Non injectivity of the q-deformed von Neumann algebra

Alexandre Nou

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Alexandre Nou
Université de Franche-Comté - Besancon
U.F.R des Sciences et Techniques
Département de Mathématiques
16 route de Gray - 25030 Besancon Cedex
nou@math.univ-fcomte.fr

Abstract. In this paper we prove that the von Neumann algebra generated by q-gaussians is not injective as soon as the dimension of the underlying Hilbert space is greater than 1. Our approach is based on a suitable vector valued Khintchine type inequality for Wick products. The same proof also works for the more general setting of a Yang-Baxter deformation. Our techniques can also be extended to the so called q-Araki-Woods von Neumann algebras recently introduced by Hiai. In this latter case, we obtain the non injectivity under some asssumption on the spectral set of the positive operator asociated with the deformation.

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1 Introduction

Let $H_{\mathbb{R}}$ be a real Hilbert space and $H_{\mathbb{C}}$ its complexification. Let T be a Yang-Baxter operator on $H_{\mathbb{C}} \otimes H_{\mathbb{C}}$ with $\|T\| < 1$. Let $\mathcal{F}_T(H_{\mathbb{C}})$ be the associated deformed Fock space and $\Gamma_T(H_{\mathbb{R}})$ the von Neumann algebra generated by the corresponding deformed gaussian random variables, introduced by Bozejko and Speicher [4] (also see [3]). In addition, we will assume that T is tracial, i.e that the vacuum expectation is a trace on $\Gamma_T(H_{\mathbb{R}})$ (cf [4]). Under these assumptions, it was proved in [4] that $\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as $\dim H_{\mathbb{R}} > \frac{16}{(1-q)^2}$, where $\|T\| = q$. Since then the problem whether $\Gamma_T(H_{\mathbb{R}})$ is not injective as soon as $\dim H_{\mathbb{R}} \geq 2$ had been left open. We emphasize that this problem remained open even in the particular case of the q-deformation, that is when $T = q\sigma$, where σ is the reflexion : $\sigma(\xi \otimes \eta) = \eta \otimes \xi$. Recall that the free von Neumann algebra $\Gamma_0(H_{\mathbb{R}})$ (corresponding to $T = 0$) is not injective as soon as $n = \dim H_{\mathbb{R}} \geq 2$, for $\Gamma_0(H_{\mathbb{R}})$ is isomorphic to the free group von Neumann algebra $VN(\mathbb{F}_n)$ (cf. [18]). The main result of this paper solves the above problem.

To explain the idea of our proof we first recall the main ingredient of the proof of the non injectivity theorem in [4]. It is the following vector-valued non-commutative Khintchine inequality. Let $(e_i)_{i\in I}$ be an orthonormal basis of $H_{\mathbb{R}}$. Let K be a complex Hilbert space and $B(K)$ the space of all bounded operators on K. Then for any finitely supported family $(a_i)_{i\in I} \subset B(K)$

$$\begin{aligned}\max\left\{\|\sum_{i\in I} a_i^* a_i\|_{B(K)}^{\frac{1}{2}}, \|\sum_{i\in I} a_i a_i^*\|_{B(K)}^{\frac{1}{2}}\right\} &\leq \left\|\sum_{i\in I} a_i \otimes G(e_i)\right\| \\ &\leq \frac{2}{\sqrt{1-q}} \max\left\{\|\sum_{i\in I} a_i^* a_i\|_{B(K)}^{\frac{1}{2}}, \|\sum_{i\in I} a_i a_i^*\|_{B(K)}^{\frac{1}{2}}\right\}\end{aligned}$$

where $G(e) = a^*(e) + a(e)$ is the deformed gaussian variable associated with a vector $e \in H_{\mathbb{R}}$. Using this Khintchine inequality and the equivalence between the injectivity and the semi-discreteness, one easily deduces the non-injectivity of $\Gamma_T(H_{\mathbb{R}})$ as soon as $\dim H_{\mathbb{R}} > \frac{16}{(1-q)^2}$.

The proof of our non-injectivity theorem follows the same pattern. We will first need to extend the preceding vector-valued non-commutative Khintchine inequality to Wick products. It is well known that for any ξ, a finite linear combination of elementary tensors, there is a unique operator $W(\xi) \in \Gamma_T(H_{\mathbb{R}})$ such that $W(\xi)\Omega = \xi$. Instead of the previous inequality, the main ingredient of our proof is the following. Let $n \geq 1$. Let $(\xi_{\underline{i}})_{|\underline{i}|=n}$ be an orthonormal basis of $H_{\mathbb{C}}^{\otimes n}$ and $(\alpha_{\underline{i}}) \subset B(K)$ a finitely supported family. Then

$$\max_{0\leq k\leq n}\{\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes R_{n,k}^* \xi_{\underline{i}}\|\} \leq \|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes W(\xi_{\underline{i}})\| \leq (n+1)C_q \max_{0\leq k\leq n}\{\|\sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes R_{n,k}^* \xi_{\underline{i}}\|\} \qquad (1)$$

where the norms in the left and right handside have to be taken in $B(K) \otimes_{\min} H_c^{\otimes n-k} \otimes_h H_c^{\otimes k}$ (see Theorem 1 below for the precise statement). Inequality (1) is the vector-valued version of Bozejko's ultracontractivity inequality proved in [2] and thus it solves a problem posed in [2]. Using (1) and a careful analysis on the norms of Wick products on a same level, we deduce our non-injectivity result.

The plan of this paper is as follows. The first section is devoted to necessary definitions and preliminaries on the deformation by a Yang-Baxter operator and the associated von Neumann algebra. In this section, we also include a brief discussion on the simplest case, the free case, i.e. when $T = 0$. All our results and arguments become very simple in this case, for instance, inequality (1) above is then easy to state and prove. The proof of the non-injectivity of $\Gamma_0(H_{\mathbb{R}})$ can be done in just a few lines. The reason why we have decided to include such a discussion on the free case is the fact that it already contains the main idea for the general case. In the second section we will establish (1) and prove the non-injectivity of $\Gamma_T(H_{\mathbb{R}})$. The last section aims at proving the non-injectivity of the Araki-Woods factors $\Gamma_q(H, U_t)$ introduced by Hiai in [12]. Note that Hiai proved a non-injectivity result with a condition on the dimension of the spectral sets of the positive generator of U_t, which is similar to that of [4]. The problem is left open whether the dimension can go down to 2. Although we cannot completely solve this, our method permits to improve in some sense the criterion for non-injectivity given in [12].

2 Preliminaries

Recall that the free Fock space associated with $H_{\mathbb{R}}$ is given by

$$\mathcal{F}_0(H_{\mathbb{C}}) = \bigoplus_{n\geq 0} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes 0}$ is by definition $\mathbb{C}\Omega$ with Ω a unit vector called the vacuum.

A Yang-Baxter operator on $H_{\mathbb{C}} \otimes H_{\mathbb{C}}$ is a self-adjoint contraction satisfying the following braid relation :

$$(I \otimes T)(T \otimes I)(I \otimes T) = (T \otimes I)(I \otimes T)(T \otimes I)$$

For $n \geq 2$ and $1 \leq k \leq n-1$ we define T_k on $H_{\mathbb{C}}^{\otimes n}$ by

$$T_k = I_{H_{\mathbb{C}}^{k-1}} \otimes T \otimes I_{H_{\mathbb{C}}^{n-k-1}}$$

Let S_n be the group of permutations on a set of n elements. A function φ is defined on S_n by quasi-multiplicative extension of :

$$\varphi(\pi_k) = T_k$$

where $\pi_k = (k, k+1)$ is the transposition exchanging k and $k+1$, $1 \leq k \leq n-1$. The symmetrizator $P_T^{(n)}$ is the following operator defined on $H_{\mathbb{C}}^{\otimes n}$ by :

$$P_T^{(n)} = \sum_{\sigma \in S_n} \varphi(\sigma)$$

$P_T^{(n)}$ is a positive operator on $H_{\mathbb{C}}^{\otimes n}$ for any Yang-Baxter operator T and is strictly positive if T is strictly contractive (cf. [4]). In the latter case we are allowed to define a new scalar product on $H_{\mathbb{C}}^{\otimes n}$ (for $n \geq 2$) by :

$$\langle \xi, \eta \rangle_T = \langle \xi, P_T^{(n)} \eta \rangle$$

The associated norm is denoted by $\|.\|_T$. The deformed Fock space associated with T is then defined by

$$\mathcal{F}_T(H_{\mathbb{C}}) = \bigoplus_{n\geq 0} H_{\mathbb{C}}^{\otimes n}$$

where $H_{\mathbb{C}}^{\otimes n}$ is now equipped with our deformed scalar product for $n \geq 2$. From now on we will only consider a strictly contractive Yang-Baxter T and $\|T\| \leq q < 1$.

For $f \in H_{\mathbb{R}}$, $a^*(f)$ will denote the creation operator associated with f, and $a(f)$ its adjoint with respect to the T-scalar product :

$$a^*(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n$$

For $f \in H_{\mathbb{R}}$ the deformed gaussian is the following hermitian operator :

$$G(f) = a^*(f) + a(f)$$

Throughout this paper we are interested in $\Gamma_T(H_{\mathbb{R}})$ which is the von Neumann algebra generated by all gaussians $G(f)$ for $f \in H_{\mathbb{R}}$:

$$\Gamma_T(H_{\mathbb{R}}) = \{G(f) : f \in H_{\mathbb{R}}\}'' \subset B(\mathcal{F}_T(H_{\mathbb{C}}))$$

Let $(e_i)_{i\in I}$ be an orthonormal basis of $H_{\mathbb{R}}$ and set

$$t_{ij}^{sr} = \langle e_s \otimes e_r, T(e_i \otimes e_j)\rangle$$

Then the following deformed commutation relations hold :

$$a(e_i)a^*(e_j) - \sum_{r,\, s\in I} t_{js}^{ir} a^*(e_r)(e_s) = \delta_{ij}$$

Moreover if the following condition holds

$$\langle e_s \otimes e_r, Te_i \otimes e_j\rangle = \langle e_r \otimes e_j, Te_s \otimes e_i\rangle$$

which is equivalent to the cyclic condition :

$$t_{ij}^{sr} = t_{si}^{rj}$$

then the vacuum is cyclic and separating for $\Gamma_T(H_{\mathbb{R}})$ and the vacuum expectation is a faithful trace on $\Gamma_T(H_{\mathbb{R}})$ that will be denoted by τ. If this cyclic condition holds we say that T is tracial, and from now on we will always assume that T has this property.

We will denote by $\Gamma_T^{\infty}(H_{\mathbb{R}})$ the subspace $\Gamma_T(H_{\mathbb{R}})\Omega$ of $\mathcal{F}_T(H_{\mathbb{C}})$. Since Ω is separating for $\Gamma_T(H_{\mathbb{R}})$, for every $\xi \in \Gamma_T^{\infty}(H_{\mathbb{R}})$ there exists a unique operator $W(\xi) \in \Gamma_T(H_{\mathbb{R}})$ such that

$$W(\xi)\Omega = \xi$$

W is called Wick product.

The right creation operator, $a_r^*(f)$, is defined by the following formula :

$$a_r^*(f)(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n \otimes f$$

We will also denote by $a_r(f)$ the right annihilation operator, which is its adjoint with respect to the T-scalar product, by $G_r(f)$ the right gaussian operator, and by $\Gamma_{T,r}(H_{\mathbb{R}})$ the von Neumann algebra generated by all right gaussians. It is easy to see that $\Gamma_{T,r}(H_{\mathbb{R}}) \subset \Gamma_T(H_{\mathbb{R}})'$. Actually, by Tomita's theory, we have

$$\Gamma_{T,r}(H_{\mathbb{R}}) = S\Gamma_T(H_{\mathbb{R}})S = \Gamma_T(H_{\mathbb{R}})'$$

where S is the anti linear operator on $\mathcal{F}_T(H_{\mathbb{C}})$ (which is actually an anti unitary) defined by

$$S(f_1 \otimes \cdots \otimes f_n) = f_n \otimes \cdots \otimes f_1$$

for any $f_1, \cdots, f_n \in H_{\mathbb{R}}$. Since Ω is also separating for $\Gamma_{T,r}(H_{\mathbb{R}})$ we can define the right Wick product, that will be denoted by $W_r(\xi)$. For any $\xi \in \Gamma_T^\infty(H_{\mathbb{R}})$ we have

$$(W(\xi))^* = W(S\xi) \qquad \text{and} \qquad SW(\xi)S = W_r(S\xi)$$

Some particular cases of deformation have been studied in the literature. Let $(q_{ij})_{i,\,j\in I}$ be a hermitian matrix such that $\sup_{i,j}|q_{ij}| < 1$. Define

$$Te_i \otimes e_j = q_{ij} e_j \otimes e_i$$

Then T is a strictly contractive Yang-Baxter operator, and it is tracial if and only if the q_{ij} are real. Our deformed Fock space is then a realisation of the following q_{ij}-relations :

$$a(e_i)a^*(e_j) - q_{ij}a^*(e_j)a(e_i) = \delta_{ij}$$

In the special case where all q_{ij} are equal, we obtain the well known q-relations.

Let us define the following selfadjoint unitary on the free Fock space :

$$\forall\, f_1, \ldots, f_n \in H_{\mathbb{C}}, \quad U(f_1 \otimes \cdots \otimes f_n) = f_n \otimes \cdots \otimes f_1$$

Since $UP_T^{(n)} = P_T^{(n)}U$ (cf. [13]), U is also a selfadjoint unitary on each T-Fock space.

Given vectors $f_1, \ldots, f_n$ in $H_{\mathbb{R}}$ we define :

$$a^*(f_1 \otimes \cdots \otimes f_n) = a^*(f_1)\ldots a^*(f_n) \quad \text{and} \quad a(f_1 \otimes \cdots \otimes f_n) = a(f_1)\ldots a(f_n)$$

For $0 \le k \le n$, let $R_{n,k}$ be the operator on $H_{\mathbb{C}}^{\otimes n}$ given by

$$R_{n,k} = \sum_{\sigma \in S_n/S_{n-k}\times S_k} \varphi(\sigma^{-1})$$

where the sum runs over the representatives of the right cosets of $S_{n-k} \times S_k$ in S_n with minimal number of inversions. Then

$$P_T^{(n)} = R_{n,k}\left(P_T^{(n-k)} \otimes P_T^{(k)}\right) \quad \text{and} \quad \|R_{n,k}\| \le C_q \tag{2}$$

where $C_q = \prod_{n=1}^{\infty}(1-q^n)^{-1}$ (cf. [2] and [13]). It follows that

$$P_T^{(n)} \le C_q P_T^{(n-k)} \otimes P_T^{(k)} \tag{3}$$

It also follows that a^*, respectively a, extend linearly, respectively antilinearly, and continuously to $H_{\mathbb{C}}^{\otimes n}$ for every $n \ge 1$. Then for each vector $\xi \in H_{\mathbb{C}}^{\otimes n}$ we have

$$\|a^*(\xi)\| \le C_q^{\frac{1}{2}}\|\xi\|_T \quad \text{and} \quad (a^*(\xi))^* = a(U\xi). \tag{4}$$

Let $n \ge 1$ and $1 \le k \le n$, $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ will be the Hilbert tensor product of the Hilbert spaces $H_{\mathbb{C}}^{\otimes k}$ and $H_{\mathbb{C}}^{\otimes n-k}$ where both $H_{\mathbb{C}}^{\otimes k}$ and $H_{\mathbb{C}}^{\otimes n-k}$ are equipped with the T-scalar product.

Lemma 1 *There is a positive constant $D_{q,n,k}$ such that*

$$P_T^{(n-k)} \otimes P_T^{(k)} \leq D_{q,n,k} P_T^{(n)}$$

Consequently for every $n \geq 1$ and $1 \leq k \leq n$, $H_{\mathbb{C}}^{\otimes n}$ and $H_{\mathbb{C}}^{\otimes k} \otimes H_{\mathbb{C}}^{\otimes n-k}$ are algebraically the same and their norms are equivalent.

Remark : It is still not known whether one can choose $D_{q,n,k}$ independent of n and k.

Proof : It was shown in [1] that there is a positive constant $\omega(q)$ such that

$$P_T^{(n-1)} \otimes I \leq \omega(q)^{-1} P_T^{(n)}$$

Since $U(P_T^{(n-1)} \otimes I)U = I \otimes P_T^{(n-1)}$ we also have

$$I \otimes P_T^{(n-1)} \leq \omega(q)^{-1} P_T^{(n)} \tag{5}$$

Fix some k, $2 \leq k \leq n-1$, using (3) and (4) we get :

$$\begin{aligned} P_T^{(n-k+1)} \otimes P_T^{(k-1)} &\leq C_q P_T^{(n-k)} \otimes I \otimes P_T^{(k-1)} \\ &\leq C_q \omega(q)^{-1} P_T^{(n-k)} \otimes P_T^{(k)} \end{aligned}$$

Thus by iteration it follows that for $0 \leq k \leq n$:

$$P_T^{(n-k)} \otimes P_T^{(k)} \leq \omega(q)^{-1} (C_q \omega(q)^{-1})^{n-k} P_T^{(n)} \tag{6}$$

Since $U(P_T^{(n-k)} \otimes P_T^{(k)})U = P_T^{(k)} \otimes P_T^{(n-k)}$ it follows from (6) that

$$P_T^{(k)} \otimes P_T^{(n-k)} \leq \omega(q)^{-1} (C_q \omega(q)^{-1})^{n-k} P_T^{(n)}$$

Combining this last inequality and (6) we finally obtain :

$$P_T^{(n-k)} \otimes P_T^{(k)} \leq \omega(q)^{-1} (C_q \omega(q)^{-1})^{\min(k,n-k)} P_T^{(n)} \tag{7}$$

Then the desired result follows from (3) and (7). $\square$

For $k \geq 0$ let us now define on the family of finite linear combinations of elementary tensors of length not less than k the following operator U_k:

$$U_k(f_1 \otimes \cdots \otimes f_n) = a^*(f_1 \otimes \cdots \otimes f_{n-k}) a(\overline{f}_{n-k+1} \otimes \cdots \otimes \overline{f}_n)$$

where $\overline{\xi + i\eta} = \xi - i\eta$ for all $\xi,\ \eta \in H_{\mathbb{R}}$.

Fix n and k with $n \geq k$. Let $\mathcal{J} :\ H_{\mathbb{C}}^{\otimes k} \to \overline{H_{\mathbb{C}}^{\otimes k}}$ be the conjugation (which is an anti isometry). For any $f_1, \cdots, f_n$, $\mathcal{J}$ is defined by $\mathcal{J}(f_1 \otimes \cdots \otimes f_n) = \overline{f_1} \otimes \cdots \otimes \overline{f_n}$. It is clear that U_k extends boundedly to $H_{\mathbb{C}}^{\otimes n-k} \otimes H_{\mathbb{C}}^{\otimes k}$ by the formula :

$$U_k = M(a^* \otimes a\mathcal{J})$$

where M is the multiplication operator from $B(\mathcal{F}_T(H_\mathbb{C})) \otimes_{\min} B(\mathcal{F}_T(H_\mathbb{C}))$ to $B(\mathcal{F}_T(H_\mathbb{C}))$ defined by $M(A \otimes B) = AB$. Moreover, by (4) we have

$$\|U_k\| \leq \|M\|.\|a^* \otimes a\mathcal{J}\| \leq C_q$$

where U_k is viewed as an operator from $H_\mathbb{C}^{\otimes n-k} \otimes H_\mathbb{C}^{\otimes k}$ to $B(\mathcal{F}_T(H_\mathbb{C}))$.

In the following lemma we state an extension of the Wick formula (Theorem 3 in [13]). We deduce it as an easy consequence of the original Wick formula and of our previous discussion.

Lemma 2 *Let $n \geq 1$ and $\xi \in H_\mathbb{C}^{\otimes n}$, then $H_\mathbb{C}^{\otimes n} \subset \Gamma_T^\infty(H_\mathbb{R})$ and we have the following Wick formula :*

$$W(\xi) = \sum_{k=0}^{n} U_k R_{n,k}^*(\xi) \tag{8}$$

Moreover

$$\|\xi\|_q \leq \|W(\xi)\| \leq C_q^{\frac{3}{2}}(n+1)\|\xi\|_q \tag{9}$$

Remark : (9) is the well known Bozejko's inequality discussed in [2] and [13], and which implies the ultracontractivity of the q-Ornstein Uhlenbeck semigroup. We include an elementary and simple proof.

Proof : The usual Wick formula is the following (cf [2] and [13]) : $\forall f_1, \ldots, f_n \in H_\mathbb{C}$ we have

$$W(f_1 \otimes \cdots \otimes f_n) = \sum_{k=0}^{n} \sum_{\sigma \in S_n/S_{n-k} \times S_k} U_k \varphi(\sigma)(f_1 \otimes \cdots \otimes f_n)$$

Hence (8) holds for every $\xi \in \mathcal{A}_n = \{$linear combinations of elementary tensors of length n$\}$. By Lemma 1 and our previous discussion, the right handside of (8) is continuous from $H_\mathbb{C}^{\otimes n}$ to $B(\mathcal{F}_T(H_\mathbb{C}))$. Since Ω is separating, it follows that $H_\mathbb{C}^{\otimes n} \subset \Gamma_T^\infty(H_\mathbb{R})$ and that (8) extends by density from $\mathcal{A}_n$ to $H_\mathbb{C}^{\otimes n}$. Actually, our argument shows that for any $\xi \in H^{\otimes n}$, $W(\xi)$ belongs to $C_T^*(H_\mathbb{R})$ which is the C^*-algebra generated by the T-gaussians.

Since for any $\xi \in H_\mathbb{C}^{\otimes n}$, $W(\xi)\Omega = \xi$, the left inequality in (9) holds. We have just showed that W is bounded from $H_\mathbb{C}^{\otimes n}$ to $B(\mathcal{F}_T(H_\mathbb{C}))$. Hence, there is a constant $B_{q,n}$ such that for any $\xi \in H_\mathbb{C}^{\otimes n}$ we have $\|W(\xi)\| \leq B_{q,n}\|\xi\|_q$. To end the proof of (9) we now give a precise estimate of $B_{q,n}$. Let $\xi \in H_\mathbb{C}^{\otimes n}$, by (8) and (3) we have

$$\|W(\xi)\| \leq \sum_{k=0}^{n} \|U_k R_{n,k}^*(\xi)\| \leq C_q \sum_{k=0}^{n} \|R_{n,k}^*(\xi)\|_{H_\mathbb{C}^{\otimes n-k} \otimes H_\mathbb{C}^{\otimes k}} \tag{10}$$

It remains to compute the norm of $R_{n,k}^*$ as an operator from $H_\mathbb{C}^{\otimes n}$ to $H_\mathbb{C}^{\otimes n-k} \otimes H_\mathbb{C}^{\otimes k}$. Let $\eta \in H_\mathbb{C}^{\otimes n}$ we have, by (2) and (3)

$$\begin{aligned} \|R_{n,k}^*\eta\|^2_{H_\mathbb{C}^{\otimes n-k} \otimes H_\mathbb{C}^{\otimes k}} &= \langle P_T^{(n-k)} \otimes P_T^{(k)} R_{n,k}^*\eta, R_{n,k}^*\eta\rangle_0 \\ &= \langle P_T^{(n)}\eta, R_{n,k}^*\eta\rangle_0 \leq \|\eta\|_T \|R_{n,k}^*\eta\|_T \end{aligned}$$

On the other hand,

$$\begin{aligned}
\|R_{n,k}^*\eta\|_T^2 &= \langle P_T^{(n)} R_{n,k}^*\eta, R_{n,k}^*\eta\rangle_0 \le C_q\langle P_T^{(n-k)}\otimes P_T^{(k)} R_{n,k}^*\eta, R_{n,k}^*\eta\rangle_0 \\
&\le C_q\langle P_T^{(n)}\eta, R_{n,k}^*\eta\rangle_0 \\
&\le C_q\|\eta\|_T\|R_{n,k}^*\eta\|_T
\end{aligned}$$

Hence it follows that $\|R_{n,k}^*\eta\|_T \le C_q\|\eta\|_T$ and $\|R_{n,k}^*\eta\|^2_{H_{\mathbb{C}}^{\otimes n-k}\otimes H_{\mathbb{C}}^{\otimes k}} \le C_q\|\eta\|_T^2$. Thus $\|R_{n,k}^*\| \le C_q^{\frac{1}{2}}$ as an operator from $H_{\mathbb{C}}^{\otimes n}$ to $H_{\mathbb{C}}^{\otimes n-k}\otimes H_{\mathbb{C}}^{\otimes k}$. From (10) and this last estimate, follows the second inequality in (9).

$\square$

The remainder of this section is devoted to a simple proof of the non-injectivity of the free von Neumann algebra $\Gamma_0(H_{\mathbb{R}})$ ($\dim H_{\mathbb{R}} \ge 2$). The main ingredient is the vector valued Bozejko inequality (Lemma 3 below), which is the free Fock space analogue of the corresponding inequality for the free groups proved by Haagerup and Pisier in [11] and extended by Buchholz in [7] (see also [6]). Note also that the inequality (11) below was first proved in [11] in the case $n = 1$ (i.e. for free gaussians) and that a similar inequality holds for products of free gaussians (see [7]).

We will need the following notations : $(e_i)_{i\in I}$ will denote an orthonormal basis of $H_{\mathbb{R}}$, and for a multi-index $\underline{i}$ of length n, $\underline{i} = (i_1, \ldots, i_n) \in I^n$, $e_{\underline{i}} = e_{i_1}\otimes\cdots\otimes e_{i_n}$. $(e_{\underline{i}})_{|\underline{i}|=n}$ is a real orthonormal basis of $H_{\mathbb{C}}^{\otimes n}$ equipped with the free scalar product and $(e_{\underline{i}})_{|\underline{i}|\ge 0}$ is a real orthonormal basis of the free Fock space.

Lemma 3 *Let $n \ge 1$, K a complex Hilbert space and $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ a finitely supported family of $B(K)$. Then :*

$$\max_{0\le k\le n}\left\{\left\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\ |\underline{l}|=k}}\right\|\right\} \le \left\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(e_{\underline{i}})\right\| \le (n+1)\max_{0\le k\le n}\left\{\left\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\ |\underline{l}|=k}}\right\|\right\} \qquad (11)$$

Remark : Since $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ is finitely supported the operator-coefficient matrix $(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\ |\underline{l}|=k}}$ is a finite matrix, say a $r\times s$ matrix, and its norm is the operator norm in $B(l_2^s(K), l_2^r(K))$.
Proof : We write

$$\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(e_{\underline{i}}) = \sum_{k=0}^{n} F_k$$

where

$$F_k = \sum_{\substack{|\underline{j}|=n-k\\ |\underline{l}|=k}}\alpha_{\underline{j},\,\underline{l}}\otimes a^*(e_{\underline{j}})a(e_{\underline{l}})$$

we have

$$F_k = (\ldots I_K\otimes a^*(e_{\underline{j}})\ldots)_{|\underline{j}|=n-k}(\alpha_{\underline{j},\,\underline{l}}\otimes I_{\mathcal{F}_0(H_{\mathbb{C}})})_{\substack{|\underline{j}|=n-k\\ |\underline{l}|=k}}\begin{pmatrix}\vdots\\ I_K\otimes a(e_{\underline{l}})\\ \vdots\end{pmatrix}_{|\underline{l}|=k}$$

that is, F_k is a product of three matrices, the first is a row indexed by $\underline{j}$, the third a column indexed by $\underline{l}$. Note that

$$\|(\dots a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|^2 = \|\sum_{|\underline{j}|=n-k} a^*(e_{\underline{j}})(a^*(e_{\underline{j}}))^*\| = \|\sum_{|\underline{j}|=n-k} a^*(e_{\underline{j}})a(Ue_{\underline{j}})\|$$

It is easy to see that $\sum\limits_{|\underline{j}|=n-k} a^*(e_{\underline{j}})a(Ue_{\underline{j}})$ is the orthogonal projection on $\bigoplus\limits_{p\geq n-k} H^{\otimes p}$.

Thus

$$\|(\dots a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\| \leq 1$$

Therefore

$$\begin{aligned}
\|F_k\| &\leq \|(\dots I_K\otimes a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|.\|(\alpha_{\underline{j},\,\underline{l}}\otimes I_{\mathcal{F}_0(H_{\mathbb{C}})})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|.\left\|\begin{pmatrix}\vdots\\ I_K\otimes a(e_{\underline{l}})\\ \vdots\end{pmatrix}_{|\underline{l}|=k}\right\| \\
&\leq \|(\dots a^*(e_{\underline{j}})\dots)_{|\underline{j}|=n-k}\|.\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|.\|(\dots a^*(Ue_{\underline{l}})\dots)_{|\underline{l}|=k}\| \\
&\leq \|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|
\end{aligned}$$

It follows that

$$\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(e_{\underline{i}})\| \leq \sum_{k=0}^{n}\|F_k\| \leq (n+1)\max_{0\leq k\leq n}\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k\\|\underline{l}|=k}}\|$$

To prove the first inequality, fix $0\leq k_0\leq n$ and consider $(v_{\underline{p}})_{|\underline{p}|=k_0}$ such that $\sum\limits_{|\underline{p}|=k_0}\|v_{\underline{p}}\|^2<+\infty$. Let $\eta=\sum\limits_{|\underline{p}|=k_0} v_{\underline{p}}\otimes Ue_{\underline{p}}$. We have :

$$\begin{aligned}
\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes W(e_{\underline{i}})\eta\|^2 &= \sum_{k=0}^{n}\|F_k\eta\|^2 \geq \|F_{k_0}\eta\|^2 \\
&= \|\sum_{\substack{|\underline{j}|=n-k_0\\|\underline{l}|=k_0}}\alpha_{\underline{j},\,\underline{l}}v_{\underline{l}}\otimes e_{\underline{j}}\|^2 \\
&= \sum_{|\underline{j}|=n-k_0}\|\sum_{|\underline{l}|=k_0}\alpha_{\underline{j},\,\underline{l}}v_{\underline{l}}\|^2 \\
&= \left\|(\alpha_{\underline{j},\,\underline{l}})_{\substack{|\underline{j}|=n-k_0\\|\underline{l}|=k_0}}\begin{pmatrix}\vdots\\ v_{\underline{l}}\\ \vdots\end{pmatrix}_{|\underline{l}|=k_0}\right\|^2
\end{aligned}$$

Then the result follows. $\square$

Using Lemma 3, it is now easy to prove that $\Gamma_0(H_{\mathbb{R}})$ is not injective as soon as $\dim H_{\mathbb{R}}\geq 2$. Suppose that $\Gamma_0(H_{\mathbb{R}})$ is injective and $\dim H_{\mathbb{R}}\geq 2$. Choose two orthonormal vectors e_1 and e_2

in $H_\mathbb{R}$. For $n \geq 1$ we have by semi-discreteness (which is equivalent to the injectivity):

$$\tau\left(\sum_{|\underline{i}|=n} W(e_{\underline{i}})^* W(e_{\underline{i}})\right) \leq \| \sum_{|\underline{i}|=n} \overline{W(e_{\underline{i}})} \otimes W(e_{\underline{i}}) \|$$

where in the above sums, the index $\underline{i} \in \{1,2\}^n$. However,

$$\begin{aligned} \tau\left(\sum_{|\underline{i}|=n} W(e_{\underline{i}})^* W(e_{\underline{i}})\right) &= \sum_{|\underline{i}|=n} \langle W(e_{\underline{i}})\Omega, W(e_{\underline{i}})\Omega\rangle_0 \\ &= \sum_{|\underline{i}|=n} \|e_{\underline{i}}\|^2 = 2^n \end{aligned}$$

On the other hand, by Lemma 3,

$$\begin{aligned} \| \sum_{|\underline{i}|=n} \overline{W(e_{\underline{i}})} \otimes W(e_{\underline{i}}) \| &\leq (n+1) \max_{0\leq k\leq n} \{ \| (\overline{W(e_{\underline{j},\underline{l}})})_{\substack{|\underline{j}|=n-k \\ |\underline{l}|=k}} \| \} \\ &\leq (n+1)\left(\sum_{|\underline{i}|=n} \|W(e_{\underline{i}})\|^2\right)^{\frac{1}{2}} \\ &\leq (n+1)(2^n(n+1)^2)^{\frac{1}{2}} \\ &\leq (n+1)^2 2^{\frac{n}{2}} \end{aligned}$$

Combining the preceding inequalities, we get $2^n \leq (n+1)^2 2^{\frac{n}{2}}$ which yields a contradiction for sufficiently large n. Therefore, $\Gamma_0(H_\mathbb{R})$ is not injective if $\dim H_\mathbb{R} \geq 2$.

3 Generalized Haagerup-Bożejko inequality and non injectivity of $\Gamma_T(H_\mathbb{R})$

In the following we state and prove the generalized inequality (1). It actually solves a question of Marek Bozejko (in [2] page 210) whether it is possible to find an operator coefficient version of the following inequality (this is inequality (9) in Lemma 2):

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} e_{\underline{i}} \| \leq \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} W(e_{\underline{i}}) \| \leq C_q^{\frac{3}{2}}(n+1) \| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} e_{\underline{i}} \| \tag{12}$$

where $(\alpha_{\underline{i}})_{\underline{i}}$ is a finitely supported family of complex numbers. Inequality (12) was proved in [2] for the q-deformation, and generalized in [13] for the Yang-Baxter deformation.

First, we need to recall some basic notions from operator space theory. We refer to [10] and [14] for more information.

Given K a complex Hilbert space, we can equip K with the column, respectively the row, operator space structure denoted by K_c, respectively K_r, and defined by

$$K_c = B(\mathbb{C}, K) \qquad \text{and} \qquad K_r = B(K^*, \mathbb{C}).$$

Moreover, we have $K_c^* = \overline{K}_r$ as operator spaces.

Given two operator spaces E and F, let us briefly recall the definition of the Haagerup tensor product of E and F. $E \otimes F$ will denote the algebraic tensor product of E and F. For $n \geq 1$ and $x = (x_{i,j})$ belonging to $M_n(E \otimes F)$ we define

$$\|x\|_{(h,n)} = \inf\{\|y\|_{M_{n,r}(E)}\|z\|_{M_{r,n}(F)}\}$$

where the infimum runs over all $r \geq 1$ and all decompositions of x of the form

$$x_{i,j} = \sum_{k=1}^{r} y_{i,k} \otimes z_{k,j}.$$

By Ruan's theorem, this sequence of norms define an operator space structure on the completion of $E \otimes F$ equipped with $\| \ . \ \|_h = \| \ . \ \|_{(h,1)}$. The resulting operator space, which is called the Haagerup tensor product of E and F is denoted by $E \otimes_h F$.
In this setting, a bilinear map $u : \ E \times F \to B(K)$ is said to be completely bounded, in short c.b, if and only if the associated linear map $\hat{u} : \ E \otimes F \to B(K)$ extends completely boundedly to $E \otimes_h F$. We define $\|u\|_{cb} = \|\hat{u}\|_{cb}$. This notion goes back to Christensen and Sinclair [9].

We will often use the following classical identities for hilbertian operator spaces :

$$K_c \otimes_{\min} H_r = K_c \otimes_h H_r = \mathcal{K}(\overline{H}, K),$$

where $\mathcal{K}$ stands for the compact operators and

$$K_c \otimes_{\min} H_c = K_c \otimes_h H_c = (K \otimes_2 H)_c$$

and similarly for rows using duality.

There is another notion of complete boundedness for bilinear maps, called jointly complete boundedness. Let E, F be operator spaces, K a complex Hilbert space, and $u : \ E \times F \to B(K)$ a bilinear map. u is said to be jointly completely bounded (in short j.c.b) if and only if for any C*-algebras B_1 and B_2, u can be boundedly extended to a bilinear map $(u)_{B_1,B_2} :$ $E \otimes_{\min} B_1 \times F \otimes_{\min} B_2 \to B(K) \otimes_{\min} B_1 \otimes_{\min} B_2$ taking $(e \otimes b_1, \ f \otimes b_2)$ to $u(e, \ f) \otimes b_1 \otimes b_2$. We put $\|u\|_{jcb} = \sup\limits_{B_1, \, B_2} \|(u)_{B_1,B_2}\|$. Observe that in this definition B_1 and B_2 can be replaced by operator spaces.

We will need the fact that every bilinear c.b map is a j.c.b map with $\|u\|_{jcb} \leq \|u\|_{cb}$. Let K be a complex Hilbert space and $u : \ B(K) \times K_c \to K_c$ the bilinear map taking $(\varphi, \ k)$ to $\varphi(k)$. Then it is easy to see that u is a norm one bilinear cb map.
To simplify our notations, $H_{\mathbb{C}}$ will be, most of the time, replaced by H in the rest of this section. For the same reason we will denote by $H_c^{\otimes n}$ (respectively $H_r^{\otimes n}$) the column Hilbert space $(H_{\mathbb{C}}^{\otimes n})_c$ (respectively the row Hilbert space $(H_{\mathbb{C}}^{\otimes n})_r$).

Lemma 4 *Let $n \geq 1$. The mappings $a^* : \ H_c^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ and $a : \overline{H}_r^{\otimes n} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ are completely bounded with cb-norms less than $\sqrt{C}_q$.*

Proof : Let us start with the proof of the statement concerning a^*. Let $n \geq 1$, K a complex Hilbert space and $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ a finitely supported family of $B(K)$ such that

$$\| \sum_{|\underline{i}|=n} \alpha_{\underline{i}} \otimes e_{\underline{i}} \|_{B(K) \otimes_{\min} H_c} < 1.$$

Then, since the maps $a^*(e_{\underline{i}})$ acts diagonally with respect to degrees of tensors in $\mathcal{F}_T(H_{\mathbb{C}})$,

$$\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes a^*(e_{\underline{i}})\|_{B(K)\otimes_{\min}B(\mathcal{F}_T(H_{\mathbb{C}}))}=\sup_{k\geq 0}\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes a^*(e_{\underline{i}})\|_{B(K)\otimes_{\min}B(H^{\otimes k},H^{\otimes n+k})}$$

To compute the right term, fix $k\geq 0$ and let $(\xi_{\underline{j}})_{|\underline{j}|=k}$ be a finitely supported family of vectors in K such that

$$\|\sum_{|\underline{j}|=k}\xi_{\underline{j}}\otimes e_{\underline{j}}\|_{K\otimes_2 H^{\otimes k}}<1.$$

By (3) we have

$$\|\sum_{\underline{i},\,\underline{j}}\alpha_{\underline{i}}(\xi_{\underline{j}})\otimes e_{\underline{i}}\otimes e_{\underline{j}}\|_{K\otimes_2 H^{\otimes n+k}}\leq C_q^{\frac{1}{2}}\|\sum_{\underline{i},\,\underline{j}}\alpha_{\underline{i}}(\xi_{\underline{j}})\otimes e_{\underline{i}}\otimes e_{\underline{j}}\|_{K\otimes_2 H^{\otimes n}\otimes_2 H^{\otimes k}}.$$

Let $u:\ B(K)\times K_c\to K_c$ given by $(\varphi,\xi)\mapsto\varphi(\xi)$. Recall that $\|u\|_{cb}=1$. Consequently, $\|u\|_{jcb}\leq 1$. Therefore, we deduce

$$\begin{aligned}\|\sum_{\underline{i},\,\underline{j}}\alpha_{\underline{i}}(\xi_{\underline{j}})\otimes e_{\underline{i}}\otimes e_{\underline{j}}\|_{K\otimes_2 H^{\otimes n}\otimes_2 H^{\otimes k}} &= \|\sum_{\underline{i},\,\underline{j}}\alpha_{\underline{i}}(\xi_{\underline{j}})\otimes e_{\underline{i}}\otimes e_{\underline{j}}\|_{K_c\otimes_{\min}H_c^{\otimes n}\otimes_{\min}H_c^{\otimes k}}\\ &= \|(u)_{H_c^{\otimes n},\,H_c^{\otimes k}}(\sum_{\underline{i}}\alpha_{\underline{i}}\otimes e_{\underline{i}},\ \sum_{\underline{j}}\xi_{\underline{j}}\otimes e_{\underline{j}})\|\\ &\leq \|u\|_{jcb}\|\sum_{\underline{i}}\alpha_{\underline{i}}\otimes e_{\underline{i}}\|_{B(K)\otimes_{\min}H_c^{\otimes n}}\|\sum_{\underline{j}}\xi_{\underline{j}}\otimes e_{\underline{j}}\|_{K_c\otimes_{\min}H_c^{\otimes k}}\\ &\leq 1\end{aligned}$$

By the result just proved, for any complex Hilbert space K and for any finitely supported family $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ of $B(K)$ we have

$$\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes a^*(e_{\underline{i}})\|_{B(K)\otimes_{\min}B(\mathcal{F}_T(H_{\mathbb{C}}))}\leq\sqrt{C_q}\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes e_{\underline{i}}\|_{B(K)\otimes_{\min}H_c^{\otimes n}}$$

Taking adjoints on both sides we get

$$\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}^*\otimes a(Ue_{\underline{i}})\|_{B(K)\otimes_{\min}B(\mathcal{F}_T(H_{\mathbb{C}}))}\leq\sqrt{C_q}\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}^*\otimes\overline{e_{\underline{i}}}\|_{B(K)\otimes_{\min}\overline{H}_r^{\otimes n}}$$

Changing $\alpha_{\underline{i}}^*$ to $\alpha_{\underline{i}}$ and using the fact that U (reversing the order of tensor) is a complete isometry on $H_r^{\otimes n}$, we get that for any finitely supported family $(\alpha_{\underline{i}})_{|\underline{i}|=n}$ of $B(K)$ we have

$$\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes a(\overline{e_{\underline{i}}})\|_{B(K)\otimes_{\min}B(\mathcal{F}_T(H_{\mathbb{C}}))}\leq\sqrt{C_q}\|\sum_{|\underline{i}|=n}\alpha_{\underline{i}}\otimes\overline{e_{\underline{i}}}\|_{B(K)\otimes_{\min}\overline{H}_r^{\otimes n}}.$$

In other words,

$$a:\overline{H}_r^{\otimes n}\to B(\mathcal{F}_T(H_{\mathbb{C}}))$$

is also completely bounded with norm less than $\sqrt{C}_q$. $\square$

Corollary 1 *For any $n \geq 0$, and any $k \in \{0...n\}$,*

$$U_k : \; H_c^{\otimes n-k} \otimes_h H_r^{\otimes k} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$$

is completely bounded with cb-norm less than C_q.

Proof : Let us denote by M the multiplication map $B(\mathcal{F}_T(H_{\mathbb{C}})) \otimes_h B(\mathcal{F}_T(H_{\mathbb{C}})) \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ given by $A \otimes B \mapsto AB$, M is obviously completely contractive. We have the formula

$$U_k = M(a^* \otimes a\mathcal{J})$$

if $\mathcal{J} : H^{\otimes k} \to \overline{H}^{\otimes k}$ is the conjugation (which is a complete isometry). By injectivity of the Haagerup tensor product and by Lemma 4 we deduce that

$$\|a^* \otimes a\mathcal{J}\|_{cb} \leq C_q$$

Then

$$\|U_k\|_{cb} \leq \|M\|_{cb} \|a^* \otimes a\mathcal{J}\|_{cb} \leq C_q$$

□

Recall that, by definition, $\Gamma_T^\infty(H_{\mathbb{R}})$ is identified with $\Gamma_T(H_{\mathbb{R}})$ by the mapping sending ξ to $W(\xi)$. Thus $\Gamma_T^\infty(H_{\mathbb{R}})$ inherits the operator space structure of $\Gamma_T(H_{\mathbb{R}})$. In particular for all $n \geq 0$, $H^{\otimes n}$ will be equipped with the operator space structure of $E_n = \{W(\xi),\ \xi \in H^{\otimes n}\}$.

Theorem 1 below was first obtained via elementary, but long, computations. In the version presented here, we have chosen to follow an approach indicated to us by Eric Ricard. This approach is much more transparent but involves some notions of operator space theory.

Theorem 1 *Let K be a complex Hilbert space. Then for all $n \geq 0$ and for all $\xi \in B(K) \otimes_{\min} H^{\otimes n}$ we have*

$$\max_{0 \leq k \leq n} \|(Id \otimes R_{n,\,k}^*)(\xi)\| \leq \|(Id \otimes W)(\xi)\|_{\min} \leq C_q(n+1) \max_{0 \leq k \leq n} \|(Id \otimes R_{n,\,k}^*)(\xi)\| \qquad (13)$$

where Id denotes the identity mapping of $B(K)$, and where the norm $\|(Id \otimes R_{n,\,k}^*)(\xi)\|$ is that of $B(K) \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}$.

Proof : For the second inequality, we use the Wick formula :

$$W\,|_{H^{\otimes n}} = \sum_{k=0}^{n} U_k\, R_{n,k}^*.$$

Let $\xi \in B(K) \otimes_{\min} H^{\otimes n}$, then by corollary 1

$$\|(Id \otimes W)(\xi)\|_{\min} \leq C_q \sum_{k=0}^{n} \|(Id \otimes R_{n,\,k}^*)(\xi)\|$$

which yields the majoration.

For the minoration, for $x \in H_c^{\otimes n-k} \otimes H_r^{\otimes k} \subset B(\overline{H}^{\otimes k}, H^{\otimes n-k})$, we claim that

$$P_{n-k} U_k(x)\,\big|_{H^{\otimes k}} = x(U\mathcal{J}) \qquad (14)$$

where P_{n-k} is the projection on tensors of rank $n-k$ in $\mathcal{F}_T(H_{\mathbb{C}})$. Assuming this claim and recalling that U and $\mathcal{J}$ are (anti)-isometry, we get that for any $x \in B(K) \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}$

$$\|x\|_{B(K)\otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}} \leq \|P_{n-k}\|_{B(\mathcal{F}_T(H_{\mathbb{C}}))} \|(Id \otimes U_k)(x)\|_{B(K)\otimes_{\min} B(\mathcal{F}_T(H_{\mathbb{C}}))}$$

The conclusion follows applying this inequality to $x = (Id \otimes R^*_{n,\,k})(\xi)$

To prove (14), it suffices to consider an elementary tensor product with entries in any basis of H, say $x = e_{\underline{i}} \otimes e_{\underline{j}}$. Consider $e_{\underline{l}} \in H^{\otimes k}$, a length argument gives that $a(\mathcal{J} e_{\underline{j}}).e_{\underline{l}}$ is of the form $\lambda\Omega$, with

$$\lambda = \langle a(\mathcal{J} e_{\underline{j}}).e_{\underline{l}}, \Omega\rangle = \langle e_{\underline{l}}, \mathcal{J} U e_{\underline{j}}\rangle$$

We deduce that

$$P_{n-k} U_k(e_{\underline{i}} \otimes e_{\underline{j}}).e_{\underline{l}} = \langle e_{\underline{l}}, U\mathcal{J} e_{\underline{j}}\rangle\, e_{\underline{i}}.$$

On the other hand, viewing x as an operator, we compute

$$x(\mathcal{J}U).e_{\underline{l}} = x.(\mathcal{J}U e_{\underline{l}}) = \langle e_{\underline{j}}, \mathcal{J}U e_{\underline{l}}\rangle\, e_{\underline{i}}$$

But since U is unitary and $\mathcal{J}$ antiunitary,

$$\langle e_{\underline{j}}, \mathcal{J}U e_{\underline{l}}\rangle = \langle e_{\underline{l}}, U\mathcal{J} e_{\underline{j}}\rangle$$

This ends the proof. □

The following theorem is our main result.

Theorem 2 $\Gamma_T(H_{\mathbb{R}})$ *is not injective as soon as* $\dim(H_{\mathbb{R}}) \geq 2$.

Proof : Let $d \leq \dim H_{\mathbb{R}}$. For all $n \geq 0$, $(\xi_{\underline{i}})_{|\underline{i}|=n}$ will denote a real orthonormal family of $H^{\otimes n}$ equipped with the T-scalar product of cardinal d^n. For example one can take $\xi_{\underline{i}} = (P_T^{(n)})^{-\frac{1}{2}} e_{\underline{i}}$.

Suppose that $\Gamma_T(H_{\mathbb{R}})$ is injective. Fix $n \geq 1$. By injectivity we have,

$$\tau(\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})^* W(\xi_{\underline{i}})) \leq \| \sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}})\|$$

It is clear that

$$\tau(\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})^* W(\xi_{\underline{i}})) = d^n$$

On the other hand, applying twice (13) consecutively

$$\| \sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})} \otimes W(\xi_{\underline{i}})\| \leq (n+1)^2 C_q^2 \max_{0\leq k,k'\leq n} \{\| \sum_{|\underline{i}|=n} \overline{R^*_{n,k'}(\xi_{\underline{i}})} \otimes R^*_{n,k}(\xi_{\underline{i}})\|\}$$

The norms are computed in $\overline{H_c}^{\otimes n-k'} \otimes_{\min} \overline{H_r}^{\otimes k'} \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}$ for fixed k and k'. We can rearrange this tensor product and use the comparison with the Hilbert Schmidt norm : Let $t = \sum_{|\underline{i}|=n} \overline{R^*_{n,k'}(\xi_{\underline{i}})} \otimes R^*_{n,k}(\xi_{\underline{i}})$,

$$\begin{aligned} \|t\|_{\overline{H_c}^{\otimes n-k'} \otimes_{\min} \overline{H_r}^{\otimes k'} \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}} &= \|t\|_{(\overline{H}^{\otimes n-k'} \otimes_2 H^{\otimes n-k})_c \otimes_{\min} (\overline{H}^{\otimes k'} \otimes_2 H^{\otimes k})_r} \\ &\leq \|t\|_{(\overline{H}^{\otimes n-k'} \otimes_2 H^{\otimes n-k}) \otimes_2 (\overline{H}^{\otimes k'} \otimes_2 H^{\otimes k})} \end{aligned}$$

$$\leq \quad \|t\|_{\overline{H}^{\otimes n-k'}\otimes_2\overline{H}^{\otimes k'}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}}$$

Finally, we use the estimates on $R^*_{n,k}$:

$$\begin{aligned}\|t\|_{\overline{H_c}^{\otimes n-k'}\otimes_{\min}\overline{H_r}^{\otimes k'}\otimes_{\min}H_c^{\otimes n-k}\otimes_{\min}H_r^{\otimes k}} &\leq \|t\|_{\overline{H}^{\otimes n-k'}\otimes_2\overline{H}^{\otimes k'}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}} \\ &\leq C_q\|\sum_{|\underline{i}|=n}\overline{\xi_{\underline{i}}}\otimes\xi_{\underline{i}}\|_{\overline{H}^n\otimes_2 H^n}\end{aligned}$$

But by the choice of $\xi_{\underline{i}}$: $\|\sum_{|\underline{i}|=n}\overline{\xi_{\underline{i}}}\otimes\xi_{\underline{i}}\|_{\overline{H}^n\otimes_2 H^n} = d^{n/2}$.

Combining all inequalities above, we deduce

$$d^n \leq C_q^3(n+1)^2 d^{n/2}$$

which yields a contradiction when n tends to infinity as soon as $d \geq 2$. $\square$

Let $C^*_T(H_{\mathbb{R}})$ be the C^*-algebra generated by all gaussians $G(f)$ for $f \in H_{\mathbb{R}}$. The preceding theorem implies directly that $C^*_T(H_{\mathbb{R}})$ is not nuclear as soon as $\dim(H_{\mathbb{R}}) \geq 2$ (cf. [8] Corollary 6.5). Actually the preceding argument can be modified to prove that $C^*_T(H_{\mathbb{R}})$ does not have the weak expectation property as soon as $\dim H_{\mathbb{R}} \geq 2$. Recall that a C^*-algebra A has the weak expectation property (WEP in short) if and only if the canonical inclusion $A \to A^{**}$ factorizes completely contractively through $B(K)$ for some complex Hilbert space K. By the results of Haagerup (cf. [14] Chapter 15) a C^*-algebra A has the WEP if and only if for all finite family $x_1, \ldots, x_n$ in A

$$\|\sum_{i=1}^n x_i\otimes\overline{x_i}\|_{A\otimes_{\max}\overline{A}} = \|\sum_{i=1}^n x_i\otimes\overline{x_i}\|_{A\otimes_{\min}\overline{A}} \tag{15}$$

Corollary 2 $C^*_T(H_{\mathbb{R}})$ *does not have the WEP as soon as* $\dim H_{\mathbb{R}} \geq 2$.

Proof : Let us use the same notations as in the preceding proof and suppose that $C^*_T(H_{\mathbb{R}})$ has the WEP. Fix $n \geq 1$, by (15) we have

$$\|\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})\otimes\overline{W(\xi_{\underline{i}})}\|_{C^*_T(H_{\mathbb{R}})\otimes_{\max}\overline{C^*_T(H_{\mathbb{R}})}} \leq \|\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})\otimes\overline{W(\xi_{\underline{i}})}\|_{C^*_T(H_{\mathbb{R}})\otimes_{\min}\overline{C^*_T(H_{\mathbb{R}})}} \tag{16}$$

To estimate from below the left handside of (16) observe that $\Phi :\ \overline{C^*_T(H_{\mathbb{R}})} \to C^*_T(H_{\mathbb{R}})'$ taking $\overline{W(\xi)}$ to $\mathcal{J}UW(\xi)\mathcal{J}U = W_r(\mathcal{J}U\xi)$ is a $*$- representation. Thus

$$\begin{aligned}\|\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})\otimes\overline{W(\xi_{\underline{i}})}\|_{C^*_T(H_{\mathbb{R}})\otimes_{\max}\overline{C^*_T(H_{\mathbb{R}})}} &= \|\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})\otimes W_r(\mathcal{J}U\xi_{\underline{i}})\|_{C^*_T(H_{\mathbb{R}})\otimes_{\max}C^*_T(H_{\mathbb{R}})'} \\ &\geq \|\sum_{|\underline{i}|=n} W(\xi_{\underline{i}})W_r(\mathcal{J}U\xi_{\underline{i}})\|_{B(\mathcal{F}_T(H_{\mathbb{C}}))} \\ &\geq \sum_{|\underline{i}|=n}\langle \mathcal{J}U\xi_{\underline{i}}, W(\xi_{\underline{i}})^*\Omega\rangle_T \\ &\geq \sum_{|\underline{i}|=n}\langle \mathcal{J}U\xi_{\underline{i}}, W(\mathcal{J}U\xi_{\underline{i}})\Omega\rangle_T \\ &\geq \sum_{|\underline{i}|=n}\|\mathcal{J}U\xi_{\underline{i}}\|_T^2 = d^n\end{aligned}$$

Then we can finish the proof as for Theorem 2. $\square$

Remark : Non nuclearity of $C_T^*(H_{\mathbb{R}})$ is equivalent to the fact that $C_T^*(H_{\mathbb{R}})$ does not have the completely positive approximation property as soon as $\dim(H_{\mathbb{R}}) \geq 2$. However it is possible to prove that $C_T^*(H_{\mathbb{R}})$ has the metric approximation property, by truncation of the Ornstein-Uhlenbeck semigroup. Arguing by duality and interpolation, it is not difficult to show that $L^p(\Gamma_T(H_{\mathbb{R}}))$ has the metric approximation property for $1 \leq p < \infty$. However, at the time of this writing, we are not able to prove that $C_T^*(H_{\mathbb{R}})$ has the completely bounded approximation property.

4 The case of the $q-$Araki-Woods algebras

For this last section we mainly refer to [12] where the $q-$Araki-Woods algebras are defined as a generalization of the $q-$ deformed case of Bożejko and Speicher on the one hand, and the quasi-free case of Shlyakhtenko (cf. [16]) on the other. More precisely, let $H_{\mathbb{R}}$ be a real Hilbert space given with U_t, a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}$. U_t can be extended to a unitary group on the complexification $H_{\mathbb{C}}$. Let A be its positive non-singular generator on $H_{\mathbb{C}}$: $U_t = A^{it}$. A new scalar product $\langle\ .\ ,\ .\ \rangle_U$ is defined on $H_{\mathbb{C}}$ by the following relation :

$$\langle \xi, \eta \rangle_U = \langle 2A(1+A)^{-1}\xi, \eta \rangle$$

We will denote by H the completion of $H_{\mathbb{C}}$ with respect to this new scalar product.

For a fixed $q \in]-1,1[$, we now consider the $q-$deformed Fock space associated with H and we denote it by $\mathcal{F}_q(H)$. Recall that it is the Fock space with the following Yang-Baxter deformation T defined by :

$$\begin{aligned} T : H \otimes H &\longrightarrow H \otimes H \\ \xi \otimes \eta &\longmapsto q\eta \otimes \xi \end{aligned}$$

Or equivalently, for every $n \geq 2$ and $\sigma \in S_n$ we have

$$\varphi(\sigma) = q^{i(\sigma)} U_\sigma$$

where $i(\sigma)$ denotes the number of inversions of the permutation σ and U_σ is the unitary on $H^{\otimes n}$ defined by

$$U_\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}$$

In this setting, the $q-$Araki-Woods algebra is the following von Neumann algebra

$$\Gamma_q(H_{\mathbb{R}}, U_t) = \{G(h), h \in H_{\mathbb{R}}\}'' \subset B(\mathcal{F}_q(H_{\mathbb{C}}))$$

Let $H'_{\mathbb{R}} = \{g \in H, \langle g, h \rangle_U \in \mathbb{R} \text{ for all } h \in H_{\mathbb{R}}\}$ and

$$\Gamma_{q,r}(H'_{\mathbb{R}}, U_t) = \{G_r(h), h \in H'_{\mathbb{R}}\}''$$

where $G_r(h)$ is the right gaussian corresponding to the right creation operator.
Since $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t) \subset \Gamma_q(H_{\mathbb{R}}, U_t)'$, $\overline{H_{\mathbb{R}} + iH_{\mathbb{R}}} = H$ and $\overline{H'_{\mathbb{R}} + iH'_{\mathbb{R}}} = H$ (cf. [16]), it is easy to

deduce that Ω is cyclic and separating for both $\Gamma_q(H_{\mathbb{R}}, U_t)$ and $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$. So Tomita's theory can apply : recall that the anti-linear operator S is the closure of the operator defined by :

$$S(x\Omega) = x^*\Omega \quad \text{for all } x \in \Gamma_q(H_{\mathbb{R}}, U_t)$$

Let $S = J\Delta^{\frac{1}{2}}$ be its polar decomposition. J and Δ are called respectively the modular conjugation and the modular operator. The following explicit formulas hold (cf. [12] and [16])

$$S(h_1 \otimes \cdots \otimes h_n) = h_n \otimes \cdots \otimes h_1 \quad \text{for all } h_1, \dots, h_n \in H_{\mathbb{R}}$$

Δ is the closure of the operator $\bigoplus_{n=0}^{\infty}(A^{-1})^{\otimes n}$ and

$$J(h_1 \otimes \cdots \otimes h_n) = A^{-\frac{1}{2}}h_n \otimes \cdots \otimes A^{-\frac{1}{2}}h_1 \quad \text{for all } h_1, \dots, h_n \in H_{\mathbb{R}} \cap \mathrm{dom}A^{-\frac{1}{2}}$$

By Tomita's theory, we have

$$\Gamma_q(H_{\mathbb{R}}, U_t)' = J\Gamma_q(H_{\mathbb{R}}, U_t)J$$

Let $h \in H_{\mathbb{R}}$, as in [16] we have $Jh \in H_{\mathbb{R}'}$, then, since Ω is separating for $\Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$, we obtain that $JG(h)J = G_r(Jh) \in \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$, so that

$$\Gamma_q(H_{\mathbb{R}}, U_t)' = \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)$$

Moreover, if $\xi \in \Gamma_q(H_{\mathbb{R}}, U_t)\Omega$, then $J\xi \in \Gamma_{q,r}(H'_{\mathbb{R}}, U_t)\Omega$ and since Ω is separating, we get $JW(\xi)J = W_r(J\xi)$.

Recall that if U_t is non trivial, the vacuum expectation φ is no longer tracial and is called the $q-$quasi-free state. In fact in most cases (cf. [12] Theorem 3.3), Araki-Woods factors are type III von Neumann algebras.

When A is bounded, it is clear that our preliminaries are still valid with minor changes. For example we should get an extra $\|A^{-1}\|^{k/2} = \|A\|^{k/2}$ in the estimation of $\|U_k\|$. Note, in particular, that the Wick formula, as stated in Lemma 2, is still true, and that the following analogue of Bożejko's scalar inequality holds : (proved in [12])
If A is bounded, $(\eta_u)_{u\in U}$ is a family of vectors in $H^{\otimes n}$ and $(\alpha_u)_{u\in U}$ a finitely supported family of complex numbers then :

$$\left\|\sum_{u\in U}\alpha_u\eta_u\right\|_q \leq \left\|\sum_{u\in U}\alpha_u W(\eta_u)\right\| \leq C_{|q|}^{\frac{3}{2}}\frac{\|A\|^{\frac{n+1}{2}}-1}{\|A\|^{\frac{1}{2}}-1}\left\|\sum_{u\in U}\alpha_u\eta_u\right\|_q \tag{17}$$

It is also a straightforward verification that Lemma 4, still hold in this setting. Observe also that U is a unitary on $\mathcal{F}_q(H)$: this follows from the fact that for every $n \geq 1$, $P_q^{(n)}$, $A^{\otimes n}$ and U commute on $H^{\otimes n}$. Note that $\mathcal{J}$ is no more an anti unitary from $H^{\otimes k}$ to $\overline{H^{\otimes k}}$, but since $U_k(I \otimes S) = M(a^* \otimes aU)$, we can deduce, as in the proof of Corollary 1, that $U_k(I \otimes S)$: $H_c^{\otimes n-k} \otimes_h \overline{H_r^{\otimes k}} \to B(\mathcal{F}_T(H_{\mathbb{C}}))$ is completely bounded with norm less than C_q, where I stands for the identity of $H_c^{\otimes n-k}$. Following the same lines as in the proof of Theorem 1 we get :

Theorem 3 *Assume A is bounded. Let K be a complex Hilbert space. Then for all $n \geq 0$ and for all $\xi \in B(K) \otimes_{\min} H^{\otimes n}$ we have*

$$\max_{0\leq k\leq n}\|(Id \otimes ((I \otimes S)R^*_{n,\,k})(\xi)\| \leq \|(Id \otimes W)(\xi)\|_{\min} \tag{18}$$

$$\leq C_q(n+1) \max_{0\leq k\leq n} \|(Id \otimes ((I\otimes S)R^*_{n,\,k})(\xi)\|$$

where Id denotes the identity mapping of $B(K)$, I the identity of $H_{\underline{c}}^{\otimes n-k}$, and where the norms of the left and right handsides are taken in $B(K) \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} \overline{H_r^{\otimes k}}$.

It is known (cf. [12]) that if U_t has a non trivial continuous part then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is not injective. Using our techniques we are able to state a non-injectivity criterion similar to that of [12] but independent of q.

Corollary 3 *If either*

$$\dim E_A\left(\{1\}\right) H_{\mathbb{C}} \geq 2$$

or for some $T > 1$

$$\frac{\dim E_A\left(]1,T]\right) H_{\mathbb{C}}}{T^2} > \frac{1}{2}$$

where E_A is the spectral projection of A, then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is non injective.

Proof : We can assume that U_t is almost periodic, then we can write

$$(H_{\mathbb{R}},\ U_t) = (\hat{H}_{\mathbb{R}},\ \mathrm{Id}_{\hat{\mathrm{H}}_{\mathbb{R}}}) \bigoplus_{\alpha\in\Lambda} (H_{\mathbb{R}}^{(\alpha)},\ U_t^{(\alpha)})$$

where

$$H_{\mathbb{R}}^{(\alpha)} = \mathbb{R}^2, \quad U_t^{(\alpha)} = \begin{pmatrix} \cos(t\ln\lambda_\alpha) & -\sin(t\ln\lambda_\alpha) \\ \sin(t\ln\lambda_\alpha) & \cos(t\ln\lambda_\alpha) \end{pmatrix}, \quad \lambda_\alpha > 1$$

Thus the eigenvalues of the generator $A^{(\alpha)}$ of $U_t^{(\alpha)}$ are λ_α and λ_α^{-1}.

If $\dim E_A\left(\{1\}\right) H_{\mathbb{C}} \geq 2$ then $\dim \hat{H}_{\mathbb{R}} \geq 2$ and since U_t is trivial on $\hat{H}_{\mathbb{R}}$, the non-injectivity follows from Theorem 2.

For the remaining case we first suppose that $\dim H_{\mathbb{R}} = 2$, U_t is not trivial and that $\Gamma_q(H_{\mathbb{R}}, U_t)$ is injective. For all $n \geq 1$, $A^{\otimes n}$ is a positive operator on $H^{\otimes n}$ equipped with the deformed scalar product, we will denote by λ and λ^{-1} the eigenvalues of A with $\lambda > 1$ and by $(\xi_{\underline{i}})_{|\underline{i}|=n}$ an orthonormal basis of eigenvectors of $A^{\otimes n}$ associated to the eigenvalues $(\lambda_{\underline{i}})_{|\underline{i}|=n}$. Since $\Gamma_q(H_{\mathbb{R}}, U_t)$ is semidiscrete we must have for every $n \geq 1$

$$\|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\| \leq \|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}}) \otimes W(\xi_{\underline{i}})\| = \|\sum_{|\underline{i}|=n} JW(\xi_{\underline{i}})J \otimes W(\xi_{\underline{i}})\|$$

It is easily seen that

$$\begin{aligned} \|\sum_{|\underline{i}|=n} W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\| &\geq \sum_{|\underline{i}|=n} \langle \Omega, W_r(J\xi_{\underline{i}})W(\xi_{\underline{i}})\Omega\rangle_q \\ &= \sum_{|\underline{i}|=n} \langle JW(\xi_{\underline{i}})^* J\Omega, W(\xi_{\underline{i}})\Omega\rangle_q \\ &= \sum_{|\underline{i}|=n} \langle \Delta^{\frac{1}{2}}\xi_{\underline{i}}, \xi_{\underline{i}}\rangle_q = \mathrm{Trace}\left(\left(A^{-\frac{1}{2}}\right)^{\otimes n}\right) = (\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}})^n \end{aligned}$$

On the other hand, the map from $J\Gamma_q(H_{\mathbb{R}},U_t)J$ to $\overline{\Gamma_q(H_{\mathbb{R}},U_t)}$ taking $JW(\xi)J$ to $\overline{W(\xi)}$ is a $*$-isomorphism, hence

$$\|\sum_{|\underline{i}|=n} JW(\xi_{\underline{i}})J\otimes W(\xi_{\underline{i}})\|_{\min} = \|\sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})}\otimes W(\xi_{\underline{i}})\|_{\min}$$

Applying (18) twice, and recalling that on $H^{\otimes k}$, $S = J\Delta^{\frac{1}{2}} = J(A^{\otimes k})^{-\frac{1}{2}}$ and that $J:\ \overline{H_r^{\otimes k}} \to H_r^{\otimes k}$ is completely isometric, we get

$$\|\sum_{|\underline{i}|=n} \overline{W(\xi_{\underline{i}})}\otimes W(\xi_{\underline{i}})\|_{\min} \le C_q^2(n+1)^2 \max_{0\le k,k'\le n} \|\sum_{|\underline{i}|=n} \overline{(I\otimes S)R^*_{n,\,k'}(\xi_{\underline{i}})}\otimes (I\otimes S)R^*_{n,\,k}(\xi_{\underline{i}})\|$$

$$\le C_q^2(n+1)^2 \max_{0\le k,k'\le n} \|\sum_{|\underline{i}|=n} \overline{(I\otimes (A^{\otimes k'})^{-\frac{1}{2}})R^*_{n,\,k'}(\xi_{\underline{i}})}\otimes (I\otimes (A^{\otimes k})^{-\frac{1}{2}})R^*_{n,\,k}(\xi_{\underline{i}})\|$$

Where the norms are computed in $\overline{H_c^{\otimes n-k'}} \otimes_{\min} \overline{H_r^{\otimes k'}} \otimes_{\min} H_c^{\otimes n-k} \otimes_{\min} H_r^{\otimes k}$. For a fixed (k,k'), let us denote by

$$t = \sum_{|\underline{i}|=n} \overline{(I\otimes (A^{\otimes k'})^{-\frac{1}{2}})R^*_{n,\,k'}(\xi_{\underline{i}})}\otimes (I\otimes (A^{\otimes k})^{-\frac{1}{2}})R^*_{n,\,k}(\xi_{\underline{i}})$$

As in the proof of Theorem 2, we have the following Hilbert-Schmidt estimate :

$$\|t\|_{\overline{H_c^{\otimes n-k'}}\otimes_{\min}\overline{H_r^{\otimes k'}}\otimes_{\min}H_c^{\otimes n-k}\otimes_{\min}H_r^{\otimes k}} \le \|t\|_{\overline{H^{\otimes n-k'}}\otimes_2\overline{H^{\otimes k'}}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}}$$

Recall that $R^*_{n,\,k}:\ H^{\otimes n}\to H^{\otimes n-k}\otimes_2 H^{\otimes k}$ is of norm less than $C_{|q|}^{\frac{1}{2}}$ and that $\|(A^{\otimes k})^{-\frac{1}{2}}\|_{B(H^{\otimes k})} = \lambda^{\frac{k}{2}}$. Hence,

$$\begin{aligned}
\|t\|_{\overline{H^{\otimes n-k'}}\otimes_2\overline{H^{\otimes k'}}\otimes_2 H^{\otimes n-k}\otimes_2 H^{\otimes k}} &\le C_{|q|}\lambda^n \|\sum_{|\underline{i}|=n} \overline{\xi_{\underline{i}}}\otimes\xi_{\underline{i}}\|_{\overline{H^{\otimes n}}\otimes H^{\otimes n}}\\
&\le C_{|q|}(\sqrt{2}\lambda)^n
\end{aligned}$$

Combining all inequalities we get

$$(\lambda^{\frac{1}{2}}+\lambda^{-\frac{1}{2}})^n \le C_{|q|}^3(n+1)^2(\sqrt{2}\lambda)^n.$$

We now return to the general case, we fix $T>1$ and we denote by $\lambda_1,\dots,\lambda_p$ the eigenvalues of A in $]1,T]$ counted with multiplicities. Thus we have $p = \dim E_A\,(]1,T])\,H_{\mathbb{C}}$. It is easy to deduce from our first step that for any $n\ge 1$ we have

$$(\sum_{i=1}^p \lambda_i^{\frac{1}{2}}+\lambda_i^{-\frac{1}{2}})^n \le C_{|q|}^3(n+1)^2(2p)^{\frac{n}{2}}T^n$$

Since for any i we have $\lambda_i^{\frac{1}{2}}+\lambda_i^{-\frac{1}{2}}\ge 2$ we deduce

$$(2p)^n \le C_{|q|}^3(n+1)^2(2p)^{\frac{n}{2}}T^n$$

So we necessarily have

$$\frac{2p}{T^2} \leq 1$$

that is to say

$$\frac{\dim E_A]1, T]H_{\mathbb{C}}}{T^2} \leq \frac{1}{2}$$

□

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