sup + inf for Riemannian surfaces and sup × inf for bounded domains of \( \mathbb{R}^n \), \( n \geq 3 \)

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Abstract
On a Riemannian surface, we give a condition to obtain a minoration of sup + inf. On an open bounded set of \( \mathbb{R}^n \) (\( n \geq 3 \)) with smooth boundary, we have a minoration of sup × inf for prescribed scalar curvature equation with Dirichlet condition.

Keywords: Riemannian Surface, sup + inf, sup × inf, Dirichlet condition.

In this paper, we study some inequalities of type sup + inf (in dimension 2) and sup × inf (in dimension \( n \geq 3 \)). We denote \( \Delta = -\nabla_i (\nabla^i) \) the geometric laplacian.

The paper is linking to the Note presented in Comptes Rendus de l’Académie des Sciences de Paris (see [B1])

In dimension 2, we work on Riemannian surface \((M, g)\) and we consider the following equation:

\[
\Delta u + f = V e^u \quad (E_1)
\]

where \( f \) and \( V \) are two functions.

We are going to prove a minoration of sup \( u \) + inf \( u \) under some conditions on \( f \) and \( V \).

Where \( f = R \), with \( R \) the scalar curvature of \( M \), we have the scalar curvature equation studied by T. Aubin, H. Brezis, YY. Li, L. Nirenberg, R. Schoen.

In the case \( f = R = 2\pi \) and \( M = S^2 \), we have a lower bound for sup + inf assuming \( V \) non negative, bounded above by a positive constant \( b \) and without condition on \( \nabla V \) (see Bahoura [B]).

The problem was studied when we suppose \( V = V_i \) uniformly lipschitzian and between two positive constants. (See Bahoura [B] and Li [L]). In fact, there exists \( c = c(a, b, A, M) \) such that for all sequences \( u_i \) and \( V_i \) satisfying:

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\( \Delta u_i + R = V_i e^{u_i}, \) \( 0 < a \leq V_i(x) \leq b \) and \( \| \nabla V \|_{\infty} \leq A, \)

we have,

\[
\sup_M u_i + \inf_M u_i \geq c \forall i.
\]

We have some results about \( L^\infty \) boundness and asymptotic behavior for the solutions of equations of this type on open set of \( \mathbb{R}^2, \) see [BM], [S], [SN 1] and [SN 2].

Here, we try to study the same problem with minimal conditions on \( f \) and \( V, \) we suppose \( 0 \leq V \leq b \) and without assumption on \( \nabla V. \)

**Theorem 1.** Assume \((M, g)\) a Riemannian surface and \( f, V \) two functions satisfying:

\[
f(x) \geq 0, \text{ and } 0 \leq V(x) \leq b < +\infty, \forall x \in M.
\]

suppose \( u \) solution of:

\[
\Delta u + f = V e^{u}.
\]

then:

if \( 0 < \int_M f \leq 8\pi, \) there exists a constant \( c = c(b, f, M) \) such that:

\[
\sup_M u + \inf_M u \geq c,
\]

if \( 8\pi < \int_M f < 16\pi, \) there exists \( C = C(f, M) \in ]0, 1[ \) and \( c = c(b, f, M) \)

such that:

\[
\sup_M u + C \inf_M u \geq c.
\]

**Remark:** In fact, we can suppose \( f \equiv k \) a constant. (See [B1]).

Now, we work on a smooth bounded domain \( \Omega \subset \mathbb{R}^n (n \geq 3). \)

Let us consider the following equations:

\[
\Delta u_\epsilon = u_\epsilon^{N-1-\epsilon}, \ u_\epsilon > 0 \text{ in } \Omega \text{ and } u_\epsilon = 0 \text{ on } \partial\Omega \quad (E_2).
\]

with \( \epsilon \geq 0, \) \( N = \frac{2n}{n-2}. \)

The existence result for those equations depends on the geometry of the domain. For example, if we suppose, \( \Omega \) starshaped and \( \epsilon = 0, \) the Pohozaev identity assure a nonexistence result. If \( \epsilon = 0, \) under assumption on \( \Omega, \) we can have an existence result. When \( \epsilon > 0 \) there exists a solutions for the previous equation.
For $\epsilon > 0$, [AP], [BP] and [H], studied some properties of the previous equation.

On unit ball of $\mathbb{R}^n$, Atkinson-Peletier (see [AP]) have proved:

$$\lim_{\epsilon \to 0} \sup_{B_1(0)} \inf_{B_k(0)} u_{\epsilon} = \left( \frac{1}{|k|} - 1 \right),$$

with $|x| = k < 1$.

In [H], Z-C Han, has proved the same estimation on a smooth open set $\Omega \subset \mathbb{R}^n$ with the following condition:

$$\lim_{\epsilon \to 0} \int_\Omega \frac{|\nabla u_{\epsilon}|^2}{\|u_{\epsilon}\|_{L^{n-1}}} = S_n \quad (1),$$

with $S_n = \pi n(n-2) \left[ \frac{\Gamma(n/2)}{\Gamma(n)} \right]^2$ the best constant in the Sobolev imbedding.

In fact, the result of Z-C Han (see [H]), is (with (1)),

$$\lim_{\epsilon \to 0} \|u_{\epsilon}\|_{L^\infty} u_{\epsilon}(x) = \sigma_n(n-2)G(x, x_0), \quad \text{with, } x \in \Omega - \{x_0\}.$$

where $x_0 \in \Omega$ and $G$ is the Green function with Dirichlet condition.

In our work, we search to know if it is possible to have a lower bound of $\sup \times \inf$, without the assumption (1).

**Theorem 2.** For all compact $K$ of $\Omega$, there exists a positive constant $c = c(K, \Omega, n) > 0$, such that for all solution $u_{\epsilon}$ of $(E_2)$ with $\epsilon \in [0, \frac{2}{n-2}]$, we have:

$$\sup_{\Omega} u_{\epsilon} \times \inf_{K} u_{\epsilon} \geq c.$$

Next, we are interesting by the following equation:

$$\Delta u = u^{N-1} + \epsilon u, \quad u > 0, \quad \text{in } \Omega, \quad \text{and } u = 0 \text{ on } \partial \Omega.$$

We know that in dimension 3, there is no radial solution for the previous equation if $\epsilon \leq \lambda_*$ with $\lambda_* > 0$, see [BN]. Next, we consider $n \geq 4$.

We set $G$ the Green function of the laplacian with Dirichlet condition. For $0 < \alpha < 1$, we denote:

$$\beta = \frac{\alpha}{\sup_{\Omega} \int_\Omega \frac{G(x, y)}{dy}}.$$

Assume $n \geq 4$, we have:
Theorem 3. For all compact $K$ of $\Omega$ and all $0 < \alpha < 1$ there is a positive constant $c = c(\alpha, K, \Omega, n)$, such that for all sequences $(\epsilon_i)_{i \in \mathbb{N}}$ with $0 < \epsilon_i \leq \beta$ and $(u_{\epsilon_i})_{i \in \mathbb{N}}$ satisfying:

$$\Delta u_{\epsilon_i} = u_{\epsilon_i}^{N-1} + \epsilon_i u_{\epsilon_i}, \quad u_{\epsilon_i} > 0 \text{ and } u_{\epsilon_i} = 0 \text{ on } \partial \Omega, \quad \forall \ i,$$

we have:

$$\forall \ i, \sup_{\Omega} u_{\epsilon_i} \times \inf_K u_{\epsilon_i} \geq c.$$
Proof of the Theorem 1:

First part \((0 < \int_M f \leq 8\pi)\):

We have:

\[
\Delta u + f = Ve^u,
\]

We multiply by \(u\) the previous equation and we integrate by part, we obtain:

\[
\int_M |\nabla u|^2 + \int_M f u = \int_M V e^u u,
\]

But \(V \geq 0\) and \(f \geq 0\), then:

\[
\int_M |\nabla u|^2 + \inf_M u \int_M f \leq \sup_M \int_M V e^u.
\]

On Riemannian surface, we have the following Sobolev inequality, (see [DJLW], \[F\]):

\[
\exists C = C(M, g) > 0, \forall v \in H^1_2(M), \log \left( \int_M e^v \right) \leq \frac{1}{16\pi} \int_M |\nabla v|^2 + \frac{1}{\text{Vol}(M)} \int_M v + \log C.
\]

Let us consider \(G\) the Green function of the laplacian such that:

\[
G(x, y) \geq 0 \quad \text{and} \quad \int_M G(x, y) dV_g(y) \equiv k = \text{constant}.
\]

Then,

\[
u(x) = \frac{1}{\text{Vol}(M)} \int_M u + \int_M G(x, y) [V(y)e^{u(y)} - f(y)] dV_g(y),
\]

and,

\[
\inf_M u = u(x_0) \geq \frac{1}{\text{Vol}(M)} \int_M u - C_1,
\]

with,

\[
\int_M [G(x_0, y)f(y)] \leq \sup_M \int_M G(x_0, y) dV_g(y) = k \sup_M f = C_1.
\]

But, \(\int_M V e^u = \int_M f > 0\), we obtain,

\[
\left( \int_M f \right) (\sup_M u + \inf_M u) \geq -2C_1 \int_M f + \frac{2}{\text{Vol}(M)} \left( \int_M u \right) \left( \int_M f \right) + \int_M |\nabla u|^2,
\]

thus,

\[
\sup_M u + \inf_M u \geq 2 \left[ \frac{1}{\text{Vol}(M)} \int_M u + \frac{1}{2 \int_M f} \int_M |\nabla u|^2 \right] - 2C_1.
\]
If we suppose, \(0 < \int_M f \leq 8\pi\), we obtain \(\frac{1}{2} \int_M u \geq \frac{1}{16\pi}\) and then:

\[
\sup_M u + \inf_M u \geq 2 \left[ \frac{1}{Vol(M)} \int_M u + \frac{1}{16\pi} \int_M |\nabla u|^2 \right] - 2C_1,
\]

We use the previous Sobolev inequality, we have:

\[
\sup_M u + \inf_M u \geq -2C_1 - 2 \log C + 2 \log \left( \int_M e^u \right),
\]

but,

\[
\int_M V e^u \leq b \int_M e^u,
\]

then,

\[
\int_M e^u \geq \frac{1}{b} \int_M f,
\]

and finally,

\[
\sup_M u + \inf_M u \geq -2C_1 - 2 \log C + 2 \log \left( \frac{1}{b} \int_M f \right).
\]

**Second part** (\(8\pi < \int_M f < 16\pi\)):

Like in the first part, we have:

a) \(\int_M |\nabla u|^2 + \inf_M u \int_M f \leq \sup u \int_M f\),

b) \(\log \left( \int_M e^u \right) \leq \frac{1}{16\pi} \int_M |\nabla u|^2 + \frac{1}{Vol(M)} \int_M u + \log C\),

c) \(\inf_M u \geq \frac{1}{Vol(M)} \int_M u - C_1\).

We set \(\lambda > 0\). We use a), b), c) and we obtain:

\[
\left( \int_M f \right) (\sup_M u + \lambda \inf_M u) \geq - (\lambda + 1)C_1 \int_M f + \frac{(1 + \lambda)}{Vol(M)} \left( \int_M u \right) \left( \int_M f \right) + \int_M |\nabla u|^2,
\]

thus,

\[
\sup_M u + \lambda \inf_M u \geq - (\lambda + 1)C_1 + (1 + \lambda) \left[ \frac{1}{Vol(M)} \int_M u + \frac{1}{(1 + \lambda) \int_M f} \int_M |\nabla u|^2 \right].
\]

We choose \(\lambda > 0\), such that, \(\frac{1}{(1 + \lambda) \int_M f} \geq \frac{1}{16\pi}\).
thus, \((1 + \lambda) \int_M f \leq 16\pi, 0 < \lambda \leq \frac{16\pi - \int_M f}{\int_M f} < 1\).

Finaly, the choice of \(\lambda\), give:

\[
\sup_M u + \lambda \inf_M u \geq -(\lambda + 1)C_1 = (1 + \lambda) \log C + (1 + \lambda) \log \left(\frac{1}{b} \int_M f\right).
\]

If we take \(\lambda = \frac{16\pi - \int_M f}{\int_M f} \in [0, 1]\), we obtain:

\[
\sup_M u + \left(\frac{16\pi - \int_M f}{\int_M f}\right) \inf_M u \geq -C_1 \frac{16\pi}{\int_M f} + \frac{16\pi}{\int_M f} \log C + \frac{16\pi}{\int_M f} \log \left(\frac{1}{b} \int_M f\right).
\]

**Proof of theorems 2 and 3:**

Here, we give two methods to prove the theorems 2 and 3, but we do the proof only for the theorem 2. In the first proof we use the Moser iterate scheme, the second proof is direct.

**Method 1:** by the Moser iterate scheme.

We argue by contradiction and we suppose:

\(\exists K \subset \subset \Omega, \forall c > 0, \exists \epsilon_c \epsilon [0, \frac{2}{n-2}]\) such that:

\[
\Delta u_{\epsilon_c} = u_{\epsilon_c}^{N-1-\epsilon_c}, \ u_{\epsilon_c} > 0 \text{ in } \Omega \text{ and } u_{\epsilon_c} = 0 \text{ on } \partial \Omega,
\]

with,

\[
\sup_{\Omega} u_{\epsilon_c} \times \inf_{K} u_{\epsilon_c} \leq c
\]

We take \(c = \frac{1}{i}\), there exists a sequence \((\epsilon_i)_{i \geq 0}\), such that \(\forall i \in \mathbb{N}, \epsilon_i \in [0, \frac{n}{n-2}]\) and

\[
\Delta u_{\epsilon_i} = u_{\epsilon_i}^{N-1-\epsilon_i}, \ u_{\epsilon_i} > 0 \text{ in } \Omega \text{ and } u_{\epsilon_i} = 0 \text{ on } \partial \Omega \ (*)
\]

with,

\[
\sup_{\Omega} u_{\epsilon_i} \times \inf_{K} u_{\epsilon_i} \leq \frac{1}{i} \to 0 \ (**).
\]

Clearly the function \(u_{\epsilon_i}\) which satisfy (*), there exists \(x_{\epsilon_i} \in \Omega\) such that:

\[
\sup_{\Omega} u_{\epsilon_i} = \max_{\Omega} u_{\epsilon_i} = u_{\epsilon_i}(x_{\epsilon_i}).
\]
Lemma:
There exists $\delta = \delta(\Omega, n) > 0$ such that for all $\epsilon > 0$ and $u_\epsilon > 0$, solution of our problem with $x_\epsilon \in \Omega$, $\sup_{\Omega} u_\epsilon = u_\epsilon(x_\epsilon)$ we have:
\[
d(x_\epsilon, \partial\Omega) \geq \delta.
\]

Proof of the lemma:
We argue by contradiction. We suppose: $\forall \delta > 0, \exists x_{\epsilon_{i,\delta}}$ such that: $d(x_{\epsilon_{i,\delta}}, \partial\Omega) \leq \delta$.

We take $\delta = \frac{1}{j}, j \rightarrow +\infty$, we have a subsequence $\epsilon_{i,j}$, noted $\epsilon_{i,j}$, such that, $d(x_{\epsilon_{i,j}}, \partial\Omega) \rightarrow 0$.

Let us consider $G$ the Green function of the laplacian with Dirichlet condition and $w$ satisfying:
\[
\Delta w = 1 \text{ in } \Omega \text{ and } w = 0 \text{ on } \partial\Omega.
\]
Using the variational method, we can prove the existence of $w$ and $w \in C^\infty(\overline{\Omega})$.

The Green representation formula and the fact $x_{\epsilon_{i,j}} \rightarrow y_0 \in \partial\Omega$ give:
\[
0 = w(y_0) = w(x_{\epsilon_{i,j}}) = \int_{\Omega} G(x_{\epsilon_{i,j}}, y)dy,
\]
we can write,
\[
\int_{\Omega} G(x_{\epsilon_{i,j}}, y)dy \rightarrow 0.
\]
The function $u_{\epsilon_{i,j}}$ satisfy (*) and thus:
\[
u_{\epsilon_{i,j}}(x_{\epsilon_{i,j}}) \leq (\max_{\Omega} u_{\epsilon_{i,j}})^{N-1-\epsilon_i} \int_{\Omega} G(x_{\epsilon_{i,j}}, y)dy,
\]
consequently,
\[
1 \leq [u_{\epsilon_{i,j}}(x_{\epsilon_{i,j}})]^{N-2-\epsilon_i} \int_{\Omega} G(x_{\epsilon_{i,j}}, y)dy.
\]
Then,
\[
u_{\epsilon_{i,j}}(x_{\epsilon_{i,j}}) \rightarrow +\infty \text{ and } x_{\epsilon_{i,j}} \rightarrow y_0 \in \partial\Omega \quad (**).\]
But, if we use the result of Z-C.Han (see [H] page 164) and [DLN] (pages 44-45 and 50-53) and the moving plane method (see [GNN]) we obtain:
if $\Omega$ is smooth bounded domain, $f$ a function in $C^1$ and $u$ is a solution of:
\[ \Delta u = f(u), \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial \Omega, \]

there exists two positive constants \( \delta \) and \( \gamma \), which depend only on the geometry of the domain \( \Omega \), such that:

\[ \forall x \in \{ z, d(z, \partial \Omega) \leq \delta \}, \exists \Gamma_x \subset \{ z, d(z, \partial \Omega) \geq \frac{\delta}{2} \} \text{ with } \text{mes}(\Gamma_x) \geq \gamma \text{ et } u(x) \leq u(\xi) \text{ for all } \xi \in \Gamma_x. \]

Thus,

\[ u(x) \leq \frac{1}{\text{mes}(\Gamma_x)} \int_{\Gamma_x} u \leq \frac{1}{\gamma} \int_{\Omega'} u \quad (\ast'), \]

with \( \Omega' \subset \subset \Omega \).

If we replace \( x \) by \( x_{\varepsilon_i} \), \( u \) by \( u_{\varepsilon_i} \) and we take \( \Omega' = \{ z \in \Omega, d(z, \partial \Omega) \geq \frac{\delta}{2} \} \), we obtain (after using the argument of the first eigenvalue like in [H]):

\[ +\infty \leftarrow u_{\varepsilon_i}(x_{\varepsilon_i}) \leq \frac{1}{\gamma} \int_{\Omega'} u_{\varepsilon_i} \leq c_2(\Omega', n) < \infty, \]

it is contradiction. The lemma is proved.

We continue the proof of the Theorem.

Without loss of generality, we can assume \( x_{\varepsilon_i} \rightarrow y_0 \). We consider \( (x_{\varepsilon_i})_{i \geq 0} \) and \( \mu > 0 \), such that \( x_{\varepsilon_i} \in B(y_0, \mu) \subset \subset \Omega \). (we take \( \mu = \frac{\delta}{2} \) for example).

We have:

\[ u_{\varepsilon_i}(x) = \int_{\Omega} G(x, y) u_{\varepsilon_i}^{N-1-\varepsilon_i}(y) dy \]

According to the properties of the Green functions and maximum principle, on the compact \( K \) of \( \Omega \):

\[ G(x, y) \geq c_3 = c(K, \Omega, n) > 0, \forall x \in K, y \in B(y_0, \mu). \]

Thus,

\[ \inf_K u_{\varepsilon_i} = u_{\varepsilon_i}(y_{\varepsilon_i}) \geq c_3 \int_{B(y_0, \mu)} u_{\varepsilon_i}^{N-1-\varepsilon_i}, \]

and then,

\[ \int_{B(y_0, \mu)} u_{\varepsilon_i}^{N-\varepsilon_i} \leq (\sup_{\Omega} u_{\varepsilon_i}) \times \int_{B(y_0, \mu)} u_{\varepsilon_i}^{N-1-\varepsilon_i} \leq \frac{(\sup_{\Omega} u_{\varepsilon_i} \times \inf_K u_{\varepsilon_i})}{c_3} \rightarrow 0. \]
Finally,

$$0 < \int_{B(y_0, 2\mu/3)} u_{\epsilon_i}^{N-\epsilon} \to 0 \quad (***) .$$

Let \( \eta \) be a smooth function such that:

$$0 \leq \eta \leq 1, \ \eta \equiv 1, \ \text{on } B(y_0, \mu/2), \ \eta \equiv 0, \ \text{on } \Omega - B(y_0, 2\mu/3).$$

Set \( k > 1 \). We multiply the equation of \( u_{\epsilon_i} \) by \( u_{\epsilon_i}^{2k-1} \eta^2 \) and we integrate by part the first member,

$$(2k-1) \int_{B(y_0, 2\mu/3)} |\nabla u_{\epsilon_i}|^2 u_{\epsilon_i}^{2k-2} \eta^2 + 2 \int_{B(y_0, 2\mu/3)} < \nabla u_{\epsilon_i} | \nabla \eta > \eta u_{\epsilon_i}^{2k-1} = \int_{B(y_0, 2\mu/3)} u_{\epsilon_i}^{N-2k-2-\epsilon} \eta^2 .$$

We compute \( |\nabla(u_{\epsilon_i}^k \eta)|^2 \) and we deduce:

$$\frac{2k - 1}{k^2} \int_{B(y_0, 2\mu/3)} |\nabla(u_{\epsilon_i}^k \eta)|^2 + \frac{2 - 2k}{k} \int_{B(y_0, 2\mu/3)} < \nabla u_{\epsilon_i} | \nabla \eta > u_{\epsilon_i}^{2k-1} - \frac{2k - 1}{k^2} \int_{B(y_0, 2\mu/3)} |\nabla \eta|^2 u_{\epsilon_i}^{2k} = \int_{B(y_0, 2\mu/3)} \eta^2 u_{\epsilon_i}^{N+2k-2-\epsilon} .$$

And,

$$\int_{B(y_0, 2\mu/3)} < \nabla u_{\epsilon_i} | \nabla \eta > u_{\epsilon_i}^{2k-1} = \frac{1}{4k} \int_{B(y_0, 2\mu/3)} < \nabla(u_{\epsilon_i}^2 \eta) | \nabla(\eta^2) > - \frac{1}{4k} \int_{B(y_0, 2\mu/3)} \Delta(\eta^2) u_{\epsilon_i}^{2k} .$$

Then,

$$\frac{2k - 1}{k^2} \int_{B(y_0, 2\mu/3)} |\nabla(u_{\epsilon_i}^k \eta)|^2 + \frac{2 - 2k}{4k^2} \int_{B(y_0, 2\mu/3)} \Delta(\eta^2) u_{\epsilon_i}^{2k} + \frac{2k - 1}{k^2} \int_{B(y_0, 2\mu/3)} |\nabla \eta|^2 u_{\epsilon_i}^{2k} + \int_{B(y_0, 2\mu/3)} u_{\epsilon_i}^{N+2k-2-\epsilon} .$$

But,

$$\int_{B(y_0, 2\mu/3)} u_{\epsilon_i}^{N+2k-2-\epsilon} = \int_{B(y_0, 2\mu/3)} (u_{\epsilon_i}^{2k} \eta^2)(u_{\epsilon_i}^{N-2-\epsilon}).$$

Using Hölder inequality with \( p = (N-\epsilon)/2 \) and \( p' = (N-\epsilon)/(N-\epsilon-2) \), we obtain:

$$\frac{2k - 1}{k^2} [||\nabla(\eta u_{\epsilon_i})||^2_{L^2(B_0)}]^2 \leq ||u_{\epsilon_i}||_{L^{N-\epsilon}(B_0)}^{N-\epsilon-2} \times ||\eta u_{\epsilon_i}^k||_{L^{N-\epsilon}(B_0)}^{N-\epsilon-2} + C [||u_{\epsilon_i}||_{L^2(B_0)}^{2k}] .$$
with $B_0 = B(y_0, 2\mu/3)$ and $C = C(k, \eta) = 2 - 2k + \frac{2k - 1}{k^2} \|\Delta \eta\|_\infty + \frac{2k - 1}{k^2} \|\nabla \eta\|_\infty$.

Hölder and Sobolev inequalities give,

$$\|\eta u_{\epsilon_i}^k\|_{L^{N-\epsilon_i}(B_0)}^2 \leq \|B_0\|^{2\epsilon_i/[N(N-\epsilon_i)]}K\|\nabla (\eta u_{\epsilon_i}^k)\|_{L^2(B_0)}^2.$$ 

We obtain:

$$\frac{2k - 1}{k^2\|B_0\|^{2\epsilon_i/[N(N-\epsilon_i)]}K} \|\eta u_{\epsilon_i}^k\|_{L^{N-\epsilon_i}(B_0)}^2 \leq \|u_{\epsilon_i}\|_{L^{N-\epsilon_i}(B_0)}^{N-2-\epsilon_i} \times \|\eta u_{\epsilon_i}^k\|_{L^{N-\epsilon_i}(B_0)}^2 +$$

$$+ C(k, \eta)\|u_{\epsilon_i}\|_{L^{2k}(B_0)}^{2k},$$

with $|B_0| = \text{mes}[B(0, 2\mu/3)]$.

We choose $k = \frac{N - \epsilon_i}{2}$ and we denote $\alpha_i = \|\eta u_{\epsilon_i}^{(N-\epsilon)/2}\|_{L^{N-\epsilon_i}(B_0)}^2 > 0$.

We have:

$$c_1 \alpha_i \leq \beta_i \alpha_i + c_2 \gamma_i,$$

with $c_1 = c_1(N, \mu) > 0$, $c_2 = c_2(N, \mu) > 0$, $\beta_i = \|u_{\epsilon_i}\|_{L^{N-\epsilon_i}}^{N-2-\epsilon_i}$ and $\gamma_i = \|u_{\epsilon_i}\|_{L^{N-\epsilon_i}}^{N-\epsilon_i}$.

with $\epsilon_i \in [0, \frac{2}{n-2})$. According to (** **), we have, $\beta_i \to 0$ and $\gamma_i \to 0$.

Thus,

$$(c_1/2) \alpha_i \leq (c_1 - \beta_i) \alpha_i \leq \gamma_i \to 0.$$ 

Finally,

$$0 < \int_{B(y_0, \mu/2)} u_{\epsilon_i}^{(N-\epsilon)/2} \leq \int_{B(y_0, 2\mu/3)} \eta u_{\epsilon_i}^{(N-\epsilon)/2} \to 0.$$ 

We iterate this process with $k = \frac{(N - \epsilon)^2}{4}$ after with $k = \frac{(N - \epsilon)^r}{2^r}$, $r \in \mathbb{N}^*$, we obtain, for all $q \geq 1$, there exists $l > 0$, such that:

$$\int_{B(y_0, l)} (u_{\epsilon_i})^q \to 0.$$ 

Using the Green representation formula, we obtain:

$$\forall \ x \in B(x, l'), \ u_{\epsilon_i}(x) = \int_{B(y_0, l)} G(x, y)u_{\epsilon_i}^{N-1-\epsilon}(y)dy + \int_{\partial B(y_0, l)} \partial_r G(x, \sigma)u_{\epsilon_i}(\sigma) \sigma_t (****).$$
where $0 < l' \leq l$.

We have,

$$\int_{B(y_0, l)} u_{\xi_i}^q = \int_0^l \int_{\partial B(y_0, r)} u_{\xi_i}^q(r \sigma_r) d\sigma_r dr \to 0,$$

We set, $s_{i, q}(r) = \int_{\partial B(y_0, r)} u_{\xi_i}^q(r \sigma_r)$. Then,

$$\int_0^l s_{i, q}(r) dr \to 0.$$

We can extract of, $s_{i, q}$, a subsequence which noted $s_{i, q}$ and which tends to 0 almost everywhere on $[0, l]$.

First, we choose, $q_1 = \frac{q(n + 2)}{n - 2}$ with $q > \frac{n}{2}$, after we choose $l_2 > 0$, such that, $\int_{B(y_0, l_2)} u_{\xi_i}^{q_1} \to 0$. Finally, we take $l_1 \in [0, l_2]$, such that, $s_{i, q_1}(l_1) \to 0$. We take $l_0 = \frac{l_1}{2} = l'$ in (***) and $l = l_1$ in (***) we obtain if we use Hölder inequality for the two integrals of (***)

$$\exists \ l_0 > 0, \ \sup_{B(y_0, l_0)} u_{\xi_i} \to 0.$$

But, $x_{\xi_i} \to y_0$, for $i$ large, $x_{\xi_i} \in B(y_0, l_0)$, which imply,

$$u_{\xi_i}(x_{\xi_i}) = \max_{\Omega} u_{\xi_i} \to 0.$$

But if we write,

$$u_{\xi_i}(x_{\xi_i}) = \int_{\Omega} G(x_{\xi_i}, y) u_{\xi_i}^{N-1-\epsilon_i}(y) dy,$$

we obtain,

$$\max_{\Omega} u_{\xi_i} = u_{\xi_i}(x_{\xi_i}) \leq \left( \sup_{\Omega} u_{\xi_i} \right)^{N-1-\epsilon_i} \int_{\Omega} G(x_{\xi_i}, y) dy = [u_{\xi_i}(x_{\xi_i})]^{N-1-\epsilon_i} w(x_{\xi_i}),$$

and finally,

$$1 \leq u_{\xi_i}(x_{\xi_i})^{N-2-\epsilon_i} w(x_{\xi_i}).$$

But, $w > 0$ on $\Omega$, $\|w\|_\infty > 0$ and $N - 2 - \epsilon_i > \frac{2}{n - 2}$, we have,

$$u_{\xi_i}(x_{\xi_i}) \geq \frac{1}{\|w\|_\infty^{1/(N-2-\epsilon_i)}} \geq c_4(n, \Omega) > 0.$$

It is a contradiction.

For the Theorem 3, we obtain a contradiction if we write:
\[
\max_{\Omega} u_{\epsilon} \leq (\max_{\Omega} u_{\epsilon})^{N-1} \|w\|_\infty + \max_{\Omega} u_{\epsilon} \epsilon \int_{\Omega} G(x,y)dy 
\leq (\max_{\Omega} u_{\epsilon})^{N-1} \|w\|_\infty + \alpha \max_{\Omega} u_{\epsilon},
\]

and finally,

\[
\max_{\Omega} u_{\epsilon} \geq \left( \frac{1-\alpha}{\|w\|_\infty} \right)^{1/(N-2)}.
\]

**Method 2:** proof of theorem 2 directly.

Suppose that:

\[
\sup_{\Omega} \times \inf_K u_i \to 0,
\]
then, for \(\delta > 0\) small enough, we have:

\[
\sup_{\Omega} \times \inf_{\{x,d(x,\partial\Omega) \geq \delta\}} u_i \to 0.
\]

Like in the first method (see [H]), for \(\delta > 0\) small,

\[
\sup_{\{x,d(x,\partial\Omega) \geq \delta\}} u_i \leq M = M(n,\Omega).
\]

We have,

\[
u_i(x) = \int_{\Omega} G(x,y)u_i^{N-1-\epsilon_i}dy.
\]

Let us consider \(K'\) another compact of \(\Omega\), using maximum principle, we obtain:

\[\exists c_1 = c_1(K,K',n,\Omega) > 0, \text{ such that } G(x,y) \geq c_1 \forall x \in K, y \in K,\]

thus,

\[\inf_K u_i = u_i(x_i) \geq c_1 \int_{K'} u_i^{N-1-\epsilon_i}dy.\]

We take, \(K' = K_\delta = \{x, d(x,\partial\Omega) \geq \delta\}\), there exists \(c_2 = c_2(\delta, n, K, \Omega) > 0\) such that:

\[\sup_{\Omega} \times \inf_K u_i \geq c_2 \int_{K_\delta} u_i^{N-\epsilon_i}dy,\]

we deduce,

\[\|u_i\|_{N-\epsilon_i}^{N-\epsilon_i} \geq c'_2 \sup_{\Omega} \times \inf_{\{x,d(x,\partial\Omega) \geq \delta\}} u_i + \max(\{x,d(x,\partial\Omega) \leq \delta\})M^{N-\epsilon_i}.\]
If we take \( \delta \) small and for \( i \) large, we have:

\[
||u_i||_{N-\epsilon_i} \rightarrow 0.
\]

Now, we use the Sobolev imbedding, \( H^1_0 \) in \( L^N \), we multiply the equation of \( u_i \) by \( u_i \), we integrate by part and finally, by Hölder inequality, we obtain:

\[
\bar{K}_1 ||u_i||_{8N-\epsilon_i}^2 \leq \bar{K}_2 ||u_i||_{N}^2 \leq \int_{\Omega} |\nabla u_i|^2 = \int_{\Omega} u_i^{N-\epsilon_i} = ||u_i||_{N-\epsilon_i}^{N-\epsilon_i},
\]

we know that, \( 0 < \epsilon_i \leq \frac{2}{n-2} \), the previous inequality:

\[
||u_i||_{N-\epsilon_i} \geq \bar{K}_3 > 0, \ \forall \ i,
\]

it is a contradiction.
References:


