2}$; otherwise it is said *abnormal.* An abnormal control is a singularity of the end-point mapping. An abnormal trajectory is said to be *strict* if it is not the projection of a normal extremal.

1.3. Abnormal minimizers and subanalyticity

In [1] it is proved that if there exists no (non trivial) abnormal minimizer starting from 0 then the sub-Riemannian distance to 0, $d_{SR}(0, .)$, is subanalytic outside 0, and hence sub-Riemannian spheres $S(0, r)$ with small radii are subanalytic. This happens generically for distributions of rank ≥ 3, see [3]. However abnormal minimizers may appear generically for rank 2 distributions, and we conjecture that in this case sub-Riemannian spheres are never subanalytic. Note that in [3] authors prove that the existence of an abnormal minimizer is responsible for non subanalyticity of the germ at 0 of $d_{SR}(0,.)$. Here the conjecture concerns sub-Riemannian spheres.

This conjecture was proved in [2] in the so-called *Martinet flat case*, but this case is not generic. Here we prove that this conjecture is true for a general gradated metric of order 0. The Martinet case is the easier sub-Riemannian case where an abnormal minimizer appears generically. The situation is the following. Consider in $\mathbb{R}^3$ the distribution $\Delta = \text{Ker}(dz - \frac{y^2}{2}dx)$; it is spanned by the two vector fields

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

Variables x, y have weight 1, and z has weight 3 (see [5]). Let g be an analytic metric on Δ ; from [2] a general gradated normal form of order 0 is :

$$g = adx^2 + cdy^2$$

where $a = (1+\alpha y)^2, c = (1+\beta x+\gamma y)^2$, and α, β, γ are real parameters. It is a perturbation of the flat case where $\alpha = \beta = \gamma = 0$. There exists an unique abnormal direction, corresponding to the axis x ; it is minimizing, see [4]. Moreover it is *strict* if and only if $\alpha \neq 0$ (see [2]).

THEOREM 1. – *If $\alpha \neq 0$ (i.e. in the strict case), the sub-Riemannian spheres $S(0,r)$ with small radii are not subanalytic.*

The main idea of the proof is the following. The interest of the previous metric of order 0 is all in the fact that the problem, formulated in the cotangent space of $\mathbb{R}^3$, projects onto the phase plane of a *one-parameter pendulum*. This fact was already used in [2], [6]. In [2], in order to prove that the sphere in the Martinet flat case is not subanalytic, the authors make a direct computation of the sphere. But this is impossible in the general case because equations of extremals given by the Maximum Principle *are no more integrable*. Actually such computations lead to asymptotic expansions of the sphere near the abnormal direction (see [6]), but these expansions are not precise enough to check non subanalyticity. The new idea here is rather *indirect* and consists in using Lagrange multipliers : then proving the theorem amounts to estimating precisely a *first return time near a saddle point* of this one-parameter pendulum and proving that it is not subanalytic.

2. Proof of Theorem 1

Consider for the system $\dot{q} = uX(q) + vY(q), q(0) = 0$, the abnormal control $u = -r, v = 0$. It is always minimizing, see [4]. In order to prove the theorem we shall prove that *the intersection of the sphere $S(0,r)$ with the plane $y = 0$ is not a subanalytic curve near the abnormal direction (Ox)*. Let $A = (-r, 0, 0)$ be the extremity of the abnormal extremal associated to $u = -r, v = 0$; it belongs to the sub-Riemannian sphere $S(0,r)$ with radius r. Consider near A the curve $\mathcal{C} = S(0,r) \cap (y = 0) \cap (z \leq 0)$. If $\alpha \neq 0$, i.e. if the abnormal is *strict*, this curve is tangent at A to $z = 0$ with a contact of order 2 (see [6] and [11] for a general result). Consider the set of Lagrange multipliers $(\psi(q), \psi^0(q))$ associated to points $q \in \mathcal{C} \backslash \{A\}$. Since these points are reached by normal minimizers, and since Lagrange multipliers are defined up to a scalar, we may suppose that their component ψ^0 is equal to $-1/2$. On the other part we easily get from the definition of Lagrange multipliers that vectors $(\psi_x(q), \psi_z(q))$, $q \in \mathcal{C}$ (where ψ_x, ψ_z denote respectively the x and z-components of ψ), are normal to the curve $\mathcal{C}$ in the plane $y = 0$. Hence to prove that the germ at A of the curve $\mathcal{C}$ is not subanalytic, it is enough to prove that the set $\mathcal{L} = \{(\psi_x(q), \psi_z(q)) \ / \ q \in \mathcal{C}\}$ is not subanalytic.

The method is the following. First note that $\psi(q)$ is equal to $p(1)$ where $p(.)$ is the adjoint vector associated to the normal minimizer steering 0 to q. Hence to compute Lagrange multipliers we have to estimate adjoint vectors at final time $t = 1$. The Maximum Principle gives us a parametrization of normal extremals and leads to the following differential system in the Martinet case :

$$\dot{x} = \frac{1}{a}(p_x + p_z\frac{y^2}{2}),\quad \dot{y} = \frac{p_y}{c},\quad \dot{z} = \frac{y^2}{2a}(p_x + p_z\frac{y^2}{2})$$
$$\dot{p_x} = \frac{p_y^2 c_x}{2c^2} + \frac{(p_x + p_z\frac{y^2}{2})^2}{2a^2}a_x,\quad \dot{p_y} = \frac{p_y^2 c_y}{2c^2} + \frac{(p_x + p_z\frac{y^2}{2})^2}{2a^2}a_y - \frac{(p_x + p_z\frac{y^2}{2})}{a}p_z y,\quad \dot{p_z} = 0 \tag{1}$$

We investigate the Lagrange multipliers $(p_x(1), p_z(1))$ as $y(1) = 0$ and $z(1) \leq 0$. Note that p_z is constant along an extremal. Near the point A, it happens that $p_z \to +\infty$; it is a phenomenon of *non-properness* due to the existence of an abnormal minimizer, see [11]. From [5], extremal controls satisfy $au^2 + cv^2 = r^2$. Then we can set : $r\cos\theta = \sqrt{a}u = \frac{p_x + \frac{y^2}{2}p_z}{\sqrt{a}}$, $r\sin\theta = \sqrt{c}v = \frac{p_y}{\sqrt{c}}$. In particular we introduce θ which is an angular coordinate of the adjoint vector. This (*analytic*) cylindric change of coordinates on the adjoint vector is classical in sub-Riemannian geometry.

Now proving that the set $\mathcal{L}$ is not subanalytic amounts to exhibiting a non subanalytic relation between $\theta(1)$ and p_z as $p_z \to +\infty$. We proceed as follows. Reparametrizing with $ds = \frac{r\sqrt{p_z}}{\sqrt{a}\sqrt{c}}dt$, first note that, due to the particular form of the metric, the previous equations (1) project onto a *pendulum equation* :

$$\frac{d^2\theta}{ds^2} + \sin\theta + \frac{\alpha^2}{p_z}\sin\theta\cos\theta - \frac{\alpha\beta}{p_z}\sin^2\theta + \frac{\beta}{\sqrt{p_z}}\cos\theta\frac{d\theta}{ds} = 0 \tag{2}$$

It is a *pendulum perturbed* by the small parameter $\varepsilon = \frac{1}{\sqrt{p_z}}$. In this pendulum representation the plane $y = 0$ projects onto the curve Σ :

$$\frac{d\theta}{ds} = \frac{\alpha}{\sqrt{p_z}}\cos\theta - \frac{\beta}{\sqrt{p_z}}\sin\theta \tag{3}$$

and moreover the abnormal direction projects onto the *saddle point* $\theta = -\pi$, $\frac{d\theta}{ds} = 0$. Near this saddle point we shall now compute a *first return time* t_f for the section Σ, in function of $\theta(t_f)$ and of the parameter p_z. Then, imposing $t_f = 1$ shall give a relation between p_z and $\theta(1)$, and we shall prove that this relation is not subanalytic.

Remark 2.1. – In the pendulum representation the branch $\mathcal{C}$ corresponds to a *local* computation near a saddle point, whereas the branch $S(0,r) \cap (y = 0) \cap (z \geq 0)$ corresponds to a *global* computation of return mapping near the separatrices of the pendulum. The interest of this branch $\mathcal{C}$ is all in the fact that computations are *localized near a saddle*, hence we shall use *normal forms* according to the following Lemma.

LEMMA 2.1. – *Let $(X_\varepsilon)_{0\leq\varepsilon<1}$ be a one-parameter analytic family of vector fields in $\mathbb{R}^2$ near 0 such that for any ε : $X_\varepsilon(0) = 0$, and let (u, v) denote standard coordinates in $\mathbb{R}^2$. Let $\mu(\varepsilon)$ and $\nu(\varepsilon)$ be the eigenvalues of $dX_\varepsilon(0)$. We suppose that $\mu(0) = -1$ and $\nu(0) = 1$. Then there exists an analytic germ at 0 from $\mathbb{R}^3$ to $\mathbb{R}^2$:*

$$\varphi\ : (u, v, \varepsilon) \longmapsto (u_1, v_1) = \varphi(u, v, \varepsilon) \qquad \text{with } \varphi(0, 0, \varepsilon) = 0$$

such that in the new coordinates (u_1, v_1) the system $(\dot{u}, \dot{v}) = X_\varepsilon(u, v)$ can be written near 0 as :

$$\dot{u}_1 = \mu(\varepsilon)u_1(1 + \mathrm{o}(\varepsilon)),\ \dot{u}_2 = \nu(\varepsilon)u_2(1 + \mathrm{o}(\varepsilon)) \qquad \text{as } \varepsilon \to 0$$

This Lemma may be proved by adding parameters in the proof of [7], see for instance [8]. To fit in this situation, set $u = \theta + \pi$, $v = \frac{d\theta}{ds}$, and $\varepsilon = \frac{1}{\sqrt{p_z}}$. Then the pendulum equation (2) can be written as the system in $\mathbb{R}^2$:

$$\frac{du}{ds} = v, \quad \frac{dv}{ds} = \sin u - \alpha^2\varepsilon^2 \sin u \cos u + \alpha\beta\varepsilon^2 \sin^2 u + \beta\varepsilon v \cos u$$

Applying Lemma 2.1, we get in new coordinates (u_1, v_1) :

$$\frac{du_1}{ds} = \mu(\varepsilon)u_1(1 + \mathrm{o}(\varepsilon)), \quad \frac{dv_1}{ds} = \nu(\varepsilon)v_1(1 + \mathrm{o}(\varepsilon))$$

where $\mu(\varepsilon) = 1 + \frac{\beta\varepsilon}{2} + \mathrm{O}(\varepsilon^2)$, $\nu(\varepsilon) = -1 + \frac{\beta\varepsilon}{2} + \mathrm{O}(\varepsilon^2)$, and actually : $u = u_1 + v_1 + \mathrm{o}(\varepsilon)$, $v = u_1 - v_1 + \mathrm{o}(\varepsilon)$. Therefore in the new coordinates the section (3) is $\Sigma \ : v_1 = u_1 + \alpha\varepsilon + \mathrm{o}(\varepsilon)$.

Now the return time s_f can be computed in a standard way :

$$s_f = \int_0^{s_f} ds = \int_{v_1(0)}^{v_1(s_f)} \frac{dv_1}{\nu(\varepsilon)v_1(1 + o(\varepsilon))} = \frac{1}{\nu(\varepsilon)}(1 + o(\varepsilon))\ln\frac{v_1(s_f)}{v_1(0)} \tag{4}$$

On the one part : $v_1(0) = \alpha\varepsilon + \mathrm{o}(\varepsilon)$; on the other part we get by integrating the system up to $\mathrm{O}(\varepsilon^{\frac{3}{2}})$: $\frac{dt}{ds} = \frac{\varepsilon}{r}(1 + \alpha y)(1 + \beta x + \gamma y) = \frac{\varepsilon}{r}\mathrm{e}^{-\beta\varepsilon s} + \mathrm{O}(\varepsilon^{\frac{3}{2}})$. Now claiming that the final t_f is equal to 1, we get :

$$1 = \int_0^{s_f} \frac{dt}{ds} ds = \frac{1 - \mathrm{e}^{-\beta\varepsilon s_f}}{\beta \mathrm{r}} + \mathrm{O}(\varepsilon^2)$$

And thus : $s_f = -\frac{\ln(1-\beta r)}{\beta\varepsilon} + \mathrm{O}(\varepsilon)$. Putting into (4) we obtain finally :

$$v_1(s_f) = \frac{\alpha\varepsilon}{\sqrt{1 - \beta r}} \mathrm{e}^{\frac{\ln(1-\beta r)}{\beta\varepsilon}}(1 + \mathrm{o}(1)) \qquad \text{as } \varepsilon \to 0$$

In particular $v_1(s_f)$ is not an analytic function in ε as $\varepsilon \to 0$.

From Lemma 2.1 we know that $v_1(s_f) = An(u(s_f), v(s_f), \varepsilon) \sim \frac{u(s_f) - v(s_f)}{2}$, where $An(.)$ denotes an analytic germ at 0. Moreover, on the section Σ, we have : $v(s_f) = -\alpha\varepsilon \cos u(s_f) + \beta\varepsilon \sin u(s_f)$. Hence : $v_1(s_f) = An(u(s_f), \varepsilon) = \frac{u(s_f)}{2} + \cdots$. From the Implicit Function Theorem in the analytic class we get : $u(s_f) = An(v_1(s_f), \varepsilon)$. Therefore $u(s_f)$ is not an analytic function in ε, which ends the proof.

References

[1] Agrachev A., Compactness for sub-Riemannian length minimizers and subanalyticity, Rend. Semin. Mat. Torino, Vol. 56, 1998.

[2] Agrachev A., Bonnard B., Chyba M., Kupka I., Sub-Riemannian sphere in the Martinet flat case, ESAIM/COCV, Vol. 2, 377-448, 1997.

[3] Agrachev A., Gauthier J.P., On subanalyticity of Carnot-Carathéodory distances, à paraître aux Annales de l'IHP.

[4] Agrachev A., Sarychev A., Strong minimality of abnormal geodesics for 2-distributions, J. of Dynamical and Control Systems, Vol. 1, No. 2, 1995, pp 139-176.

[5] Bellaïche A., Tangent space in sub-Riemannian geometry, Sub-Riemannian geometry, Birkhäuser, 1996.

[6] Bonnard B., Trélat E., On the role of abnormal minimizers in SR-geometry, Preprint Dijon, 1998.

[7] Hadamard J., Sur l'itération et les solutions asymptotiques des équations différentielles, Bulletin de la SMF, tome 29, 1901 - Oeuvres complètes, tome 3, 1023–1026.

[8] Moussu R., Sur la dynamique des gradients, Existence de variétés invariantes, Math. Ann. 307, 1997, 445–460.

[9] Pontryagin L. et al., Théorie mathématique des processus optimaux, Eds Mir, Moscou, 1974.

[10] Roussarie R., On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields, Bol. Soc. Br. Mat., Vol. 17, No. 2, 1986.

[11] Trélat E., Some properties of the value function and its level sets for affine control systems with quadratic cost, J. of Dynamical and Control Systems, Vol. 6, No. 4 (Oct. 2000), pp 511-541.