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# Lazard's elimination (in traces) is finite-state recognizable

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## Abstract

We prove that the codes issued from the elimination of any sub-alphabet in a trace monoid are finite-state recognizable. This implies in particular that the transitive factorizations of the trace monoids are recognizable by (boolean) finite-state automata.

**Keywords:** Trace monoid; Lazard's elimination; automata with multiplicities.

## 1 Introduction

Sch utzenberger ([8] Chapter 5) introduced the notion of a factorization of a monoid  $M$

$$M = \prod_{i \in I}^{\rightarrow} M_i \quad (1)$$

where  $(M_i)_{i \in I}$  is a subfamily of submonoids of the given monoid  $M$ . When  $M = A^*$  is a free monoid, at the both ends of the chain, one has complete factorizations like Lyndon and Hall factorizations [11] and the bisections  $|I| = 2$  [7].

A nice way to produce factorizations is to start with a bisection  $M = M_1 M_2$  and refine the factors using a uniform process. Doing this, we could obtain a complete factorization for every trace monoid [3]. Trace monoids are defined as follows. Consider an alphabet  $\Sigma = \{x_1, \dots, x_n\}$  and a commutation relation  $\vartheta$  (*i.e.* a reflexive and symmetric relation) on  $\Sigma$ . The trace monoid  $\mathbf{M}(\Sigma, \vartheta)$  is the quotient

$$\mathbf{M}(\Sigma, \vartheta) = \Sigma^* / \equiv_{\vartheta} \quad (2)$$

where  $\equiv_{\vartheta}$  is the congruence generated by the relators  $ab \equiv ba$  where  $(a, b) \in \vartheta$ .

Later on, we addressed the question of bisecting a trace monoid so that the left factor be generated by a subalphabet (Lazard bisection) and the right factor be a trace monoid [5]. Doing so, we obtained a complete description of the factors and graph-theoretical criteria for the factorization. We conjectured that the trace codes so obtained could be recognized by finite-state automata [5].

In this paper, we prove that the answer to the conjecture is positive. This will be a consequence of the more general result that if a trace monoid  $M(\Sigma, \vartheta)$  is bisected as

$$M(\Sigma, \vartheta) = L.M(B, \vartheta_B) \quad (3)$$

with  $B \subset \Sigma$  and  $\vartheta_B = \vartheta \cap (B \times B)$ , then the minimal generating set  $\beta(L)$  of  $L$  is recognizable by a finite-state, effectively constructible automaton. Here, we prove this fact and give the construction of the automaton.

The paper is organised as follows:

In section 2, we recall basic notions related to trace monoids and recognizability. In section 3, we prove that the left factor of a Lazard bisection is a recognizable set and we describe the construction of a deterministic automaton recognizing it in section 4. To end with, we explain in section 5 how to construct a deterministic automaton which recognizes the generating set of the left factor of such a bisection.

## 2 Trace Monoids

Trace monoids were introduced by Cartier and Foata with the purpose of studying some combinatorial problems linked with rearrangements (see [2]).

Next, this notion has been studied by Mazurkiewicz and many schools of Computer Sciences in the context of concurrent program schemes (see [9, 10]).

Let  $x \in \Sigma$  be a letter and denote  $\text{Com}(x)$  the set of letters which commute with  $x$

$$\text{Com}(x) = \{z \mid (x, z) \in \vartheta\}. \quad (4)$$

In particular, one has  $x \in \text{Com}(x)$ . Let  $w \in \mathbf{M}(\Sigma, \vartheta)$  be a trace, we will denote

$$TA(w) = \{x \in \Sigma \mid w = ux, u \in \mathbf{M}(\Sigma, \vartheta)\} \quad (5)$$

the *terminal alphabet* of  $w$ .

As it is shown in [3], Lazard elimination occurs in the context of traces. Let  $B$  be a subalphabet of  $\Sigma$  and  $\vartheta_B = \vartheta \cap (B \times B)$ . The trace monoid splits into two submonoids

$$\mathbf{M}(\Sigma, \vartheta) = L.\mathbf{M}(B, \vartheta_B) \quad (6)$$

where  $L$  is the submonoid consisting in the traces whose terminal alphabet is a subset of  $\Sigma \setminus B$ . Furthermore the decomposition is unique, which suggests that the following equality occurs in  $\mathbf{Z}\langle \Sigma, \vartheta \rangle = \mathbf{Z}[\mathbf{M}(\Sigma, \vartheta)]$ , the algebra of series corresponding to  $\mathbf{Z}[\mathbf{M}(\Sigma, \vartheta)]$  [4]. Thus,

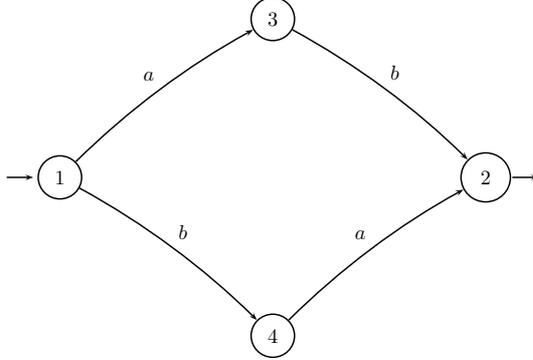
$$\underline{\mathbf{M}(\Sigma, \vartheta)} = \underline{L}.\underline{\mathbf{M}(B, \vartheta_B)} \quad (7)$$

where  $\underline{S}$  denotes the characteristic series of a subset  $S \subset \mathbf{M}(\Sigma, \vartheta)$  *i.e.*

$$\underline{S} = \sum_{w \in S} w \in \mathbf{Z}\langle \langle \Sigma, \vartheta \rangle \rangle. \quad (8)$$

Let  $\phi$  be the natural surjection  $\Sigma^* \rightarrow \mathbf{M}(\Sigma, \vartheta)$ , the *set of the representative words* of a trace  $t$  is defined as  $\text{Rep}(t) = \phi^{-1}(t)$ . We can extend this definition to trace langages  $\text{Rep}(L) = \phi^{-1}(L) = \bigcup_{t \in L} \phi^{-1}(t)$ . A trace langage is said *recognizable* if and only if its representative set is, and we say that an automaton recognizes  $L$  if and only if it *recognizes*  $\text{Rep}(L)$ .

**Example 1** Let  $a$  and  $b$  be two commuting letters, then the set  $\text{Rep}(\{ab\})$  is recognized by the automaton



In fact, one can prove that a rational language is a set of representatives (i.e. it is saturated w.r.t. the congruence  $\equiv_\theta$ ) if and only if the corresponding minimal automaton shows complete squares as above.

We will denote  $\text{Rec}(\Sigma, \vartheta)$  the set of recognizable sets of traces.

### 3 Recognizing the left factor

The  $\mathbf{Z}$ -rationality of the left factor  $L$  is a direct consequence of the unicity of the decomposition, which, in term of formal series, reads

$$\underline{\mathbf{M}}(\Sigma, \vartheta) = \underline{L} \cdot \underline{\mathbf{M}}(B, \vartheta_B). \quad (9)$$

where  $\underline{S}$  denotes the  $\mathbf{Z}$ -characteristic series of the set  $S$  (i.e.  $\underline{S} = \sum_{x \in S} x$ ). Indeed, by a classical result due to Cartier and Foata ([2] Theorem 2.4) the  $\mathbf{Z}$ -characteristic series of  $\underline{\mathbf{M}}(\Sigma, \vartheta)$  is rational when the alphabet  $\Sigma$  is finite<sup>a</sup>:

$$\underline{\mathbf{M}}(\Sigma, \vartheta) = \frac{1}{\sum_{\{a_1, \dots, a_n\} \in \mathbf{Cliques}(\Sigma)} (-1)^n a_1 \cdots a_n}. \quad (10)$$

where the sum at the denominator is taken over the set  $\mathbf{Cliques}(\Sigma)$  of the cliques of  $\Sigma$  (i.e. commutative sub-alphabets). Hence, one obtains the

<sup>a</sup>The formula holds also when the alphabet is infinite but the denominator is then a series.

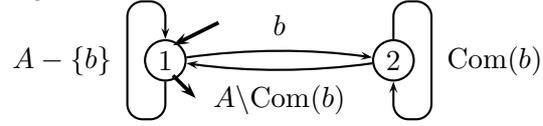
rational equality

$$\underline{L} = \frac{1}{\sum_{\{a_1, \dots, a_n\} \in \mathbf{Cliques}(\Sigma)} (-1)^n a_1 \cdots a_n} \times \left( \sum_{\{b_1, \dots, b_n\} \in \mathbf{Cliques}(B)} (-1)^n b_1 \cdots b_n \right) \quad (11)$$

Nevertheless, this remark is not sufficient to show that  $L$  is recognizable as a language. Furthermore, for traces, one has the strict inclusion  $\text{Rec}(\Sigma, \vartheta) \subset \text{Rat}(\Sigma, \vartheta)$ . To prove that  $L$  is recognizable it suffices to find a construction of  $\text{Rep}(L)$  using only recognizable operations. For each letter  $x \in \Sigma$ , let  $\mathbf{TN}_x$  be the set of representative words of traces whose terminal alphabet does not contain  $x$ . Remarking that  $\text{Rep}(L)$  is the representative set of the traces whose terminal alphabet contains no letter of  $B$ , one has

$$\text{Rep}(L) = \bigcap_{b \in B} \mathbf{TN}_b. \quad (12)$$

Hence,  $\text{Rep}(L)$  is recognizable if each  $\mathbf{TN}_b$  is. But, one can easily verify that automaton  $\mathcal{A}_b$ :



recognizes  $\mathbf{TN}_b$ . Thus, we have the proposition

**Proposition 1**  $L$  is a recognizable submonoid of  $M(\Sigma, \vartheta)$ .

## 4 A deterministic automaton for a terminal condition

One can compute a deterministic automaton recognizing  $L$  generalizing the construction of  $\mathcal{A}_b$ . We consider an automaton  $\mathcal{A}_B = (S_B, I_B, F_B, T_B)$  such that:

1. The set  $S_B$  of its states is the set of all the sub-alphabets of  $B$ ,
2. There a unique initial state  $I_B = \{\emptyset\}$ ,
3. There a unique final state  $F_B = \{\emptyset\} = I_B$ ,
4. The transitions are

$$T_B = \{(B', x, ((B' \cup \{x\}) \cap \text{Com}(x) \cap B))\}_{B' \subset B, x \in \Sigma}.$$

One has

**Proposition 2** *The automaton  $\mathcal{A}_B$  is a complete deterministic automaton recognizing  $\text{Rep}(L)$ .*

**Proof** It is straightforward to see that such an automaton is complete and deterministic. Now, let us prove that it recognizes  $\text{Rep}(L)$ . As  $\mathcal{A}_L$  is complete deterministic, for each word  $w = a_1 \cdots a_n$  we can consider a state  $s_w$  which is the state of  $\mathcal{A}_B$  after reading  $w$ . More precisely, we can define  $s_w$  as  $s_w = s_n$  in the following chain of transitions

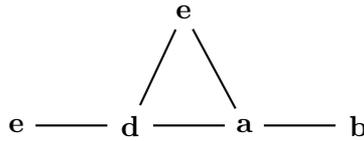
$$(\emptyset, a_1, s_1), (s_1, a_2, s_2), \dots, (s_{n-1}, a_n, s_n = s_w). \quad (13)$$

We first prove that if  $w$  is a word then  $s_w = TA(t_w) \cap B$  where  $t_w$  denotes the trace admitting  $w$  as representative word. We use an induction process, considering as starting point:  $(\emptyset, x, \{x\} \cap B)$  where  $x \in \Sigma$ . Let  $w = a_1 \cdots a_n$  be a word of length  $n$ , such that  $s_w$  is the intersection between  $B$  and the terminal alphabet of trace  $t_w$ . Let  $a_{n+1} \in \sigma$  be an other letter. One has,  $(s_w, a_{n+1}, s_{wa_{n+1}}) \in T_B$ . Hence, the set  $s_{wa_{n+1}}$  is

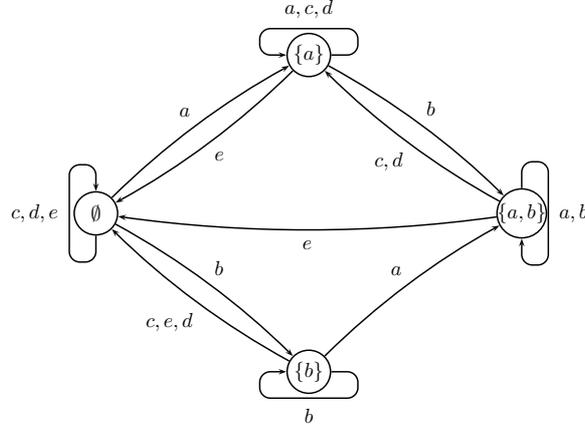
$$s_{wa_{n+1}} = (s_w \cup \{a_{n+1}\}) \cap \text{Com}\{a_{n+1}\} \cap B = TA(t_w a_{n+1}) \cap B = TA(t_{wa_{n+1}}) \cap B. \quad (14)$$

This proves our assertion. Then, the set of words  $w$  such that  $s_w = \emptyset$  is exactly the set of representative words of  $L$ .  $\square$

**Example 2** We consider the trace alphabet given by the following commutation graph



If we set  $B = \{a, b\}$ , then  $L$  is recognized by the following automaton (in the figure the only initial state and the only final state is  $\emptyset$ ):



## 5 A deterministic automaton for the generating set of the left factor

Each submonoid  $M$  of a trace monoid has an unique *generating set* which is the subset  $G(M) = M \setminus M^2$ .<sup>b</sup>

In this section, we prove that  $G(L)$  is recognizable and we construct an automaton  $A_\beta$  which recognizes it. The automaton  $A_\beta$  is obtained from  $A_B$  by adding two states  $F, H$ , choosing  $F$  as final state instead of  $\emptyset$  and modifying the transitions in such a way that if a letter of  $Z = A - B$  is read, the state reached belongs in  $F, H$  and the other states become unreachable.

More precisely, one considers the automaton  $\mathcal{A}_\beta = (S_\beta, I_\beta, F_\beta, T_\beta)$  obtained from the automaton  $A_B = (S_B, I_B, F_B, T_B)$  computed in the previous section as follows:

1. The set of its states  $S_\beta$ , is the set of the sub-alphabets of  $B$  plus two states  $F$  and  $H$ ,
2. There is a unique initial state  $I_\beta = \{\emptyset\}$ ,
3. There is a unique final state  $F_\beta = \{F\}$ ,
4. The transitions are

$$T_\beta = T_{B \rightarrow B} \cup T_{B \rightarrow F} \cup T_{B \rightarrow H} \cup T_{F \rightarrow H} \cup T_{H \rightarrow H}$$

<sup>b</sup>The fact that  $G(M)$  generates  $M$  is straightforward and the unicity comes from that the  $\mathbf{Z}$ -characteristic series of  $G(M)$  is the inverse of the  $\mathbf{Z}$ -characteristic series of  $M$  in  $\mathbf{Z}\langle\langle A \rangle\rangle$ .

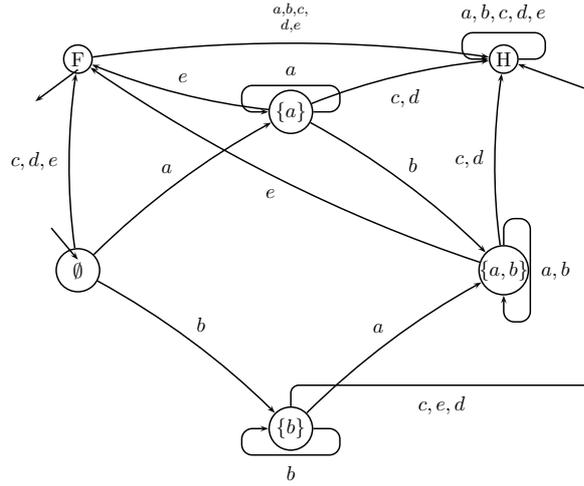
where

- (a)  $T_{B \rightarrow B} = \{(B', b, B'')\}_{B', B'' \subset B, b \in B, (B', b, B'') \in T_B}$ ,
- (b)  $T_{B \rightarrow F} = \{(B', z, F)\}_{(B', z, \emptyset) \in T_B, B' \subset B, B' \neq \emptyset, z \in Z}$ ,
- (c)  $T_{B \rightarrow H} = \{(B', z, H)\}_{(B', z, B'') \in T_B, B', B'' \notin \{\emptyset, F, H\}, z \in Z}$ ,
- (d)  $T_{F \rightarrow H} = \{(F, x, H)\}_{x \in \Sigma}$ ,
- (e) and  $T_{H \rightarrow H} = \{(H, x, H)\}_{x \in \Sigma}$ .

**Proposition 3** *The automaton  $\mathcal{A}_\beta$  recognizes  $\text{Rep}(G(L))$ .*

**Proof** The automaton is almost the same as  $\mathcal{A}_B$ . As for  $\mathcal{A}_B$ , if a word of  $B^*$  is read, the automaton is in the state corresponding to its terminal alphabet. The difference appears when a letter of  $Z$  is read, if it is read from the  $\emptyset$  state the automaton goes to the state  $F$ . Consider now a word  $w = w'z$  with  $w' \in B^+$ ,  $z \in Z$ . We denote  $\delta_w$  the state of the automaton after reading  $w$  (this definition makes sense as, like  $\mathcal{A}_B$ ,  $\mathcal{A}_\beta$  is deterministic). Now, if  $\{z\} = TA(w'z)$ , then  $(\delta'_w, z, F) \in T_{B \rightarrow F}$  which means that  $w$  is recognized by  $\mathcal{A}_\beta$ , otherwise  $(\delta'_w, z, H) \in T_{B \rightarrow H}$  and  $w$  is not recognized by  $\mathcal{A}_\beta$ . Furthermore, for each  $z \in Z$  and  $b \in B$ ,  $\delta_{w'zaw''} = H$  (for each  $w', w'' \in \Sigma^*$ ). This ends the proof.  $\square$

**Example 3** Consider again the example (2). Then,  $\beta$  is recognized by the automaton



## 6 Conclusion

The factorisations of free monoids (or in a more general setting of a monoid constructed by generators and relations) is a relevant topic in the context of the theory of codes [1]. Lazard bisections, or more generally rational bisections [7], play a role in the construction of bases of free Lie algebras [11] and the study of circular codes [1, 11]. A natural question asks if it is possible to generalize these properties to other monoids in particular when the free module over these monoids can be endowed with a shuffle coproduct [6]. The results contained in the paper consist in a step in the study of these problems for the trace monoids. The role played by the Lazard bisections in this context is not still completely known (see [3, 5] for some results).

## References

- [1] J. Berstel and D. Perrin , Theory of codes (Pure and applied Mathematics Acad. Press. Inc., 1995).
- [2] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements*, Lect. Not. In Math., **85** (1969).
- [3] G. Duchamp and D. Krob , *Factorisations dans le monoïde partiellement Commutatif libre*, C.R. Acad. Sci. Paris, **312**, série I (1991), 189-192.
- [4] Duchamp G., Krob D., *Combinatorics on traces*, Ch II du "Book of traces" (Ed. G. Rozenberg, V. Dieckert), World Scientific (1995).
- [5] G. Duchamp and J.-G. Luque, *Transitive factorisations of partially Commutative free monoids and Lie algebras.*, Discrete Math., **246** (2002).
- [6] G. Duchamp and J.G. Luque, *Congruences compatible with the shuffle product*, proceeding of Formal Power series and algebraic combinatorics FPSAC'00, Moscow, 2000
- [7] G. Duchamp and J.-Y. Thibon, *Bisections reconnaissables*, Theoretical Informatics and Applications, **22** (1988), 113-128, (Gauthiers-Villars).
- [8] N. Lothaire, *Combinatorics on words*, Cambridge University Press (1997).

- [9] A. Mazurkiewicz, *Concurrent program schemes and their interpretation*, (DAIMI Rep., PB 78, Aarhus University, 1977).
- [10] A. Mazurkiewicz, *Traces, histories and graph: instances of process monoid*, Lect. Not. in Comp. Sci., **176**, 115-133 (1984).
- [11] C. Reutenauer, *Free Lie Algebras* (Oxford University Press, New-York, 1993)
- [12] G. Rozenberg, V. Dieckert, *The Book of traces* (World Scientific ,1995).
- [13] G.X. Viennot. *Heaps of pieces*, Lect. Notes in Math., **1234** (1986), 321-350.