Study of a transport operator with unbounded coefficients

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Abstract

In this paper we study the transport operator given by the sum of a free-streaming term and of an absorption term. The velocity coefficient appearing in the definition of the operator is unbounded ($v \in \mathbb{R}$) while the position coordinate belongs to the one-dimensional bounded set $[-b, +b]$. The boundary condition associated to the transport operator are given by means of a bounded and positive operator Λ. We prove that an evolution problem associated to the above transport operator has a unique positive solution whose explicit form is given by means of a C_0-semigroup in the case $\|\Lambda\| \leq 1$ and by means of an integrated semigroup or a C-semigroup in the case $\|\Lambda\| > 1$.

1 Introduction

In [10], [11] and [12] we have studied the generation properties of a linear transport operator equipped with *general* boundary conditions by means of classical semigroup techniques.

In particular, in [10] we have proved the generation of a C_0-semigroup when the position coordinate x belongs to the one-dimensional set $[-b, +b]$ and the velocity has a fixed modulus. On the other hand, the *classical* models for charged particles transport consider unbounded velocities ([13]). Thus, in [11] and [12] we have considered the free-streaming transport operator with velocity $v \in \mathbb{R}$. In particular, in [12] the generation of a C_0-semigroup was proved in the case of boundary conditions given by means of a positive linear bounded operator Λ with $\|\Lambda\| \leq 1$ (dissipative and conservative "general" boundary conditions).

It is not possible to extend the techniques used to study dissipative and conservative boundary conditions to the case of multiplying boundary conditions ($\|\Lambda\| > 1$) . In fact, the assumption $\|\Lambda\| \leq 1$ plays a fundamental role in order to prove the generation of the semigroup by using the classical theorem of Hille-Yosida.

On the other hand, if multiplying boundary conditions are taken into account, an *absorption* term must be added to the free-streaming operator in order to avoid the 'blow up' of the system of particles. Hence, in this paper we study the properties of the sum of the free-streaming operator with an absorption term proportional to the modulus of the velocity. We shall call it the *transport* operator. We consider general boundary conditions given by means of a linear, positive and bounded operator Λ which relates the *outgoing flux* of particles with the *incoming* one. The norm of Λ can be smaller, equal or bigger than one.

In section 2, we give the explicit form of the transport operator and we introduce the functional spaces needed in order to study its properties. Moreover, we prove that the resolvent operator $R(\lambda, L)$ is a positive operator and we give estimates of $\|R(\lambda, L)\|$ both in the cases $\|\Lambda\| \leq 1$ and $\|\Lambda\| > 1$. Section 3 is devoted to a summary of the definitions and the theorems related to the generation of C_0-semigroups, intergrated semigroups and C-semigroups. In section 4, we prove that in the case $\|\Lambda\| \leq 1$ the transport operator generates a C_0-semigroup, while in the case $\|\Lambda\| > 1$ the transport operator is the generator of an integrated semigroup and a C-semigroup for a particular operator C. We justify the study of such an operator describing a physical application. Moreover, under a specific assumption on the initial distribution function (i.e. the density function at time $t = 0$), we shall prove that the proposed evolution problem has a unique positive solution which can be expressed by means of the C_0-semigroup in the case $\|\Lambda\| \leq 1$ and of the integrated semigroup or the C-semigroup in the case $\|\Lambda\| > 1$. Finally, we give the relation between the integrated semigroup and the C-semigroup.

2 The transport operator

We consider the set $\Omega = [-b, +b] \times \mathbb{R}$ and introduce the *incoming* and *outgoing* sets:

$$\Omega^{in} = (\{-b\} \times (0, +\infty)) \cup (\{+b\} \times (-\infty, 0)),$$

and

$$\Omega^{out} = (\{-b\} \times (-\infty, 0)) \cup (\{+b\} \times (0, +\infty)).$$

Further, we introduce the functional space:

$$X = L^1(\Omega; dxdv) \ , \quad \|f\| = \int_\Omega |f(x,v)| \, dx \, dv,$$

and denote by X^+ the positive cone of X.

We also define the *incoming* and *outgoing* functional spaces:

$$X^{in} = L^1(\Omega^{in}; |v|dv), \quad X^{out} = L^1(\Omega^{out}; |v|dv), \tag{1}$$

with norms respectively given by:

$$\|f^{in}\|_{in} = \int_0^{+\infty} v|f(-b,v)| \, dv + \int_{-\infty}^0 |v| \, |f(b,v)| \, dv, \tag{2}$$

and

$$\|f^{out}\|_{out} = \int_{-\infty}^0 |v| \, |f(-b,v)| \, dv + \int_0^{+\infty} v|f(b,v)| \, dv, \tag{3}$$

where f^{in}, f^{out} are the traces of the function f on the spaces X^{in} and X^{out}, respectively:

$$f^{in} = f|_{X^{in}} , \quad f^{out} = f|_{X^{out}}.$$

We shall denote by f_i^{in} and f_i^{out}, where $i = 1, 2$, the i-th *component* of the functions f^{in} and f^{out} belonging to the component spaces X_i^{out} and X_i^{in}, and by $\|\cdot\|_{i,in}$ and $\|\cdot\|_{i,out}$, $i = 1, 2$, the associated norms. For instance,

$$\|f_1^{in}\|_{1,in} = \int_0^{+\infty} v|f(-b, v)|\, dv,$$

denotes the first term on the right hand side of the norm $\|\cdot\|_{in}$ given by (2). In other words, we may interpret X^{in} as $X_1^{in} \times X_2^{in}$ and X^{out} as $X_1^{out} \times X_2^{out}$. Thus:

$$X^{in} = L^1((\{-b\} \times (0, +\infty)); vdv) \times L^1(\{+b\} \times (-\infty, 0)); |v|dv) = X_1^{in} \times X_2^{in},$$

$$X^{out} = L^1((\{-b\} \times (-\infty, 0)); |v|dv) \times L^1(\{+b\} \times (0, +\infty)); vdv) = X_1^{out} \times X_2^{out},$$

$$f^{in} = \begin{pmatrix} f_1^{in} \\ f_2^{in} \end{pmatrix}, \quad f^{out} = \begin{pmatrix} f_1^{out} \\ f_2^{out} \end{pmatrix}, \tag{4}$$

where $f_1^{in} = f(-b, v)$, for $v > 0$, $f_2^{in} = f(b, v)$, for $v < 0$, and $f_1^{out} = f(-b, v)$, for $v < 0$, $f_2^{out} = f(b, v)$, for $v > 0$.

Our goal is the study of the transport operator:

$$Lf = -v\frac{\partial f}{\partial x} - |v|\sigma f,$$

$$D(L) = \left\{ f \in X,\ v\frac{\partial f}{\partial x} \in X,\ |v|\sigma f \in X,\ f^{in} \in X^{in},\ f^{out} \in X^{out},\ f^{in} = \Lambda f^{out} \right\}, \tag{5}$$

$$R(L) \subset X,$$

where σ is a suitable positive constant and Λ is a bounded positive operator such that

$$D(\Lambda) = X^{out} , \quad R(\Lambda) \subset X^{in}.$$

Remark 2.1 We note that, by using a suitable operator Λ, we can describe any kind of boundary conditions. For example the reflection conditions:

$$\begin{aligned} f(-b, v) &= \alpha f(-b, -v),\ v > 0, \\ f(b, v) &= \alpha f(b, -v),\ v < 0, \end{aligned} \tag{6}$$

where $\alpha > 0$ is a suitable constant, can be written in the following way:

$$f^{in} = \Lambda f^{out}$$

with

$$\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad D(\Lambda) = X^{out} , \quad R(\Lambda) \subset X^{in}. \tag{7}$$

Now we prove some results on the resolvent operator of L which we shall use in order to study the semigroup generation properties of L.

Let $\lambda > 0$ and define the following linear operator :

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}, \quad D(A) = X^{in}, \quad R(A) \subset X^{out}, \tag{8}$$

where the operators A_{12} and A_{21} are defined by:

$$\begin{aligned} A_{12} f_2^{in} &= e^{2b\lambda/v}\, e^{-2b\sigma} f_2^{in} \quad , \quad D(A_{12}) = X_2^{in} \quad , \quad R(A_{12}) \subset X_1^{out}, \\ A_{21} f_1^{in} &= e^{-2b\lambda/v}\, e^{-2b\sigma} f_1^{in} \quad , \quad D(A_{21}) = X_1^{in} \quad , \quad R(A_{21}) \subset X_2^{out}. \end{aligned} \tag{9}$$

Let also define:

$$\mathcal{A} = \begin{pmatrix} 0 & \mathcal{A}_{12} \\ \mathcal{A}_{21} & 0 \end{pmatrix}, \quad D(\mathcal{A}) = X^{in}, \quad R(\mathcal{A}) \subset X, \tag{10}$$

where the operators $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ are defined by:

$$\begin{aligned} \mathcal{A}_{12} f_2^{in} &= e^{(x+b)\delta} f_2^{in} \;, \quad D(\mathcal{A}_{12}) = X_2^{in} \;, \quad v < 0, \\ \mathcal{A}_{21} f_1^{in} &= e^{(x-b)\delta} f_1^{in} \;, \quad D(\mathcal{A}_{21}) = X_1^{in} \;, \quad v > 0, \end{aligned} \tag{11}$$

with $\delta = \delta(v) = -(\lambda + |v|\sigma)/v$.

Lemma 2.1 The operator A defined by (8) is bounded and:

$$\|A\| < \exp(-2b\sigma). \tag{12}$$

Proof:
From the definition of the spaces X^{in} and X^{out} we obtain, for each $\lambda > 0$, that:

$$\|Af^{in}\|_{out} = \int_{-\infty}^{0} |v||f_2^{in}(v)| \exp(2b\lambda/v)\, \exp(-2b\sigma)\, dv$$

$$+ \int_{0}^{+\infty} |v||f_1^{in}(v)| \exp(-2b\lambda/v) \exp(-2b\sigma)\, dv$$

$$\leq \exp(-2b\sigma) \left(\int_{-\infty}^{0} |v||f_2^{in}(v)|\, dv + \int_{0}^{\infty} |v||f_1^{in}(v)|\, dv \right) = \exp(-2b\sigma)\|f^{in}\|_{in}.$$

$\blacksquare$

Lemma 2.2 The operator $\mathcal{A}$ given by (10) is bounded and:

$$\|\mathcal{A}\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Proof:
We first remark that, if $v > 0$:

$$\int_{-b}^{+b} \exp\left(-(x+b)\frac{\lambda+|v|\sigma}{v}\right) dx = \frac{-v}{\lambda+|v|\sigma}\left(\exp\left(\frac{-2b}{v}(\lambda+|v|\sigma)\right)-1\right) \leq \frac{v}{\lambda+|v|\sigma}.$$

Analogously, if $v < 0$:

$$\int_{-b}^{+b} \exp\left(-(x-b)\frac{\lambda+|v|\sigma}{v}\right) dx \leq -\frac{v}{\lambda+|v|\sigma} = \frac{|v|}{\lambda+|v|\sigma}.$$

Since f_1^{in} and f_2^{in} do not depend on x, we have that:

$$\begin{aligned}\|\mathcal{A}f\| &= \int_{-b}^{+b}\int_{-\infty}^{0} |f_2^{in}| \exp((x-b)\delta)\, dx\, dv + \int_{-b}^{+b}\int_{0}^{+\infty} |f_1^{in}| \exp((x+b)\delta)\, dx\, dv \\ &\leq \int_{-\infty}^{0} \frac{|v|}{\lambda+|v|\sigma}|f_2^{in}|\, dv + \int_{0}^{+\infty} \frac{v}{\lambda+|v|\sigma}|f_1^{in}|\, dv \leq \frac{1}{\lambda}\|f^{in}\|_{in},\end{aligned}$$

and the lemma is proved. ■

In the next theorem we prove that the resolvent operator $R(\lambda, L)$ of the operator L exists for each $\lambda > 0$ and maps the positive cone of X into itself, that is $R(\lambda, L)$ is a positive operator.

Theorem 2.1 If
(i) the norm of Λ is smaller or equal to 1,
or
(ii) the norm of Λ is larger than 1, and

$$\sigma > \frac{\ln\|\Lambda\|}{2b}, \tag{13}$$

then the resolvent $R(\lambda, L)$ of the operator L exists for each $\lambda > 0$ and is a positive operator.

Proof:
Consider the resolvent equation for the operator L:

$$\lambda f - Lf = g,$$

that is:

$$v\frac{\partial f}{\partial x} + (\lambda + |v|\sigma)f = g, \tag{14}$$

where g is a given element of X and the unknown f must be sought in $D(L)$. Integrating equation (14) with respect to x, we obtain:

$$f(x,v) = \begin{cases} \exp((x+b)\delta)C_1 + \dfrac{1}{v}\displaystyle\int_{-b}^{x} \exp((x-x')\delta)g(x',v)\, dx'\ , \ v > 0 \\ \\ \exp((x-b)\delta)C_2 - \dfrac{1}{v}\displaystyle\int_{x}^{+b} \exp((x-x')\delta)g(x',v)\, dx'\ , \ v < 0 \end{cases} \tag{15}$$

where $\delta = \delta(v) = -(\lambda + |v|\sigma)/v$ and $C_1 = C_1(v)$ and $C_2 = C_2(v)$ are to be determined by means of the boundary conditions defined in (5).

Evaluating $f(x,v)$ on the boundary we get:

$$f_1^{in} = f(-b,v) = C_1(v), \quad v > 0, \tag{16}$$

$$f_2^{in} = f(+b,v) = C_2(v), \quad v < 0, \tag{17}$$

$$f_2^{out} = f(+b,v) = \exp(2b\delta)C_1(v) + \frac{1}{v}\int_{-b}^{+b} \exp((b-x')\delta)g(x',v)\,dx', \quad v > 0,$$

$$f_1^{out} = f(-b,v) = \exp(-2b\delta)C_2(v) - \frac{1}{v}\int_{-b}^{+b} \exp(-(b+x')\delta)g(x',v)\,dx', \quad v < 0.$$

Hence,

$$f^{out} = Af^{in} + G, \tag{18}$$

where A is defined in (8) and G is defined as follows:

$$G = G(v) = \begin{pmatrix} g_1(v) \\ g_2(v) \end{pmatrix} \in X^{out}, \tag{19}$$

with :

$$g_1(v) = -\frac{1}{v}\int_{-b}^{+b} \exp(-(b+x')\delta)g(x',v)\,dx', \quad v < 0$$

and

$$g_2(v) = \frac{1}{v}\int_{-b}^{+b} \exp((b-x')\delta)g(x',v)\,dx', \quad v > 0.$$

Taking into account the boundary conditions, we have from (18):

$$f^{in} = \Lambda f^{out} = \Lambda A f^{in} + \Lambda G,$$

and so:

$$f^{in} = (I - \Lambda A)^{-1}\Lambda G, \tag{20}$$

provided that the operator $(I - \Lambda A)^{-1}$ exists. This is the case if $\|\Lambda A\| < 1$.

The above condition is satisfied when $\|\Lambda\| \leq 1$, because $\|A\| < 1$ (see lemma 2.1). On the other hand, if $\|\Lambda\| > 1$ and the parameter σ fulfills conditions (13) we also have that $\|\Lambda A\| \leq \|\Lambda\| \exp(-2b\sigma) < 1$.

Hence, we can obtain f^{in} and so C_1 and C_2, from (16) and (17) as a function of g (see (19)). By substituting C_1 and C_2 into (15) we get the esplicit form of the solution of (14).

By taking into account that the integral term appearing in the righthand-side of (15) is the resolvent operator of the operator L_0 defined as follows:

$$L_0 f = Lf \;, \quad D(L_0) = \left\{ f \in X,\ v\frac{\partial f}{\partial x} \in X,\ |v|\sigma f \in X,\ f^{in} \in X^{in},\ f^{in} = 0 \right\}, \tag{21}$$

the solution f of the resolvent equation (14) reads:

$$f = R(\lambda, L)g = \mathcal{A}(I - \Lambda A)^{-1}\Lambda G + R(\lambda, L_0)g, \tag{22}$$

where the operator $\mathcal{A}$ is defined by (10).

It is not difficult to prove that $R(\lambda, L)$ is a positive operator because $\mathcal{A}, \Lambda, A, R(\lambda, L_0)$ are positive operators. ■

Now we give some estimates on the resolvent operator $R(\lambda, L)$.

Theorem 2.2 Let $||\Lambda|| \leq 1$, then the resolvent operator $R(\lambda, L)$ of the operator L is a bounded operator and satisfies the following conditions*:*

$$||R(\lambda, L)g|| \leq \frac{1}{\lambda}||g||, \quad \lambda > 0.$$

Proof:

Let $g \in X^+$ and $\lambda > 0$. By integrating (14) with respect to x and v and by taking into account that the solution f belongs to X^+ according to the above Theorem, we obtain

$$\lambda||f|| + \int_{-\infty}^{+\infty} [vf(b, v) - vf(-b, v)]dv + \int_{-b}^{b} dx \int_{-\infty}^{+\infty} |v|\sigma f(x, v)\, dv = ||g||. \tag{23}$$

Since $||f^{in}|| \leq ||f^{out}||$ because $||\Lambda|| \leq 1$, we obtain that

$$\int_{-\infty}^{+\infty} [vf(b, v) - vf(-b, v)]dv = -||f^{in}|| + ||f^{out}|| \geq 0.$$

Hence, we get from (23)

$$\lambda||f|| \leq ||g||,$$

and the theorem is proved. ■

Theorem 2.3 Let $||\Lambda|| > 1$, then the resolvent operator $R(\lambda, L)$ of the operator L is a bounded operator and satisfies the following conditions*:*

$$||R(\lambda, L)g|| \leq \frac{M}{\lambda}||g||,$$

with

$$M = \frac{||\Lambda||(1 - exp(-2b\sigma)) + 1}{1 - ||\Lambda||exp(-2b\sigma)},$$

where the parameter σ satisfies (13).

Proof:

The resolvent operator is bounded since is a positive operator defined on the whole space X ([15]).

To evaluate its norm it is enough to consider relation (22) and use Lemma 2.1 and Lemma 2.2. We have from (22) that:

$$
\begin{aligned}
\|R(\lambda, L)g\| &= \|\mathcal{A}(I - \Lambda A)^{-1}\Lambda G + R(\lambda, L_0)g\| \\
&\leq \frac{1}{\lambda}\frac{1}{1 - \|\Lambda\|\exp(-2b\sigma)}\|\Lambda\|\|g\| + \frac{1}{\lambda}\|g\| \\
&\leq \frac{1}{\lambda}\|g\|\left(\frac{\|\Lambda\|(1 - exp(-2b\sigma)) + 1}{1 - \|\Lambda\|exp(-2b\sigma)}\right) = \frac{M}{\lambda}\|g\|.
\end{aligned}
\tag{24}
$$

To get relation (24) we used the fact that $\|G\|_{out} \leq \|g\|$ as one can easily check and the well known inequality $\|R(\lambda, L_0)\| \leq \dfrac{1}{\lambda}, \forall\lambda > 0$. ■

3 Integrated semigroups and C-semigroups

Throughout this section X will be a Banach space and A a linear operator with domain $D(A) \subset X$ and range $R(A) \subset X$.

We shall write $\rho(A)$ for the resolvent set of A.

The space of bounded operators from X into X will be denoted by $\mathcal{B}(X)$. When the operator A generates a strongly continuous semigroup (C_0- semigroup) we shall write such a semigroup $\{exp(tA), t \geq 0\}$

We recall some definitions and theorems we need to solve the abstract Cauchy problem

$$
\begin{cases}
\dfrac{du(t)}{dt} = Au(t)\ , \quad t > 0, \\
u(0) = u_0\ .
\end{cases}
\tag{25}
$$

In (25) $u(t)$ is a function from $[0, +\infty)$ into X and the derivative is in the strong sense.

According to [6] we give the following definitions.

Definition 3.1 *A strong solution of (25) is a map $t \to u(t)$ such that $u(t) \in C^1([0, +\infty), X)$, $u(t) \in D(A)$, for $t \geq 0$ and $u(t)$ satisfies (25).*

Definition 3.2 *A mild solution of (25) is a map $t \to u(t)$ such that $u \in C([0, +\infty), X)$,*

$$
v(t) = \int_0^t u(s)\,ds \ \in D(A)\ , \quad t \geq 0,
$$

and

$$
\frac{dv}{dt} = Av(t) + u_0\ , \quad t \geq 0.
$$

Definition 3.3 *If $n \in \mathbb{N}$, an n-times integrated semigroup is a strongly continuous family of operators $\{S(t),\ t \geq 0\}$ such that:*

$$
S(0) = 0
$$

and

$$S(t)S(s) = \frac{1}{(n-1)!}\left(\int_t^{s+t} (s+t-r)^{n-1} S(r)\, dr \right.$$
$$\left. - \int_0^s (s+t-r)^{n-1} S(r)\, dr \right) , \quad \forall s,t \geq 0. \tag{26}$$

If $n = 1$, $S(t)$ is called a once integrated semigroup (or simply an integrated semigroup). $S(t)$ is called "non-degenerate" if:

$$S(t)f = 0 \,, \ \forall\, t \geq 0 \ \text{ implies } f = 0. \tag{27}$$

$S(t)$ is called "increasing" if:

$$0 = S(0) \leq S(s) \leq S(t) \,, \quad \forall\, t \geq s \geq 0. \tag{28}$$

$S(t)$ is called "exponentially bounded" if there exist M and $w \in \mathbb{R}$ such that:

$$\|S(t)\| \leq M e^{wt} \,, \quad \forall t \geq 0. \tag{29}$$

The generator A of the integrated semigroup $\{S(t),\, t \geq 0\}$ is defined as follows. If:

$$D(A) = \left\{ f \,:\, \exists\, g \text{ such that } S(t)f = \frac{t^n}{n!} f + \int_0^t S(r) g\, dr \,, \quad \forall t \geq 0 \right\},$$

then $Af = g$ by definition.

Definition 3.4 *Let $C \in \mathcal{B}(X)$ be an injective operator. A family $\{W(t),\, t \geq 0\} \subset \mathcal{B}(X)$ is called a C-regularized semigroup (briefly a C-semigroup) if:*

$$\begin{array}{ll} a) & W(0) = C, \\ b) & W(t)W(s) = CW(t+s) \,, \quad \forall\, s,t \geq 0. \end{array}$$

We say that an operator A generates $\{W(t),\, t \geq 0\}$ if

$$Af = C^{-1} \left[\lim_{t \to 0} \frac{1}{t} (W(t)f - Cf) \right],$$

with

$$D(A) = \{f :\ \text{the limit exists and belongs to } R(C)\}.$$

Note that in general, C^{-1} does not belong to $\mathcal{B}(X)$.

Finally, we give some definitions and theorems which ensure the existence of C_0-semigroup, integrated semigroup and C-semigroup and the relations among all these families of semigroups and the solution of (25).

Definition 3.5 ([1]) *We say that the positive cone X^+ of X is generating and normal if $X = X^+ - X^+$ (that is every $f \in X$ can be written as the sum of two elements g and h with $g \in X^+$ and $-h \in X^+$), and $X' = X'^+ - X'^+$ where X'^+ denotes the dual cone.*

Definition 3.6 ([1]) *We say that X is an ideal in X'' (the bidual of X) if for $f \in X$, $g \in X''$ such that $0 \leq g \leq f$ we have $g \in X$.*

Remark 3.1 From now on, we assume that the Banach space X has generating and normal cone and it is an ideal in X''. We note that the Banach space X defined in section 2 has generating and normal cone and it is an ideal in X''.

Definition 3.7 ([1]) *A linear operator A is resolvent positive if there exists $w \in \mathbb{R}$ such that $(w, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for every $\lambda > w$.*

If A is resolvent positive we introduce the following notation:

$$s(A) = inf\{w \in \mathbb{R},\ (w, \infty) \subset \rho(A),\ R(\lambda, A) \geq 0,\ \forall\, \lambda > w\}.$$

Theorem 3.1 ([8]) If A generates a C_0-semigroup in X and $u_0 \in D(A)$, then the solution of (25) is given by $u(t) = exp(tA)u_0$, $t \geq 0$.

Theorem 3.2 (Hille-Yosida theorem) ([8], [14]) If the following hold:
a) A is a closed operator;
b) A is densely defined ($\overline{D(A)} = X$);
c) there exist $w \in \mathbb{R}$ and $M \geq 0$ such that $(w, \infty) \subset \rho(A)$ and

$$\|R(\lambda, A)^k\| \leq \frac{M}{(\lambda - w)^k}\ , \quad k = 1, 2....;$$

then A generates a C_0-semigroup, $\{exp(tA), t \geq 0\}$ such that:

$$\|exp(tA)\| \leq Me^{wt}\ , \quad t \geq 0$$

and conversely.

Remark 3.2 The inf of the family of the numbers w appearing in c) of Theorem (3.2) is called $\omega(A)$ the type (or the growth bound) of the semigroup generated by A.

Corollary 3.1 If A satisfies the assumption of theorem 3.2 and is resolvent positive, then the semigroup exp(tA) is a positive operator.

Theorem 3.3 ([1]) Let A be a densely defined and resolvent positive operator. If there exists $\lambda_0 > s(A)$ and $c > 0$ such that

$$\|R(\lambda_0, A)f\| \geq c\|f\|\ , \quad \forall\, f \in X^+,$$

then A is the generator of a positive C_0- semigroup and $s(A) = \omega(A)$.

Remark 3.3 The converse of Theorem 3.3 is not true, see [1] for examples. Anyway, Theorem 3.3 is useful in applications when multiplying boundary conditions are considered and it is not possible to prove the estimate c) of the Hille-Yosida Theorem (see [10]).

Theorem 3.4 ([1]) Suppose that X is an ideal in X''. Let A be a resolvent positive operator, then there exists a unique strongly continuous family $\{S(t),\, t \geq 0\}$ of operators on X that is an increasing integrated semigroup. Moreover:

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}\, dS(t)\ , \quad \lambda > s(A).$$

The family $\{S(t),\, t \geq 0\}$ is called the integrated semigroup generated by A.

Theorem 3.5 ([1]) Let $\{S(t),\, t \geq 0\}$ be an increasing integrated semigroup. Then $S(t)$ is exponentially bounded.

Theorem 3.6 ([1]) Let A be a resolvent positive operator and let $\{S(t),\, t \geq 0\}$ the integrated semigroup generated by A, then:
a) if $s(A) \geq 0$ then $s(A) = inf\{w > 0 {:} \exists\ M \geq 0,\ \|S(t)\| \leq Me^{wt}\ \forall\, t \geq 0\}$;
b) if $s(A) < 0$ then $s(A) = inf\{w > 0 {:} \exists\ M \geq 0, \|R(0, A) - S(t)\| \leq Me^{wt}\ \forall\, t \geq 0\}$, and

$$\lim_{t\to\infty} S(t) = R(0, A) = -A^{-1}.$$

Theorem 3.7 ([1]) Assume that A is resolvent positive and either $D(A)$ dense in X or X is an ideal in X''. Then for every $u_0 \in D(A^2)$ there exists a unique solution of the problem (25). Furthermore, if $S(t)$ is the integrated semigroup generated by A such a solution reads:

$$u(t) = S(t)Au_0 + u_0\ , \quad t \geq 0.$$

Finally, if $u_0 \in X^+$ also $u(t) \in X^+$ for every $t \geq 0$.

Theorem 3.8 ([4], [5], [6]) If $\lambda \in \rho(A)$ and $n \in \mathbb{N}$, then the following statements are equivalent:
a) The problem (25) has a unique strong solution for all $u_0 \in D(A^{n+1})$;
b) The problem (25) has a unique mild solution for all $u_0 \in D(A^n)$;
c) The operator A generates an $(\lambda - A)^{-n}$ regularized semigroup $\{W(t),\, t \geq 0\}$;
d) The operator A generates a n-times integrated non-degenerate semigroup $\{S(t), t \geq 0\}$.
Furthermore, we have:

$$W(t)x = \left(\frac{d}{dt}\right)^n S(t)(\lambda - A)^{-n}x,$$

$$S(t)x = (\lambda - A)^n J^n W(t)x\ , \quad \forall\, x \in X,$$

where

$$Jf(t) = \int_0^t f(s)\, ds.$$

Theorem 3.9 ([2]) Let X be a Banach space with generating and normal cone X^+. Let the operator L has dense domain in X. If the resolvent set of L contains the interval $(a, +\infty)$ for some positive a and the resolvent $R(\lambda, L)$ is positive for $\lambda > a$, then L is the generator of an integrated semigroup $S(t)$, for every $t \geq 0$.

Finally, we quote this result from [4] (theorem 2.4).

Theorem 3.10 Assume that A generates a C-semigroup W(t), then if $f \in D(A), W(t)f$ is a differentiable function of t and $\frac{d}{dt}W(t)f = AW(t)f = W(t)Af$.

Remark 3.4 From the above theorem it follows that if we know that a solution of (25) exists, and $u_0 \in R(C)$, then we can write $u(t) = W(t)C^{-1}u_0$.

4 The transport problem

Let us now consider the following evolution model for charged or uncharged particles (see [3], [13]):

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} + |v|\sigma n = 0, \tag{30}$$

where the function $n = n(x, v, t)$ represents the density of the particles which at time t are in $x \in [-b, +b]$ with velocity $v \in \mathbb{R}$ and $\sigma > 0$ is the absorption cross section (i.e. the probability of a particle to be absorbed by the host medium). Equation (30) describes the evolution of a system of particles moving in a host medium capable to capture particles.

Equation (30) is usually equipped with the initial condition:

$$n(x, v, 0) = n_0(x, v), \tag{31}$$

and with the boundary conditions:

$$n^{in} = \Lambda n^{out}, \tag{32}$$

where $\|\Lambda\|$ may be smaller, equal or larger than 1 and Λ represents the interaction of the *outgoing* flux of particles with the boundaries (i.e. reflection, diffusion). Introducing the linear operator L, given by (5), the evolution problem (30)-(32) reads:

$$\begin{cases} \dfrac{dn(t)}{dt} = Ln(t)\ , \quad t > 0, \\ n(0) = n_0\ , \end{cases} \tag{33}$$

where now $n(t) = n(\cdot, \cdot, t)$ is a function from $[0, +\infty)$ with values in X, the Banach space defined in Section 2 and we assume that $n_0 \in D(L)$ or $n_0 \in D(L^2)$ with

$$D(L^2) = \{n \in D(L),\ Ln \in D(L)\}.$$

We remark that, if n is a particle density, then $n \in X^+$ and $\|n\|$ gives the total number of particles in the considered region. Moreover, $\|n^{in}\|_{in}$ and $\|n^{out}\|_{out}$ respectively are the total ingoing and outgoing fluxes at the boundaries, ([7]).

Now, we use the results of the preceding sections to prove that (33) has a unique solution and to write it.

We note that the operator L is resolvent positive from theorem 2.1 and has dense domain because $C_0^\infty \subset D(L)$.

Theorem 4.1 If $n_0 \in D(L)$ and $\|\Lambda\| \leq 1$, there exists a unique solution of the Cauchy problem (33) given by:

$$n(t) = exp(tL)n_0 \ , \quad \forall\, t \geq 0, \tag{34}$$

where $\{exp(tL), t \geq 0\}$ is the C_0 -semigroup generated by L. Moreover, if $n_0 \in X^+$ then $n(t) \in X^+$ for every $t \geq 0$.

Proof:
The Theorem directly follows from Theorem 2.2, Theorem 3.2 and Corollary 3.1. ■

In the case $\|\Lambda\| > 1$ we can apply Theorem 3.4 , Theorem 3.5 and Theorem 3.9. Thus, we obtain that L generates an integrated semigroup $S(t)$ that is increasing and exponentially bounded from Remark 3.1. Hence, the following Theorem follows by Theorem 3.7.

Theorem 4.2 Assume that $n_0 \in D(L^2)$, then there exists a unique solution of the Cauchy problem (33) given by:

$$n(t) = S(t)Ln_0 + n_0 \ , \quad \forall\, t \geq 0, \tag{35}$$

where $\{S(t), t \geq 0\}$ is the integrated semigroup generated by L. Moreover, if $n_0 \in X^+$ then $n(t) \in X^+$ for every $t \geq 0$.

Theorem 4.3 If $n_0 \in D(L)$ and $\|\Lambda\| > 1$ then system (33) has a unique mild solution.

Proof:
The theorem follows from b) of Theorem 3.8 with $n = 1$ because L generates an integrated non-degenerate semigroup. ■

By using Theorem 3.8 it is possible to relate the integrated semigroup generated by L and the solution of (33) with the C-semigroup generated by L with $C = (\lambda I - L)^{-1}$, $\lambda > 0$.

Theorem 4.4 Let $\lambda > 0, \|\Lambda\| > 1$ and $n_0 \in D(L^2)$, then system (33) has a unique strong solution given by:

$$n(t) = S(t)Ln_0 + n_0 = W(t)\lambda n_0 - W(t)Ln_0 \tag{36}$$

where $S(t), W(t)$ are the integrated semigroup and the $(\lambda I - L)^{-1}$-semigroup generated by L respectively.

Proof:
We know that L generates an integrated semigroup from Theorem 3.9. Hence, by Theorem 3.8 we have that L generates an $(\lambda I - L)^{-1}$-semigroup $\{W(t), t \geq 0\}$ for $\lambda > 0$. Relation (36) follows from Theorem 3.10 and Remark 3.4. ■

If $\|\Lambda\| > 1$ by using a result of [1] (Theorem 9.1) it is possible to construct a new Banach space E such that the Banach space X is an ideal in E. Then, L is closable in

E and generates a positive C_0-semigroup in E. This approach on one side gives better results than those obtained in Theorem 4.3, but on the other makes us to work in a space that is not useful from a physical point of view. Thus, we have preferred to deal with integrated semigroup and C-semigroup in the space $X = L^1(\Omega)$ that we have chosen for physical reasons.

In [9] the existence and uniqueness of the solution of problem (33) has been recently proved by means of a B-bounded semigroup (see [9] and references therein). Hence, we have also the following relation between integrated semigroups, C-semigroups and B-bounded semigroups:

$$n(t) = S(t)Ln_0 + n_0 = W(t)\lambda n_0 - W(t)Ln_0 = B^{-1}Z(t)n_0. \tag{37}$$

In (37), $Z(t) = exp(\hat{L}t)B$ is the B-semigroup generated by the operators $\hat{L}$ and B, where B is a suitable bounded positive operator depending on $0 < \beta < 1/\|\Lambda\|$ and $\hat{L}$ acts like L with boundary conditions given by $f^{in} = \beta\Lambda f^{out}$.

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