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Let $p$ be an odd prime number. It was noticed by Iwasawa that the $p$-adic behavior of Jacobi sums in $\mathbb{Q}(\zeta_p)$ is linked to Vandiver’s Conjecture (see [Iw]). This result has been generalized by various authors for the cyclotomic $\mathbb{Z}_p$-extensions of abelian fields (see for example [HI], [I1], [B]). In this paper we consider the module of Weil numbers (see §2) for the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\zeta_p)$, and we get some results quite similar to those for Jacobi sums. In particular we establish a connection between the $p$-adic behavior of Weil numbers and a weak form of Greenberg Conjecture (see [N2], [BN]).

1 Notations

Let $p$ be a fixed odd prime number. For any $n \in \mathbb{N}$ we denote by $k_n$ the $p^{n+1}$-th cyclotomic field $\mathbb{Q}(\mu_{p^{n+1}})$, where $\mu_{p^{n+1}}$ is the group of $p^{n+1}$-th roots of unity. We note $\Delta = \text{Gal} (k_0/\mathbb{Q})$, $\Gamma_n = \text{Gal} (k_n/k_0)$ and $G_n = \text{Gal} (k_n/\mathbb{Q})$, so $G_n \simeq \Delta \times \Gamma_n$. Let $\zeta_p \in \mu_p \setminus \{1\}$ and take for any $n \in \mathbb{N} \zeta_p^{n+1} \in \mu_{p^{n+1}}$ such that $\forall n \geq 1 \zeta_p^{p^{n+1}} = \zeta_p^{n+1}$. We note $\pi_n = 1 - \zeta_p^{n+1}$.

We shall also use the following more or less standard notations:
- $k_{n,p}$ the $p$-completion of $k_n$;
- $U_n = 1 + \pi_n \mathbb{Z}_p[\zeta_p^{n+1}]$ principal units in $k_{n,p}$;
- $\Gamma = \varprojlim \Gamma_n \simeq \mathbb{Z}_p$, $\gamma_0$ its topological generator, where $\forall \varepsilon \in \mu_{p^{\infty}}$, $\gamma_0(\varepsilon) = \varepsilon^{1+p}$;
- $\Lambda = \mathbb{Z}_p[[\Gamma]]$ the Iwasawa algebra of the profinite group $\Gamma$, $\Lambda \simeq \mathbb{Z}_p[[T]]$ by sending $\gamma_0 - 1$ to $T$ ([W, Theorem 7.1]);
- $A_n$ is the Sylow $p$-subgroup of $\text{Cl}(k_n)$, where $\text{Cl}(k_n)$ is the ideal class group of $k_n$;
- $X = \varprojlim A_n$ be the projective limit of $A_n$ for the norm maps;
- $I_n$ the group of prime-to-$p$ ideals of $k_n$;
- $k_\infty = \bigcup_{n \in \mathbb{N}} k_n$, $\text{Gal} (k_\infty/k_0) = \Gamma$;
- $L_n/k_n$ the maximal abelian unramified $p$-extension of $k_n$; $\text{Gal} (L_n/k_n) \simeq A_n$ by class field theory;
- $M_n/k_n$ the maximal abelian $p$-extension of $k_n$ unramified outside of $p$;
- $\mathfrak{X}_n = \text{Gal} (M_n/k_n)$;
- $L_\infty = \bigcup L_n$; $X \simeq \text{Gal} (L_\infty/K_\infty)$;
- $M_\infty = \bigcup M_n$;
- $\mathfrak{X}_\infty = \text{Gal} (M_\infty/k_\infty)$.

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Let $\psi$ be a fixed odd character of $\Delta$, different from Teichmüller character $\omega$. We note $e_\psi$ the associated idempotent defined by

$$e_\psi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \psi(\delta) \delta^{-1} \in \mathbb{Z}_p[\Delta].$$

Let $M \in \Lambda$ be the distinguished polynomial of smallest degree such that $M(T)e_\psi.X = \{0\}$. We call it the minimal polynomial of $e_\psi.X$. It is well known to be prime to $\omega_n = (T+1)^{p^n}-1$ for any $n$ (cf. [W, §13.6, Theorem 7.10, Theorem 5.11 and Theorem 4.17]).

2 Weil numbers and Jacobi sums.

Fix an $n$ for a moment.

Definition 1 We call Weil module of $k_n$ the module $W_n$ defined by

$$W_n = \{ f \in \text{Hom}_{\mathbb{Z}[G_n]}(I_n, k_n^+) \mid \exists \beta(f) \in \mathbb{Z}[G_n] \text{ such that } \forall a \in I_n \ a = (\alpha) \Rightarrow f(a) \equiv \alpha^{\beta(f)} \mod \mu_{2p^{n+1}} \} \quad (1)$$

Observe that $\forall f \in W_n$, $f(I_n) \subset \mu_{2p^{n+1}}U_n$.

Definition 2 So we define the module of Weil numbers $W_n$

$$W_n = \{ f(a) \mid f \in W_n, \ a \in I_n \}.$$

Observe that $W_n$ is a submodule of $\mu_{2p^{n+1}}U_n$.

Let $k_n^+$ be the maximal totally real subfield of $k_n$ and let $G_n^+$ stay for $\text{Gal}(k_n^+/\mathbb{Q})$. Let $N_n$ be the norm element in $\mathbb{Z}[G_n]$. Let $N_n^+ \in \mathbb{Z}[G_n]$ be such that its image by the restriction map $\mathbb{Z}[G_n] \rightarrow \mathbb{Z}[G_n^+]$ is $\sum_{\sigma \in G_n^+} \sigma$.

Lemma 1 Let $f \in W_n$. Then $\beta(f) \in \mathbb{Z}[G_n]$ is unique and

$$\beta(f) \in N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-. $$

Proof: Let $f$ be in $W_n$ and suppose $\beta(f)$ and $\beta'(f)$ verify the required condition. Let $p$ be a split prime ideal in $I_n$. Let $m \geq 1$ be such that $p^m = \alpha\mathcal{O}_{k_n}$. Then

$$f(p^m) \equiv \alpha^{\beta(f)} \equiv \alpha^{\beta'(f)} \mod \mu_{2p^{n+1}}.$$

Thus $p^{m\beta(f)} = p^{m\beta'(f)}$, that implies $p^{m(\beta(f)-\beta'(f))} = \mathcal{O}_{k_n}$, so $\beta(f) = \beta'(f)$. Furthermore:

$$\beta(f) \in \text{Ann}_{\mathbb{Z}[G_n]}(\mathcal{O}_{k_n}^+/\mu_{2p^{n+1}}) = N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^-.$$


Proposition 1  The map $\beta : W_n \rightarrow N_n^+ \mathbb{Z} + \mathbb{Z}[G_n]^{-}$ defined by $f \mapsto \beta(f)$ gives rise to the exact sequence of $\mathbb{Z}[G_n]$-modules.

$$0 \rightarrow \mathrm{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^n+1}) \rightarrow W_n^{-} \rightarrow (\mathrm{Ann}_{\mathbb{Z}[G_n]} \mathrm{Cl}(k_n))^{-} \rightarrow B_n \rightarrow 0$$

where $B_n$ is a finite abelian elementary 2-group.

Proof:

By the definition of $W_n$ one has

$$\mathrm{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^n+1}) = \{ f \in W_n | \beta(f) = 0 \} = \ker \beta.$$ 

Note that $f \in W_n^-$ implies $\beta(f) \in \mathbb{Z}[G_n]^-$ . Take $f \in W_n^-$ and $p$ a prime ideal in $I_n$. Let $\mathfrak{p}$ be a split prime ideal in $I_n$. Let $m \geq 1$ be such that $\mathfrak{p}^m$ is principal. Then

$$\mathfrak{p}^{m\beta(f)} = f(\mathfrak{p})^m \mathcal{O}_{k_n},$$

that implies

$$\mathfrak{p}^{\beta(f)} = f(\mathfrak{p})\mathcal{O}_{k_n}.$$ 

Thus $\beta(f) \in (\mathrm{Ann}_{\mathbb{Z}[G_n]} \mathrm{Cl}(k_n))^-$.

Let $\beta$ be in $(\mathrm{Ann}_{\mathbb{Z}[G_n]} \mathrm{Cl}(k_n))^-$ and $\mathfrak{p}$ a prime ideal in $I_n$. Then there exists $\gamma \in \mathcal{O}_{n}^*$ such that $\mathfrak{p}^{\beta} = \gamma \mathcal{O}_{k_n}$. Let $\bar{\gamma}$ be the complex conjugate of $\gamma$. Then $\bar{\gamma} = \gamma^{-1} \varepsilon$ for some $\varepsilon \in \mathcal{O}_{k_n}^*$. Thus $\varepsilon = \gamma \bar{\gamma}$, i.e. $\varepsilon$ is a real unit. Consider $\gamma_1 = \varepsilon^{-1} \gamma^2$. One has: $\gamma_1^{-1} = \bar{\gamma}_1$ and

$$\mathfrak{p}^{2\beta} = \gamma^2 \mathcal{O}_{k_n} = \varepsilon^{-1} \gamma^2 \mathcal{O}_{k_n} = \gamma_1 \mathcal{O}_{k_n}.$$ 

Let $\gamma_2 \in \mathcal{O}_{n}^*$ such that $\mathfrak{p}^{2\beta} = \gamma_1 \mathcal{O}_{k_n} = \gamma_2 \mathcal{O}_{k_n}$ and $\bar{\gamma}_2 = \gamma_2^{-1}$. Then $\gamma_1 = \gamma_2 \eta$ for some $\eta \in \mathcal{O}_{k_n}^*$. Then $\gamma_1^{-1} = \gamma_2^{-1} \bar{\eta}$. That implies $\eta \bar{\eta} = 1$, i.e. $\eta$ is a root of unity.

Now one can choose, for any $\mathfrak{p} \in I_n$, $\gamma_{\mathfrak{p}} \in \mathcal{O}_{n}^*$ such that $\mathfrak{p}^{2\beta} = \gamma_{\mathfrak{p}} \mathcal{O}_{k_n}$, $\bar{\gamma}_{\mathfrak{p}} = \gamma_{\mathfrak{p}}^{-1}$ and $\gamma_{\mathfrak{p}^\sigma} = \gamma_{\mathfrak{p}}^\sigma \forall \sigma \in \mathcal{G}_n$. We set:

$$f(\mathfrak{p}) = \gamma_{\mathfrak{p}}$$

and one can verify that $f \in W_n^-$ and $\beta(f) = 2\beta$. Thus

$$2(\mathrm{Ann}_{\mathbb{Z}[G_n]} \mathrm{Cl}(k_n))^- \subset \beta(W_n^-) \subset (\mathrm{Ann}_{\mathbb{Z}[G_n]} \mathrm{Cl}(k_n))^-, $$

that completes the proof. $\square$

Let $l \neq p$ be a prime number. Let $l$ be the prime ideal of $k_n$ above $l$ and $q = |\mathcal{O}_{k_n}/l|$. Fix a primitive $l$-th root of unity $\zeta_l$. The Gauss sum $\tau_n(l)$ associated to $l$ is defined by

$$\tau_n(l) = - \sum_{a \in \mathbb{F}_q^*} \chi_l(a) \zeta_l^{\mathrm{Tr}_{\mathbb{F}_q/l}(a)}$$

where $\chi_l$ is a character on $\mathbb{F}_q^*$ of order $p^{n+1}$ defined by

$$\chi_l(x) \equiv x^{-\frac{q-1}{p^{n+1}}} \mod l.$$ 

One can show that $\forall \delta \in \mathcal{G}_n$ on has $\tau_n(l^\delta) = \tau_n(l)^\delta$ (see [W, §6.1]. So we have a well defined morphism of $\mathbb{Z}[G_n]$-modules

$$\tau_n : I_n \longrightarrow \Omega(\zeta_{p^{n+1}})^{\ast},$$

where $\Omega$ is the compositum of all the $\mathbb{Q}(\zeta_m)$, $m$ prime to $p$.  

3
Definition 3 The Jacobi module $J_n$ associated to $k_n$ is defined by

$$J_n = \mathbb{Z}[G_n] \tau_n \cap \text{Hom}_{\mathbb{Z}[G_n]}(I_n, k_n^*),$$

and the module of Jacobi sums $J_n$ is defined by

$$J_n = \{ f(\mathbf{a}) \mid f \in J_n, \mathbf{a} \in I_n \}.$$ 

Let us denote by $\sigma_a$ the image of $a \in \mathbb{Z}$, prime to $p$, via the standard isomorphism $(\mathbb{Z}/p^n+1\mathbb{Z})^* \simeq G_n$. Let

$$\theta_n = \frac{1}{p^{n+1}} \sum_{a=1, (a,p)=1}^{p^{n+1}} a \sigma_a^{-1}$$

be the Stickelberger element of $k_n$. Set

$$S_n' = \sum_{(t,p)=1} \mathbb{Z}[G_n](t - \sigma_t).$$

Definition 4 The Stickelberger ideal of $k_n$ is defined by

$$S_n = S_n' \theta_n,$$

(see [W, Lemma 6.9]).

Theorem 1 (Stickelberger’s Theorem [S, Theorem 3.1]) Let $p$ be a prime ideal in $I_n$, and $\beta \in S_n'$. Then $\tau(p)^\beta \in k_n^*$, $\beta \theta_n \in \mathbb{Z}[G_n]$ and

$$\tau(p)^\beta \mathcal{O}_{k_n} = p^{\beta \theta_n}.$$ 

Moreover, $\tau_n^\beta(p) \in U_n$.

Lemma 2

$$S_n' = \{ \beta \in \mathbb{Z}[G_n] \mid \tau_n^\beta \in J_n \}.$$ 

Proof: The inclusion $S_n' \subset \{ \beta \in \mathbb{Z}[G_n] \mid \tau_n^\beta \in J_n \}$ is obvious by Stickelberger’s theorem.

To prove the inverse inclusion it suffices to show that $\tau_n^\beta \in J_n$ implies $\beta \theta_n \in \mathbb{Z}[G_n]$.

Let $p \in I_n$ be a split prime ideal and $\tilde{p}$ the unique prime ideal of $\mathbb{Z}[\zeta_{p^n+1}, \zeta_l]$ above $p$, where $l = p \cap \mathbb{Q}$, $l \equiv 1 \pmod{p^{n+1}}$. Then

$$\tau_n(p) \mathbb{Z}[\zeta_{p^n+1}, \zeta_l] = \tilde{p}^{(l-1)\theta_n}.$$ 

Thus

$$\tau_n(p)^\beta \mathbb{Z}[\zeta_{p^n+1}, \zeta_l] = \tilde{p}^{(l-1)\beta \theta_n}.$$ 

On the other hand,

$$\tau_n(p)^\beta \mathcal{O}_{k_n} = p^z$$ 

for some $z \in \mathbb{Z}[G_n]$, that implies

$$\tilde{p}^{(l-1)z} = \tilde{p}^{(l-1)\theta_n \beta}$$ 

and since $l \equiv 1 \pmod{p^{n+1}}$, one has $(l - 1)z = (l - 1)\theta_n \beta$. Thus $z = \theta_n \beta$ that implies $\beta \theta_n \in \mathbb{Z}[G_n]$. □
Proposition 2

(1) \( \mathcal{J}_n \subset \mathcal{W}_n \)

(2) \( \mathcal{J}_n \cong \mathcal{S}_n \).

Proof:

(1) Using the lemma \([2]\) one can easily verify that

\[ \mathcal{J}_n = \{ \tau_n^\delta \mid \delta \in \mathcal{S}'_n \} \]

Then for any \( f \in \mathcal{J}_n \) there exists \( \delta \in \mathcal{S}'_n \) such that \( f = \tau_n^\delta \).

Let \( f \in \mathcal{J}_n \) and let \( a \in \mathcal{I}_n \) be a principal ideal, \( a = \alpha \mathcal{O}_k \). Then by the Stickelberger Theorem one has

\[ f(a) = \tau_n^\delta(a) = \varepsilon \alpha^{\delta \theta_n} \]

for some unit \( \varepsilon \). But

\[ \tau_n(a) \tau_n(a) = N_n(a) = N_n(\alpha), \]

so \( \varepsilon \in \mu_{2p^n+1} \). That means \( f(a) \equiv \alpha^{\delta \theta_n} \mod \mu_{2p^n+1} \), i.e. \( f \in \mathcal{W}_n \).

(2) As \( \mathcal{J}_n \subset \mathcal{W}_n \) by (1), the map \( \beta|_{\mathcal{J}_n} \) is well defined. On the other hand, for any \( f \in \mathcal{J}_n \)

\[ \beta(f) = \beta(\tau_n^\delta) = \delta \theta_n \]

for some \( \delta \in \mathcal{S}'_n \). Thus one has a well defined map

\[ \mathcal{J}_n \to \mathcal{S}_n \]

\[ \tau_n^\delta \mapsto \delta \theta_n \]

This map is obviously surjective (by the definition of \( \mathcal{S}_n \)). Its kernel is a submodule of \( \text{Hom}_{\mathbb{Z}[G_n]}(\mathcal{I}_n, \mu_{p^n+1}) \) by proposition \([3]\). Let \( \delta \in \mathbb{Z}[G_n] \) such that \( \delta \theta_n = 0 \). Then we have \( \sigma_{-1}^\delta = \delta \) (\( \sigma_{-1} \) being the complex conjugation in \( G_n \)) and \( \delta N_n = 0 \).

Now let \( \delta \in \mathcal{S}'_n \) such that \( \delta \theta_n = 0 \). Then

\[ \tau_n^{\sigma_{-1}^\delta} = \tau_n^\delta, \]

and

\[ \tau_n^{\delta \sigma_{-1}^\delta} = (\tau_n^{\sigma_{-1}^\delta})^\delta = \tau_n^{-\delta} \]

as \( \theta_n \theta_n^{\sigma_{-1}^\delta} = N_n \). Thus \( \tau_n^{2\delta} = 1 \). Therefore \( \tau_n^\delta = 1 \) as \( \tau_n^\delta \equiv 1 \mod \pi_n \).

Lemma 3 Let \( N_{n,n-1} \) be the norm map in the extension \( k_n/k_{n-1} \) and \( \mathfrak{L} \in \mathcal{I}_n \) a prime ideal. Then

\[ N_{n,n-1}(\tau_n(\mathfrak{L})) = \tau_{n-1}(N_{n,n-1}(\mathfrak{L})) \zeta_n^a b, \]

for some \( a, b \in \mathbb{Z} \) and some \( \zeta \in \mu_{p^n+1} \).

For a proof see \([4]\) Lemma 2].

Remark 1 The composition \( N_{n,n-1} \circ \tau_n \) is well defined because

\[ \text{Gal}(k_n/k_{n-1}) \cong \text{Gal}(\Omega(\zeta_{p^n+1})/\Omega(\zeta_{p^n})), \forall \geq 1. \]
Lemma 4 ([W, Proposition 7.6 (c)]) The restriction map \( Res : \mathbb{Z}[G_n] \rightarrow \mathbb{Z}[G_{n-1}] \) induces the surjective map
\[
Res : S_n^- \longrightarrow S_{n-1}^-.
\]

Proposition 3
\[
\forall n \geq 1, \quad N_{n,n-1} \circ J_n^\beta \equiv I_n \circ N_{n,n-1} \mod \text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^n+1})
\]

Proof: Let \( f \in J_n^\beta \). By the proposition 2 there exists some \( \beta \in S_n^- \) such that \( f = \tau_n^\beta \). Let \( \mathfrak{L} \in I_n \) be a prime ideal and \( l = N_{n,n-1} \). Then by the lemmas 3 and 4
\[
N_{n,n-1}(\tau_n^\beta(\mathfrak{L})) \equiv \tau_{n-1}^{\text{Res}(\beta)}(l) \mod \mu_{p^n+1}(\mathfrak{L}).
\]
The Proposition follows. \( \square \)

3 Annihilators

We recall that \( \psi \) is an odd \( \mathbb{Q}_p \)-valued character of \( \Delta \), irreducible over \( \mathbb{Q}_p \), different from Teichmüller character \( \omega \).

Lemma 5 Let \( M \in \Lambda \) be the minimal polynomial of \( e_\psi X \). Then
\[
\lim_{\leftarrow} e_\psi(\text{Ann}_{\mathbb{Z}_p[G_n]}A_n) = M(T)\Lambda,
\]
the projective limit being taken for the restriction maps.

Proof: First we remark that
\[
e_\psi \text{Ann}_{\mathbb{Z}_p[G_n]}A_n = \text{Ann}_{\mathbb{Z}_p[\Gamma_n]}e_\psi A_n.
\]
We set \( A_n,\psi = e_\psi A_n \) for simplicity.

Let \( M = (M_n)_{n \geq 0} \in \Lambda \simeq \lim_{\leftarrow} \mathbb{Z}_p[\Gamma_n] \), the limit being taken with respect for restriction maps. As \( X^- \) has no nontrivial finite submodule (see [W, Proposition 13.28]), \( M_n \) annihilates \( A_n,\psi \), that means \( M_n Z_p[\Gamma_n] A_n,\psi \subset \text{Ann}_{\mathbb{Z}_p[\Gamma_n]}A_n,\psi \). Thus
\[
M(T)\Lambda \subset \lim_{\leftarrow} \text{Ann}_{\mathbb{Z}_p[\Gamma_n]}A_n,\psi.
\]

Let \( \delta = (\delta_n)_{n \geq 0} \in \lim_{\leftarrow} \text{Ann}_{\mathbb{Z}_p[\Gamma]}A_n,\psi \). Then for any \( n \geq 0 \), \( \delta_n A_n,\psi = \{0\} \). On the other hand,
\[
e_\psi X = \lim_{\leftarrow} A_n,\psi.
\]
Then \( \delta e_\psi X = \{0\} \), that implies
\[
\delta \in \text{Ann}_\Lambda e_\psi X = M(T)\Lambda,
\]
that completes the proof. \( \square \)

Let \( \overline{W}_n = W_n \otimes \mathbb{Z}_p \) the \( p \)-adic adherence of \( W_n \). The map \( \beta \) of Proposition 1 induces the map
\[
\overline{W}_n \longrightarrow (\text{Ann}_{\mathbb{Z}[G_n]}Cl(k_n))^- \otimes_{\mathbb{Z}_p} \mathbb{Z}_p = (\text{Ann}_{\mathbb{Z}_p[G_n]}A_n)^- \quad \quad \quad w \otimes a \longmapsto a \beta(w)
\]
that we shall always note \( \beta \). Thus we have the short exact sequence

\[
0 \longrightarrow \text{Hom}_{\mathbb{Z}[G_n]}(I_n, \mu_{2p^{n+1}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow \overline{W}_n \longrightarrow (\text{Ann}_{\mathbb{Z}_p[G_n]} A_n)^- \longrightarrow 0.
\]

Applying \( e_\psi \) to all the terms of this sequence we get an isomorphism of \( \mathbb{Z}_p[G_n] \)-modules

\[
e_\psi \overline{W}_n \cong e_\psi \text{Ann}_{\mathbb{Z}_p[G_n]} A_n = \text{Ann}_{\mathbb{Z}_p[G_n]} A_n, \psi.
\] (2)

Let \( z \in \overline{W}_{n,\psi} = e_\psi \overline{W}_n \). Then \( z \) induces naturally by class field theory (see [Iw, p.455]) a morphism of \( \mathbb{Z}_p[\Gamma_n] \)-modules:

\[
z : \mathfrak{x}_{n,\psi} \longrightarrow \mathcal{U}_{n,\psi}.
\] (3)

**Lemma 6** Let \( z \in \overline{W}_{n,\psi} \) such that \( \beta(z) \in \mathbb{Q}_p[\Gamma_n]^* \). Then the kernel of \( z \) is \( \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{x}_{n,\psi} \).

**Proof:** We have \( z(\mathcal{U}_{n,\psi}) = \mathcal{U}_{n,\psi}^{\beta(z)} \). As \( \beta \in \mathbb{Q}_p[\Gamma_n]^* \), the quotient \( \mathcal{U}_{n,\psi}/\mathcal{U}_{n,\psi}^{\beta(z)} \) if finite. Thus

\[
\text{rank}_{\mathbb{Z}_p} z(\mathfrak{x}_{n,\psi}) = \text{rank}_{\mathbb{Z}_p} \mathfrak{x}_{n,\psi} = \text{rank}_{\mathbb{Z}_p} \mathcal{U}_{n,\psi}.
\]

Thus \( \text{ker}(z : \mathfrak{x}_{n,\psi} \longrightarrow \mathcal{U}_{n,\psi}) = \text{Tor}_{\mathbb{Z}_p[\Gamma_n]} \mathfrak{x}_{n,\psi} \). \( \square \)

For any \( n \in \mathbb{N} \) let \( \mathcal{M}_n \in \mathbb{Z}_p[G_n] \) such that

\[
\mathcal{M}_n \equiv \mathcal{M} \mod \omega_n.
\]

Then \( (\mathcal{M}_n)_{n \geq 0} = \mathcal{M} \) in \( \Lambda \). Let \( w_n \in \overline{W}_{n,\psi} = e_\psi \overline{W}_n \) be the element of \( \overline{W}_{n,\psi} \) corresponding to \( \mathcal{M}_n \) via the homomorphism (2).

**Remark 2** The Lemma [2] is applicable to \( w_n \), and to the map that consists in multiplication by \( e_\psi \theta_n \).

**Lemma 7** Let \( \overline{J}_n = J_n \otimes_{\mathbb{Z}} \mathbb{Z}_p \) be the \( p \)-adic adherence of \( J_n \) in \( \mathcal{U}_n \) and \( \overline{W}_n = W_n \otimes_{\mathbb{Z}} \mathbb{Z}_p \) the \( p \)-adic adherence of \( W_n \) in \( \mathcal{U}_n \). Then

\[
e_\psi \overline{J}_n \subset w_n(\mathfrak{x}_{n,\psi}) \subset e_\psi \overline{W}_n.
\]

**Proof:** One can verify that \( \beta(e_\psi \overline{J}_n) = e_\psi \theta_n \mathbb{Z}_p[\Gamma_n] \) (see [W, Chap. 7]). By the Main Conjecture (see [W], §13.6)

\[
e_\psi \theta_n \mathbb{Z}_p[\Gamma_n] \subset \mathcal{M}_n \mathbb{Z}_p[\Gamma_n].
\]

Set \( \overline{W}_{n,\psi} \) the sub-\( \mathbb{Z}_p[\Gamma_n] \)-module of \( \overline{W}_{n,\psi} \) generated by \( w_n \). Then \( \beta(\overline{W}_{n,\psi}) = \mathcal{M}_n \mathbb{Z}_p[\Gamma_n] \).

Thus

\[
\beta(e_\psi \overline{J}_n) \subset \beta(\overline{W}_{n,\psi}).
\]

As \( \beta \) is an isomorphism, this is equivalent to

\[
e_\psi \overline{J}_n \subset \overline{W}_{n,\psi},
\]

That implies

\[
e_\psi \overline{J}_n \subset w_n(\mathfrak{x}_{n,\psi}). \quad \square
\]
Take \((z_n)_{n \geq 1}, z_n \in \tilde{W}_{n, \psi}\) such that \(\forall n \geq 1, \text{Res}_{n+1, n}(z_{n+1}) = \beta(z_n)\). By the class field theory the following diagram is commutative:

\[
\begin{array}{ccc}
X_{n+1, \psi} & \xrightarrow{z_{n+1}} & U_{n+1, \psi} \\
\downarrow \text{Res}_{n+1, n} & & \downarrow N_{n+1, n} \\
X_{n, \psi} & \xrightarrow{z_n} & U_{n, \psi}
\end{array}
\]

so the map

\[z_\infty : X_{\infty, \psi} \rightarrow U_{\infty, \psi}\]

is naturally well defined and

\[z_\infty(X_{\infty, \psi}) = \lim_{\rightarrow} z_n(X_{n, \psi}) \subseteq U_{\infty, \psi}\]

**Lemma 8** The kernel of \(z_\infty\) is isomorphic to \(\alpha(e_{\omega \psi^{-1}}X)\), where \(\alpha(e_{\omega \psi^{-1}}X)\) is the Iwasawa adjoint module of \(e_{\omega \psi^{-1}}X\).

**Proof:** By the definition of \(z_\infty\), \(\ker z_\infty = \lim \ker z_n = \lim \text{Tor} z_p[\Gamma_n]X_{n, \psi}\). But \(\lim \text{Tor} z_p[\Gamma_n]X_{n, \psi} \simeq \alpha(e_{\omega \psi^{-1}}X)\) (see [N1, Proposition 3.1]).

Take \(z_n = w_n \ \forall n \geq 1\). Then \(e_{\psi} J_\infty \subseteq w_\infty(X_{\infty, \psi})\) by the Lemma 7.

**Lemma 9**

\[w_\infty(e_{\psi} U_\infty) = e_{\psi} \mathcal{M} U_\infty\]

**Proof:** Obvious as \(e_{\psi} U_\infty\) is free of rank 1.

**Lemma 10** The module \(W_{\infty, \psi} = \lim \overline{W}_{n, \psi}\) is pseudo-isomorphic to \(w_\infty(X_{\infty, \psi})\).

**Proof:** Let \(E\) be the elementary \(\Lambda\)-module such that

\[0 \rightarrow e_{\psi} X \rightarrow E \rightarrow B \rightarrow 0,
\]

where \(B\) is a finite \(\Lambda\)-module. Then \(\forall n \gg 0, \omega_n B = \{0\}\), and by the snake lemma we obtain the exact sequence

\[0 \rightarrow B \rightarrow e_{\psi} A_n \rightarrow E/\omega_n E \rightarrow B \rightarrow 0.
\]

Let \(Y_n = \text{Ann}_{Z_p[\Gamma_n]} E/\omega_n E = \mathcal{M} n Z_p[\Gamma_n]\). It is a submodule of \(Z_n = \text{Ann}_{Z_p[\Gamma_n]} e_{\psi} A_n \simeq e_{\psi} \overline{W}_n\), so there exists a submodule \(\overline{W}_n\) of \(e_{\psi} \overline{W}_n\) such that \(\overline{W}_n \simeq Y_n\) as \(Z_p[\Gamma_n]\)-modules. \(Y_n\) being monogenous, the same is for \(\overline{W}_{n, \psi}\), so it is generated by \(w_n\). Thus \(\overline{W}_n = \overline{W}_{n, \psi}\).

There exists \(\delta \in \Lambda\), prime to \(\mathcal{M}\), such that \(\delta B = \{0\}\). Then \(\delta Z_n \subseteq Y_n\), i.e. \(\delta e_{\psi} \overline{W}_n \subseteq \overline{W}_{n, \psi}\). In particular that means

\[\delta e_{\psi} \overline{W}_n \subseteq \overline{W}_{n, \psi} \subseteq e_{\psi} \overline{W}_n,\]

(5)
where \( \widetilde{W}_n = w_n(\mathfrak{x}_{n,\psi}) \). So, taking the projective limit in (6) we obtain

\[
\delta e_\psi \overline{W}_\infty \subset w_\infty(e_\psi \mathfrak{x}_\infty) \subset e_\psi \overline{W}_\infty.
\]

Thus the quotient module \( e_\psi \overline{W}_\infty/w_\infty(e_\psi \mathfrak{x}_\infty) \) is annihilated by two relatively prime polynomials \( \delta \) and \( M \), i.e. is finite (see [W, §13.2]). □

The classical class field theory sequence

\[
0 \longrightarrow \overline{O}_{k_n} \cap \mathcal{U}_n \longrightarrow \mathcal{U}_n \longrightarrow \mathfrak{x}_n \longrightarrow A_n \longrightarrow 0
\]
gives by taking the \( \psi \)-parts the short exact sequence

\[
0 \longrightarrow e_\psi \mathcal{U}_n \longrightarrow e_\psi \mathfrak{x}_n \longrightarrow e_\psi A_n \longrightarrow 0,
\]

as \( \psi \neq \omega \).

Passing to the projective limit in this sequence we obtain the short exact sequence

\[
0 \longrightarrow e_\psi \mathcal{U}_\infty \longrightarrow e_\psi \mathfrak{x}_\infty \longrightarrow e_\psi X \longrightarrow 0.
\]

**Theorem 2**

\[
\text{char}_\Lambda (e_\psi \overline{W}_\infty/e_\psi \overline{J}_\infty) = (\text{char}_\Lambda e_\psi X_\infty)/\mathcal{M}(T).
\]

**Proof:** By the Lemma 9, the map \( w_\infty \) gives rise to the map

\[
\overline{w}_\infty : \frac{e_\psi \mathfrak{x}_\infty}{e_\psi \mathcal{U}_\infty} \longrightarrow \frac{e_\psi \mathcal{U}_\infty}{\mathcal{M} e_\psi \mathcal{U}_\infty}.
\]

So in virtue of the sequence (4), we have the map

\[
\overline{w}_\infty : e_\psi X \longrightarrow \frac{e_\psi \mathcal{U}_\infty}{\mathcal{M} e_\psi \mathcal{U}_\infty}.
\]

The kernel of the map \( w_\infty \) being the \( \Lambda \)-torsion module isomorphic to \( \alpha(e_{\omega \psi^{-1}} X) \) and \( e_\psi \mathcal{U}_\infty \) being a free \( \Lambda \)-module, \( \ker(w_\infty) \cap e_\psi \mathcal{U}_\infty = \{0\} \). So \( \ker(\overline{w}_\infty) \simeq \alpha(e_{\omega \psi^{-1}} X) \). And we have the following exact sequence

\[
0 \longrightarrow \alpha(e_{\omega \psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow e_\psi \mathcal{U}_\infty/\mathcal{M} e_\psi \mathcal{U}_\infty.
\]

Let \( F = \text{char}_\Lambda e_\psi X \) the characteristic polynomial of \( e_\psi X \) and \( \theta_{\infty, \psi} = (e_\psi \theta_n)_{n \geq 0} \). Then by the Main Conjecture and by the Lemma 9 we obtain the second exact sequence

\[
0 \longrightarrow \alpha(e_{\omega \psi^{-1}} X) \longrightarrow e_\psi X \xrightarrow{\times \theta_{\infty, \psi}} e_\psi \mathcal{U}_\infty/F e_\psi \mathcal{U}_\infty
\]

These two sequences give two short exact sequences

\[
0 \longrightarrow \alpha(e_{\omega \psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow w_\infty(\mathfrak{x}_{\infty, \psi})/\mathcal{M} e_\psi \mathcal{U}_\infty \longrightarrow 0
\]

and

\[
0 \longrightarrow \alpha(e_{\omega \psi^{-1}} X) \longrightarrow e_\psi X \longrightarrow e_\psi \overline{J}_\infty/F e_\psi \mathcal{U}_\infty \longrightarrow 0.
\]
as \( \theta_\infty x_{\infty, \psi} = e\psi \overline{J}_{\infty} \) (see \( \mathbb{B} \) Lemme 8)). Thus

\[
\text{char}_{\Lambda} w_\infty(x_{\infty, \psi})/Me_\psi U_\infty = \text{char}_{\Lambda} e\psi \overline{J}_{\infty}/Fe_\psi U_\infty.
\] (8)

Set \( e\psi \overline{W}_{\infty} = w_\infty(x_{\infty, \psi}) \). The tautological short exact sequence

\[
0 \rightarrow e\psi \overline{J}_{\infty} \rightarrow e\psi \overline{W}_{\infty} \rightarrow e\psi \overline{W}_{\infty}/e\psi \overline{J}_{\infty} \rightarrow 0
\]
gives rise to the short exact sequence

\[
0 \rightarrow \frac{e\psi \overline{J}_{\infty}}{FU_{\infty, \psi}} \rightarrow \frac{e\psi \overline{W}_{\infty}}{FU_{\infty, \psi}} \rightarrow \frac{e\psi \overline{W}_{\infty}}{e\psi \overline{J}_{\infty}} \rightarrow 0.
\]

Thus

\[
\text{char}_{\Lambda} \frac{e\psi \overline{J}_{\infty}}{FU_{\infty, \psi}} = \text{char}_{\Lambda} \frac{e\psi \overline{W}_{\infty}}{FU_{\infty, \psi}} \left( \frac{\text{char}_{\Lambda} e\psi \overline{W}_{\infty}}{\text{char}_{\Lambda} e\psi \overline{J}_{\infty}} \right)^{-1}.
\] (9)

In the same way, the sequence

\[
0 \rightarrow Me_\psi U_\infty \rightarrow e\psi \overline{W}_{\infty} \rightarrow e\psi \overline{W}_{\infty}/Me_\psi U_\infty \rightarrow 0
\]
gives rise to the sequence

\[
0 \rightarrow \frac{Me_\psi U_{\infty}}{FU_{\infty, \psi}} \rightarrow \frac{e\psi \overline{W}_{\infty}}{FU_{\infty, \psi}} \rightarrow \frac{e\psi \overline{W}_{\infty}}{Me_\psi U_{\infty}} \rightarrow 0
\]

Thus

\[
\text{char}_{\Lambda} \frac{e\psi \overline{W}_{\infty}}{Me_\psi U_{\infty, \psi}} = \text{char}_{\Lambda} \frac{e\psi \overline{W}_{\infty}}{FU_{\infty, \psi}} \left( \frac{\text{char}_{\Lambda} Me_\psi U_{\infty, \psi}}{\text{char}_{\Lambda} FU_{\infty, \psi}} \right)^{-1}.
\] (10)

Comparing the equalities (8), (9) and (10) we obtain the equality

\[
\text{char}_{\Lambda} e\psi \overline{W}_{\infty}/Me_\psi U_\infty = \text{char}_{\Lambda} e\psi \overline{W}_{\infty}/e\psi \overline{J}_{\infty}.
\]

In virtue of the Lemma (K),

\[
\text{char}_{\Lambda} e\psi \overline{W}_{\infty}/Me_\psi U_\infty = \text{char}_{\Lambda} e\psi \overline{W}_{\infty}/e\psi \overline{J}_{\infty}.
\]

As \( U_{\infty, \psi} \) is free of rank 1,

\[
\text{char}_{\Lambda} Me_\psi U_{\infty, \psi} = \frac{F(T)}{M(T)}.
\]

So

\[
\text{char}_{\Lambda} e\psi \overline{W}_{\infty}/e\psi \overline{J}_{\infty} = \frac{F(T)}{M(T)} \quad \square
\]

Corollary 1 (cf. \( \mathbb{B} \) Théorème 1)]

\[
\text{char}_{\Lambda} \frac{U_{\infty, \psi}}{e\psi \overline{J}_{\infty}} = \text{char}_{\Lambda} e\psi \overline{J}_{\infty}(e\psi \omega^{1-1} X).
\]
Corollary 2. The module $e_\psi X$ is pseudo-monogenous if and only if the quotient module $e_\psi W_\infty / e_\psi J_\infty$ is finite.

By the corollary 1, we see that Greenberg Conjecture implies that

$$\frac{e_\psi U_\infty}{e_\psi W_\infty}$$

is finite.

So it is natural to ask the following question.

**Question:** Let $p$ be an odd prime number. Let $\psi$ be an odd character of $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, $\psi \neq \omega$, then, is it true that

$$\frac{e_\psi U_\infty}{e_\psi W_\infty}$$

is finite?

**Remark 3** Note that the positive answer to this question is equivalent to

$$\text{char } e_\psi X = \text{char } e_{\omega\psi^{-1}} X \times M,$$

so it implies weak Greenberg Conjecture (see [BN], [N2]).

**References**


