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A SMOOTH COUNTEREXAMPLE TO NORI’S CONJECTURE ON THE FUNDAMENTAL GROUP SCHEME

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Abstract. We show that Nori’s fundamental group scheme $\pi(X, x)$ does not base change correctly under extension of the base field for certain smooth projective ordinary curves $X$ of genus 2 defined over a field of characteristic 2.

1. Introduction

In the paper [N] Madhav Nori introduced the fundamental group scheme $\pi(X, x)$ for a reduced and connected scheme $X$ defined over an algebraically closed field $k$ as the Tannaka dual group of the Tannakian category of essentially finite vector bundles over $X$. In characteristic zero $\pi(X, x)$ coincides with the étale fundamental group, but in positive characteristic it does not (see e.g. [MS]). By analogy with the étale fundamental group, Nori conjectured that $\pi(X, x)$ base changes correctly under extension of the base field. More precisely:

Nori’s conjecture (see [MS] page 144 or [N] page 89) If $K$ is an algebraically closed extension of $k$, then the canonical homomorphism

$$h_{X,K} : \pi(X_K, x) \longrightarrow \pi(X, x) \times_k \text{Spec}(K)$$

is an isomorphism.

In [MS] V.B. Mehta and S. Subramanian show that Nori’s conjecture is false for a projective curve with a cuspidal singularity. In this note (Corollary 4.2) we show that certain smooth projective ordinary curves of genus 2 defined over a field of characteristic 2 also provide counterexamples to Nori’s conjecture.

The proof has two ingredients: the first is an equivalent statement of Nori’s conjecture in terms of $F$-trivial bundles due to V.B. Mehta and S. Subramanian (see section 2) and the second is the description of the action of the Frobenius map on rank-2 vector bundles over a smooth ordinary curve $X$ of genus 2 defined over a field of characteristic 2 (see section 3). In section 4 we explicitly determine the set of $F$-trivial bundles over $X$.

I would like to thank V.B. Mehta for introducing me to these questions and for helpful discussions.

2. Nori’s conjecture and $F$-trivial bundles

Let $X$ be a smooth projective curve defined over an algebraically closed field $k$ of characteristic $p > 0$. Let $F : X \to X$ denote the absolute Frobenius of $X$ and $F^n$ its $n$-th iterate for some positive integer $n$.

2.1. Definition. A rank-$r$ vector bundle $E$ over $X$ is said to be $F^n$-trivial if

$E$ stable and $F^n E \cong \mathcal{O}_X^n$.

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2.2. Proposition ([MS] Proposition 3.1). If the canonical morphism $h_{X,K}$ ([1]) is an isomorphism, then any $F^n$-trivial vector bundle $E_K$ over $X_K := X \times_k Spec(K)$ is isomorphic to $E_k \otimes_k K$ for some $F^n$-trivial vector bundle $E_k$ over $X$.

3. The action of the Frobenius map on rank-2 vector bundles

We briefly recall some results from [LP1] and [LP2].

Let $X$ be a smooth projective ordinary curve of genus 2 defined over an algebraically closed field $k$ of characteristic 2. By [LP2] section 2.3 the curve $X$ equipped with a level-2 structure can be uniquely represented by an affine equation of the form

$$y^2 + x(x+1)y = x(x+1)(ax^3 + (a+b)x^2 + cx + c),$$

for some scalars $a, b, c \in k$. Let $\mathcal{M}_X$ denote the moduli space of $S$-equivalence classes of semistable rank-2 vector bundles with trivial determinant over $X$ — see e.g. [LeP]. We identify $\mathcal{M}_X$ with the projective space $\mathbb{P}^3$ (see [LP1] Proposition 5.1). We denote by $V : \mathbb{P}^3 \rightarrow \mathbb{P}^3$ the rational map induced by pull-back under the absolute Frobenius $F : X \rightarrow X$. There are homogeneous coordinates $(x_0 : x_1 : x_10 : x_{11})$ on $\mathbb{P}^3$ such that the equations of $V$ are given as follows (see [LP2] section 5)

$$V(x_0 : x_1 : x_10 : x_{11}) = (\sqrt{abcP^2_{00}(x)} : \sqrt{bP^2_{01}(x)} : \sqrt{cP^2_{10}(x)} : \sqrt{aP^2_{11}(x)}),$$

with

$$P_{00}(x) = x_0^2 + x_1^2 + x_{10}^2 + x_{11}^2, \quad P_{10}(x) = x_0x_10 + x_1x_{11}, \quad P_{11}(x) = x_0x_{11} + x_1x_{10}.$$

Given a semistable rank-2 vector bundle $E$ with trivial determinant, we denote by $[E] \in \mathcal{M}_X = \mathbb{P}^3$ its $S$-equivalence class. The semistable boundary of $\mathcal{M}_X$ equals the Kummer surface $Kum_X$ of $X$. Given a degree 0 line bundle $N$ on $X$, we also denote the point $[N \oplus N^{-1}] \in \mathbb{P}^3$ by $N$.

3.1. Proposition ([LP1] Proposition 6.1 (4)). The preimage $V^{-1}(N)$ of the point $N \in Kum_X \subset \mathcal{M}_X = \mathbb{P}^3$ with coordinates $(x_0 : x_1 : x_10 : x_{11})$

- is a projective line, if $x_{00} = 0$.
- consists of the 4 square-roots of $N$, if $x_{00} \neq 0$.

4. Computations

In this section we prove the following

4.1. Proposition. Let $X = X_{a,b,c}$ be the smooth projective ordinary curve of genus 2 given by the affine model ([3.1]). Suppose that

$$a^2 + b^2 + c^2 + a + c = 0.$$

Then there exists a nontrivial family $\mathcal{E} \rightarrow X \times S$ parametrized by a 1-dimensional variety $S$ (defined over $k$) of $F^4$-trivial rank-2 vector bundles with trivial determinant over $X$. Moreover any $F^4$-trivial rank-2 vector bundle $E$ with trivial determinant appears in the family $\mathcal{E}$, i.e., is of the form $(id_X \times s)^* \mathcal{E}$ for some $k$-valued point $s$ : $Spec(k) \rightarrow S$.

We therefore obtain a counterexample to Nori’s conjecture.

4.2. Corollary. Let $X = X_{a,b,c}$ be a curve satisfying ([1.1]). Then for any algebraically closed extension $K$, the morphism $h_{X,K}$ is not an isomorphism.

Proof. Since $S$ is 1-dimensional, there exists a $K$-valued point $s : Spec(K) \rightarrow S$, which is not a $k$-valued point. Then the bundle $E_K = (id_X \times s)^* \mathcal{E}$ over $X_K$ is not of the form $E_k \otimes_k K$. Now apply Proposition 2.2. □
Proof of Proposition 4.1. The method of the proof is to determine explicitly all \( F^n \)-trivial rank-2 vector bundles \( E \) over \( X \) for \( n = 1, 2, 3, 4 \). Taking tensor product of \( E \) with \( 2^{n+1} \)-torsion line bundles allows us to restrict attention to \( F^n \)-trivial vector bundles with trivial determinant.

We first compute the preimage under iterates of \( V \) of the point \( A_0 \in \mathbb{P}^3 \) determined by the trivial rank-2 vector bundle over \( X \). We recall (see e.g. [LP], Lemma 2.10 (i)) that the coordinates of \( A_0 \in \mathbb{P}^3 \) in the coordinate system \((x_0 : x_1 : x_10 : x_{11})\) are \((1 : 0 : 0 : 0)\). It follows from Proposition 3.1 and equations (3.2) that \( V^{-1}(A_0) \) consists of the 4 points
\[
(1 : 0 : 0 : 0), \quad (0 : 1 : 0 : 0), \quad (0 : 0 : 1 : 0) \quad \text{and} \quad (0 : 0 : 0 : 1),
\]
which correspond to the 2-torsion points of the Jacobian of \( X \). Abusing notation we denote by \( A_1 \) both the 2-torsion line bundle on \( X \) and the point \((0 : 1 : 0 : 0)\) \( \in \mathbb{P}^3 \).

Both points \( A_0 \) and \( A_1 \) correspond to \( S \)-equivalence classes of semistable rank-2 vector bundles. The set of isomorphism classes represented by the two \( S \)-equivalence classes \( A_0 \) and \( A_1 \) equal \( \mathbb{P} \text{Ext}^1(A_1, A_1) \cup \{0\} \) and \( \mathbb{P} \text{Ext}^1(O_X, O_X) \cup \{0\} \) respectively, where \( 0 \) denotes the trivial extensions \( A_1 \oplus A_1 \) and \( O_X \oplus O_X \). Note that the two cohomology spaces \( \text{Ext}^1(A_1, A_1) \) and \( \text{Ext}^1(O_X, O_X) \) are canonically isomorphic to \( H^1(O_X) \). The pull-back by the absolute Frobenius \( F \) of \( X \) induces a rational map
\[
F^*: \mathbb{P} \text{Ext}^1(A_1, A_1) \longrightarrow \mathbb{P} \text{Ext}^1(O_X, O_X),
\]
which coincides with the projectivized \( p \)-linear map on the cohomology \( H^1(O_X) \to H^1(O_X) \) induced by the Frobenius map \( F \). Since we have assumed \( X \) ordinary, this \( p \)-linear map is bijective. Hence we obtain that there is only one (strictly) semistable bundle \( E \) such that \([E] = A_1\) and \( F^* E \cong O_X\), namely \( E = A_1 \oplus A_1 \). In particular there are no \( F^1 \)-trivial rank-2 vector bundles over \( X \).

By Proposition 3.1 and using the equations (3.2), we easily obtain that the preimage \( V^{-1}(A_1) \) is a projective line \( L \cong \mathbb{P}^1 \), which passes through the two points
\[
(1 : 1 : 1 : 1) \quad \text{and} \quad (0 : 0 : 1 : 1).
\]
We now determine the bundles \( E \) satisfying \( F^* E \cong A_1 \oplus A_1 \). Given \( E \) with \([F^*E] = A_1 \in \mathbb{P}^3 \) we easily establish the equivalence
\[
F^* E \cong A_1 \oplus A_1 \quad \iff \quad \dim \text{Hom}(F^* E, A_1) = \dim \text{Hom}(E, F_\ast A_1) = 2.
\]
Suppose that \( E \) is stable and \( F^* E \cong A_1 \oplus A_1 \). The quadratic map
\[
\det : \text{Hom}(E, F_\ast A_1) \longrightarrow \text{Hom}(\det E, \det F_\ast A_1) = H^0(O_X(w))
\]
has nontrivial fibre over \( 0 \), since \( \dim \text{Hom}(E, F_\ast A_1) = 2 \). Hence there exists a nonzero \( f \in \text{Hom}(E, F_\ast A_1) \) not of maximal rank. We consider the line bundle \( N = \text{im} f \subset F_\ast A_1 \). Since \( F_\ast A_1 \) is stable (see [LaP], Proposition 1.2), we obtain the inequalities
\[
0 = \mu(E) < \deg N < \frac{1}{2} = \mu(F_\ast A_1),
\]
a contradiction. Therefore \( E \) is strictly semistable and \([E] = [A_2 \oplus A_2^{-1}]\) for some 4-torsion line bundle \( A_2 \) with \( A_2^{\otimes 2} = A_1 \). The \( S \)-equivalence class \([A_2 \oplus A_2^{-1}]\) contains three isomorphism classes and a standard computation shows that only the decomposable bundle \( A_2 \oplus A_2^{-1} \) is mapped by \( F^* \) to \( A_1 \oplus A_1 \). In particular there are no \( F^2 \)-trivial rank-2 bundles.

We now determine the coordinates of \( A_2 \) by intersecting the line \( L \), which can be parametrized by \((r : r : s : s)\) with \( r, s \in k \), with the Kummer surface, whose equation is (see [LP], Proposition 3.1)
\[
c(x_{00}^2 x_{10} + x_{01}^2 x_{11}) + b(x_{00}^2 x_{01}^2 + x_{10}^2 x_{11}^2) + a(x_{00}^2 x_{11}^2 + x_{10}^2 x_{01}^2) + x_{00} x_{01} x_{10} x_{11} = 0.
\]
The computations are straightforward and will be omitted. Let \( u \in k \) be a root of the equation
\[
(4.3) \quad u^2 + u = b.
\]
Then \( u + 1 \) is the other root. The coordinates of the two 4-torsion line bundles (modulo the canonical involution of the Jacobian of \( X \)) \( A_2 \) such that \( A_2^{\otimes 2} = A_1 \) are
\[
(u : u : \sqrt{b} : \sqrt{b}) \quad \text{and} \quad (u + 1 : u + 1 : \sqrt{b} : \sqrt{b}).
\]
Now the equation \( u = 0 \) (resp. \( u + 1 = 0 \)) implies by (4.3) \( b = 0 \), which is excluded because we have assumed \( X \) smooth. So by Proposition 3.1 the preimage \( V^{-1}(A_2) \) consists of the 4 line bundles \( A_3 \) such that \( A_3^{\otimes 2} = A_2 \). In particular there are no \( F^3 \)-trivial rank-2 bundles.

One easily verifies that the image under the rational map \( V \) given by (5.2) of the hyperplane \( x_{00} = 0 \) is the quartic surface given by the equation
\[
(4.4) \quad b x_{11} x_{10}^2 + c x_{12} x_{01}^2 + a x_{12} x_{01}^2 + x_{00} x_{10} x_{01} x_{11} = 0.
\]
When we replace \( (x_{00} : x_{01} : x_{10} : x_{11}) \) with \( (u : u : \sqrt{b} : \sqrt{b}) \) in (4.4) we obtain the equation
\[
(4.5) \quad b^2 + u^2(1 + a + c) = 0.
\]
Similarly replacing \( (x_{00} : x_{01} : x_{10} : x_{11}) \) with \( (u + 1 : u + 1 : \sqrt{b} : \sqrt{b}) \) in (4.4) we obtain the equation
\[
(4.6) \quad b^2 + (u^2 + 1)(1 + a + c) = 0.
\]
Finally the product of (4.3) with (4.6) equals (here one uses (4.3)) equation (4.1) up to a factor \( b^3 \), which we can drop since \( b \neq 0 \) — note that we have assumed \( X \) smooth, hence \( b \neq 0 \) by [LP2] Lemma 2.1. To summarize we have shown that if (4.1) holds, then by Proposition 3.1 there exists an 8-torsion line bundle \( A_3 \) with \( A_3^{\otimes 4} = A_1 \) and such that the preimage \( V^{-1}(A_3) \) is a projective line \( \Delta \subset \mathbb{P}^3 \).

Consider a point \([E] \in \Delta \) away from the Kummer surface — note that \( \Delta \) is not contained in the Kummer surface \( \text{Kum}_X \) because its intersection is contained in the set of 16-torsion points. Then \( E \) is stable and \([F^* E] = [A_3 \oplus A_3^{-1}] \). There are three isomorphism classes represented by the \( S \)-equivalence class \([A_3 \oplus A_3^{-1}] \), namely the trivial extension \( A_3 \oplus A_3^{-1} \) and two nontrivial extensions (for the details see [LP2] Remark 6.2). Since \( E \) is invariant under the hyperelliptic involution we obtain \( F^* E = A_3 \oplus A_3^{-1} \) and finally that \( E \) is \( F^4 \)-trivial. Hence any stable point on \( \Delta \) is \( F^4 \)-trivial.

Therefore, assuming (4.1), there exists a 1-dimensional subvariety \( \Delta_0 \subset \mathcal{M}_X \setminus \text{Kum}_X \) parametrizing all \( F^4 \)-trivial rank-2 bundles. Passing to an étale cover \( S \rightarrow \Delta_0 \) ensures existence of a “universal” family \( \mathcal{E} \rightarrow X \times S \) and we are done.

\[ \square \]

**Remark.** Note that equation (4.1) depends on the choice of a nontrivial 2-torsion line bundle \( A_1 \). If one chooses the 2-torsion line bundle \((0 : 0 : 1 : 0)\) or \((0 : 0 : 0 : 1)\) — see (1.2) — the corresponding equations are
\[
(a^2 + b^2 + c^2 + a + b = 0) \quad \text{or} \quad (a^2 + b^2 + c^2 + b + c = 0).
\]

## References


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