Diffusion driven by collision with the boundary

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Abstract

We present a mathematically rigorous derivation of a diffusion model previously introduced by the first author to model the diffusion of charged-particles moving in the gap between two plane parallel plates. The particles are subject to crossed electric and magnetic fields and to collisions against the surface of the solid plates. The surface collisions are supposed to be elastic. Under appropriate scaling assumptions, the particle distribution function converges to a function of the energy and of the longitudinal position coordinates only, which evolves in time according to a diffusion equation. A rigorous convergence proof is given. The proof relies on precise estimates on the trace of the distribution function at the boundary.

Key Words: diffusion approximation, transport equation, accomodation boundary condition, transport-diffusion equation, Spherical Harmonics Expansion model, moment method, Hilbert expansion, Onsager relation.

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1 Introduction

Our goal is the mathematically rigorous derivation of a diffusion model which has formally been introduced by the first author in [11], [12]. This model describes the diffusion of charged-particles (say electrons to fix the ideas), moving in the gap between two solid plane parallel plates. The particles are subject to crossed electric and magnetic fields and to diffusion by collisions against the solid plates. This situation arises for instance in certain kind of ion propellors for satellites (see [9], [16] for more details about these devices, [13], [20], for numerical applications of the model and [21], [22] for related physical approaches).

It has first been demonstrated in [1] (by probablistic arguments) and in [2] (by functional theoretic arguments) that collisions with the boundary can drive a particle system towards a diffusion regime. To some extent, the present work is a follow-up of [2] in a different physical framework but it also departs from [2] in important mathematical aspects.

In [2], the authors consider a collisionless neutral gas flowing in a thin domain (e.g. the gap between two plane parallel plates) and subject to a combination of specular reflection (i.e. reflection like a billiard ball) and pure diffusion (i.e. reemission according to a given distribution function) at the solid plates. If the ratio (denoted by α) of the distance between the two plates to the typical longitudinal length scale (i.e. along the planes) is small, they show that the large time behaviour (on time scales of order α^{-2}) of the particle distribution function is, to leading order, given by $n(\underline{\xi}, t)M(x, v)$, where $\underline{\xi}$ denotes the longitudinal position variable while x is the transversal coordinate, v is the velocity and t the time. M is the normalized distribution function in transversal sections of the domain (typically, M is the Maxwellian at the plates temperature) and n is a solution of a diffusion equation, which describes how the flow evolves between the plates.

The associated diffusion constant is the most important characteristic parameter of the flow at this scale. It is related to the typical distance that a particle travels between two encounters with the plates. Particles with grazing velocities (i.e. almost tangent to the plates) travel a very large distance before hitting them, giving rise to a large diffusivity. In fact, for the present plane parallel geometry, the diffusivity of [2] is infinite. It has been shown in [8] (by probabilistic tools) and in [18] (by functional theoretic arguments) that a logarithmic time rescaling restores a finite diffusivity.

In our case, motivated by the physical context [12], we consider elastically diffusive collisions at the plates: the particles are reemitted with the same energy as their incident one and with a random velocity direction. As a consequence, the large time behaviour of the distribution function is given by an 'energy distribution function' $F(\underline{\xi}, \varepsilon, t)$, where $\varepsilon = |v|^2/2$ is the particle kinetic energy and F satisfies a diffusion equation in both position and energy. Energy diffusion is caused by the combined effects of the electric field and of the collisions with the plates. Position and energy diffusions are not independent (in other words, the diffusion is degenerate) because total energy is preserved during both free flight and collisions. The resulting diffusion model is known in semiconductor physics as the SHE model (for Spherical Harmonics Expansion, a terminology originating from its early derivation, see [5] and references therein) and is also used in plasma physics [15] and gas discharge physics [26].

Another very important fact in our model is the presence of a strong magnetic field directed transversally to the plates. Particle motion then does not occur along straight lines, but rather, along helices whose axis are parallel to the magnetic field lines. The distance that a particle travels between two encounters with the plates cannot be larger than the radii of these helices (the so-called 'Larmor' radius). This limitation results in the finiteness of the diffusivity without time rescaling.

Mathematically, the present problem belongs to the class of diffusion approximation problems for kinetic equations (see e.g. [7], [4] in the context of neutron transport, [23], [19] for semiconductors and [14] for plasmas). Two methods are usually developped: the Hilbert expansion method [4] and the moment method [19]. The latter, although providing only weak convergence results without explicit rates, is more flexible as it requires only mild regularity assumptions on the solution. It proceeds in three steps:
(i) prove that the time asymptotic profile of the distribution function is of the form $F(\underline{\xi}, \varepsilon, t)$, with the macroscopic variable F still to be determined at this stage.
(ii) by using the conservation properties of the collision operator, show that the macroscopic variable obeys a continuity equation, i.e. that its time derivative is balanced by the divergence of a current to be determined.
(iii) find an expression for the current. This is the delicate point of the proof, as the current depends on the first (order α) correction to the asymptotic profile. It is determined through the so-called 'auxilliary function', which essentially provides the response of the microscopic system to gradients of the macroscopic variable. The equation for the current is found as a moment of the kinetic equation through the auxiliary function (hence justifying the terminology 'moment method').

Point (ii) of the proof is actually straightforward. For (i), we first derive trace estimates (that to our knowledge are original) of the distribution function at the boundary (the plates). These estimates allow us to show that, in the limit $\alpha \to 0$, the traces converge in an L^2 sense to a function which is independent of the velocity direction $\omega = v/|v|$. It is then easy to prove that this property 'propagates' inside the domain and that the limiting distribution function is independent of the transversal coordinate x and of ω, the result to be proved. The trace estimate directly follows from the dissipative character of the boundary operator measured in terms of the maximal eigenvalue of its projection onto the orthogonal space to the equilibrium states. This technique has also proved its usefullness in other applications as well (see [6] for an application to semiconductor superlattices).

Concerning point (iii), the auxiliary equation takes the form $\mathcal{A}_0^* f = g$, where $\mathcal{A}_0^*$ is the adjoint of the leading order operator in the kinetic equation and g are special functions which measure the microscopic effects of gradients in the macroscopic variable F. In the present case, the auxiliary equation is not solvable in f for arbitrary data g. Instead, we need to use the fact that, for our specific data, the action of the magnetic field balances the singularity which appears at grazing velocities. Note that we cannot apply the theory of [2]: indeed, in our case, hypothesis (52) of [2], which would mean the integrability of the function $\omega \in \mathbb{S}^2 \to |\omega_x|^{-1}$ on the sphere, breaks down due to grazing velocities.

This paper is organized as follows. First, an introduction to the kinetic model which serves as starting point to the present work is given in section 2. Then, in section 3, we focus on the study of the boundary collision operator. In section 4, we deduce both an existence result and fundamental trace estimates for the initial kinetic model. Section 5 is the core of the paper: it develops the convergence proof of the kinetic model towards the diffusion model. The proof is divided in several steps corresponding to points (i) to (iii) mentionned above. Finally, in section 6, we state and prove some properties of the diffusivites.

2 Setting of the problem

We consider a domain which consists of the gap between two plane parallel plates (see [12] for a more realistic geometry). The distance between the plates is normalized to 1. We denote by $X = (x, y, z)$, $x \in [0, 1]$, $(y, z) \in \mathbb{R}^2$ and by $v = (v_x, v_y, v_z) \in \mathbb{R}^3$ the position and the velocity vectors of an electron in between the planes, where the x coordinate is perpendicular and the vector $\underline{\xi} = (y, z)$, parallel to the plates. Similarly, we denote by $\underline{v} = (v_y, v_z)$ the velocity coordinates that are parallel to the planes. Accordingly, the gradient operators are decomposed : $\nabla_X = (\partial/\partial x, \nabla_{\underline{\xi}})$, $\nabla_v = (\partial/\partial v_x, \nabla_{\underline{v}})$.

Electrons moving between the two planes are subject to a given time-independent electric field directed parallel to the (y, z) directions $E = E(y, z) = (0, E_y, E_z)$ and to a given time-independent magnetic field directed along the x direction $B = B(y, z) = (B_x, 0, 0)$. For simplicity we note $\underline{E} = (E_y, E_z)$ and $B = B_x$.

The electrons are supposed to move in between the planes according to a scaled collisionless transport equation for the electron distribution function $f(X, v, t)$ written as follows:

$$\alpha^2 \frac{\partial f^\alpha}{\partial t} + \alpha \left(\underline{v} \cdot \nabla_{\underline{\xi}} f^\alpha - \underline{E} \cdot \nabla_{\underline{v}} f^\alpha \right) + v_x \frac{\partial f^\alpha}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f^\alpha = 0, \qquad (2.1)$$

where $X \in \Omega = [0, 1] \times \mathbb{R}^2$, $v \in \mathbb{R}^3$ and $\alpha \ll 1$ is a small parameter. This scaling, which is exposed in details in [12] corresponds to a small distance between the plates, a large magnetic field and a long (diffusion) time scale.

The collisions of the electrons against the solid plates are modeled by a combination of specular and diffuse reflections (see [12] and references therein). We introduce the set $\Theta = \Omega \times \mathbb{R}^3$, its boundary $\Gamma = \gamma \times \mathbb{R}^3$, where $\gamma = \{0, 1\} \times \mathbb{R}^2$, and the following *incoming* and *outgoing* subsets of Γ (respectively representing incoming and outgoing particles to the domain Ω):

$$\Gamma_- = \left(\{0\} \times \mathbb{R}^2 \times \{v \in \mathbb{R}^3, v_x > 0\} \right) \cup \left(\{1\} \times \mathbb{R}^2 \times \{v \in \mathbb{R}^3, v_x < 0\} \right),$$

and Γ_+ obtained by reversing the inequalities.

Denoting by f^α_- and f^α_+ the traces of the distribution function f^α on the sets Γ_- and Γ_+, the boundary conditions at $x = 0$ and $x = 1$ read as follows:

$$f^\alpha_-(X, v) = \beta f^\alpha_+(X, v_*) + (1 - \beta)\mathcal{K}(f^\alpha_+)(X, v), \quad (X, v) \in \Gamma_-, \tag{2.2}$$

where the accommodation coefficient $\beta = \beta(x, y, z, |v|)$ is such that $0 \le \beta < 1$, v_* is the specularly reflected velocity $v_* = (-v_x, v_y, v_z)$ and $\mathcal{K}$ is the diffuse reflection operator. $\mathcal{K}$ maps the outgoing trace f^α_+ to the incoming one f^α_- and is defined for $(X, v) \in \Gamma_-$ by

$$\mathcal{K}(f^\alpha_+)(X, v) = \int K(X, |v|; \omega' \to \omega)\, f^\alpha_+(X, |v|\omega')\, |\omega'_x|\, d\omega', \tag{2.3}$$

where $\omega = v/|v|$ belongs to the unit sphere $\mathbb{S}^2$. The integration is carried over the set of directions ω' that are incoming with respect to the boundary at the point X.

The dependence of the kernel K with respect to $X = (x, y, z)$, $x \in \{0, 1\}$, $(y, z) \in \mathbb{R}^2$ and $|v|$ will be omitted, otherwise specified. The quantity $K(\omega' \to \omega)|\omega_x| d\omega$ is the probability for an electron impinging on the plate at a position (y, z) with velocity $|v|\omega'$ to be reflected with a new velocity $|v|\omega$ such that ω belongs to the solid angle $d\omega$.

The aim of the present work is to show that the limit of f^α when α goes to 0 is a function of the longitudinal coordinate $\underline{\xi}$, of the energy $\varepsilon = |v|^2/2$ and of the time, $F(\underline{\xi}, \varepsilon, t)$, which obeys a diffusion equation in the position-energy space. This equation is often referred to in the literature as the SHE model (see introduction for references). More precisely, we prove:

Theorem 2.1 *Under some hypotheses listed later on (namely hypotheses 3.1, 3.2, 3.3, 4.1, 5.1, 5.2), f^α converges to f^0 as $\alpha \to 0$ in the weak star topology of $L^\infty([0, T], L^2(\Theta))$ for any $T > 0$, where $f^0(X, v, t) = F(\underline{\xi}, |v|^2/2, t)$ and $F(\underline{\xi}, \varepsilon, t)$ is a distributional solution of the problem:*

$$4\pi\sqrt{2\varepsilon}\frac{\partial F}{\partial t} + \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right) J = 0, \tag{2.4}$$

$$J(\underline{\xi}, \varepsilon, t) = -\mathbb{D}\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right) F(\underline{\xi}, \varepsilon, t), \tag{2.5}$$

$$F|_{t=0} = F_I, \tag{2.6}$$

in the domain $(x, \underline{\xi}) \in \mathbb{R} \times (0, \infty)$. The diffusion tensor $\mathbb{D} = \mathbb{D}(\underline{\xi}, \varepsilon)$ is given by

$$\mathbb{D}(\underline{\xi}, \varepsilon) = (2\varepsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \underline{D}(x, \omega; \underline{\xi}, \varepsilon)\underline{\omega}\, dx\, d\omega, \tag{2.7}$$

where $\underline{\omega} = (\omega_y, \omega_z)$, $\underline{D} = (D_y, D_z)$, $\underline{D}\,\underline{\omega}$ is the tensor product $(D_i\omega_j)_{i,j\in\{y,z\}}$ and finally $D_i(x, \omega; \underline{\xi}, |v|^2/2)$, $(i = y, z)$ is a solution of the problem

$$-v_x\frac{\partial D_i}{\partial x} + (\underline{v} \times B) \cdot \nabla_{\underline{v}} D_i = \omega_i\,, \text{ in } \Theta\,, \tag{2.8}$$

$$(D_i)_+ = \mathcal{B}^*(D_i)_-\,, \text{ on } \Gamma\,, \tag{2.9}$$

unique, up to an additive function of $\underline{\xi}$ and ε.

Remark 2.1 Hypothesis 5.2 supposes that the diffusion tensor vanishes for vanishing energy $\mathbb{D}(\underline{\xi}, \varepsilon = 0) = 0$, $\forall \underline{\xi} \in \mathbb{R}^2$. Therefore, the diffusion problem in ε is degenerate and does not require additional boundary conditions at $\varepsilon = 0$. This hypothesis is satisfied in practice [12]

We introduce the functional setting and prove the existence of solutions of (2.1) in the following sections.

3 The boundary operator: assumptions and properties

We begin with some notations. We denote $\mathcal{S}_\pm(x)$, $x = 0, 1$ the following half-spheres:

$$\mathcal{S}_+(0) = \mathcal{S}_-(1) = \{\omega \in \mathbb{S}^2, \omega_x < 0\} \, , \; \mathcal{S}_-(0) = \mathcal{S}_+(1) = \{\omega \in \mathbb{S}^2, \omega_x > 0\}, \tag{3.1}$$

We introduce the domain $\mathcal{S} = [0,1]_x \times \mathbb{S}^2_\omega$, with its associated *incoming* and *outgoing* boundaries defined by:

$$\mathcal{S}_- = (\{0\} \times \mathcal{S}_-(0)) \cup (\{1\} \times \mathcal{S}_-(1)) \, , \, \mathcal{S}_+ = (\{0\} \times \mathcal{S}_+(0)) \cup (\{1\} \times \mathcal{S}_+(1)).$$

We define the inner products on $L^2(\Theta)$ and on $L^2(\Gamma_\pm)$ respectively by:

$$(f,g)_\Theta = \int_\Theta f g \, d\theta \, , \quad (f,g)_{\Gamma_\pm} = \int_{\Gamma_\pm} f g \, |v_x| \, d\Gamma.$$

where $d\theta = dx d\underline{\xi} dv$ is the volume element in phase space, and $d\Gamma = \sum_{x=0,1} d\underline{\xi} dv$ is the surface element. The inner products on $L^2(\mathcal{S})$, $L^2(\mathcal{S}_\pm)$ are defined analogously:

$$(f,g)_\mathcal{S} = \int_0^1 \int_{\mathbb{S}^2} f g(x,\omega) \, dx \, d\omega, \tag{3.2}$$

$$(f,g)_{\mathcal{S}_\pm} = \int_{\mathcal{S}_\pm(0)} |\omega_x| \, f g(0,\omega) \, d\omega + \int_{\mathcal{S}_\pm(1)} |\omega_x| \, f g(1,\omega) \, d\omega, \tag{3.3}$$

We now introduce the orthogonal projection $Q_\pm$ of $L^2(\mathcal{S}_\pm)$ on the space $\mathcal{C}^\pm$ of constant functions on each connected component of $\mathcal{S}_\pm$, i.e.:

$$Q_\pm f(x,\omega) = \frac{1}{\pi} \int_{\mathcal{S}_\pm(x)} |\omega'_x| \, f(x,\omega') \, d\omega' \, , \; \omega \in \mathcal{S}_\pm(x) \, , x = 0, 1 \, , \tag{3.4}$$

and its orthogonal complement $P_\pm = \mathrm{I}_{\mathcal{S}_\pm} - Q_\pm$, where $\mathrm{I}_{\mathcal{S}_\pm}$ is the identity.

We now define the boundary operator $\mathcal{B}$ the following way: for $\phi \in L^2(\mathcal{S}_+)$, $\mathcal{B}\phi \in L^2(\mathcal{S}_-)$ and

$$\mathcal{B}(\phi) = \beta J\phi + (1-\beta)\mathcal{K}(\phi), \tag{3.5}$$

where the mirror reflection operator J from $L^2(\mathcal{S}_+)$ to $L^2(\mathcal{S}_-)$ is defined by:

$$J\phi(x,\omega) = \phi(x,\omega_*), \tag{3.6}$$

with $\omega_* = (-\omega_x, \omega_y, \omega_z)$. Moreover, its adjoint J^* from $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$ is also the mirror reflection operator, and so J and J^* satisfy $J^*J = Id_{\mathcal{S}_+}$, $JJ^* = Id_{\mathcal{S}_-}$.

We now list the required assumptions on the operator $\mathcal{K}$. We recall that we omit the dependence of $\mathcal{K}$ on $(x, \underline{\xi}, |v|)$ when the context is clear.

Hypothesis 3.1 *We assume that the kernel K satisfies the following properties:*
(i) Positivity:

$$K(\omega' \to \omega) > 0, \tag{3.7}$$

for almost all $(\omega,\omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x)$, $x = 0,1$.
(ii) Flux conservation:

$$\int_{\mathcal{S}_-(x)} K(\omega' \to \omega)\, |\omega_x|\, d\omega = 1\,,\ x = 0,1\,. \tag{3.8}$$

(iii) Reciprocity relation:

$$K(\omega' \to \omega) = K(-\omega \to -\omega'), \quad \forall\, (\omega,\omega') \in \mathcal{S}_-(x) \times \mathcal{S}_+(x)\,,\ x = 0,1\,. \tag{3.9}$$

Positivity and flux conservation are natural physical assumptions. Reciprocity is a translation of the time reversibility of the microscopic interaction process which occurs at the boundary. Its relevance is discussed in [10]. From relations (3.8) and (3.9), the *normalization identity* easily follows:

$$\int_{\mathcal{S}_+(x)} K(\omega' \to \omega)\, |\omega'_x|\, d\omega' = 1\,, \quad x = 0,1. \tag{3.10}$$

Moreover, from Hypothesis 3.1, we derive the *Darrozes-Guiraud inequality*:

Lemma 3.1 *(i) Let $f_+ \in L^2(\mathcal{S}_+)$ and $f_- = \mathcal{B}f_+$. Then,*

$$\int_{\mathcal{S}_-(x)} |f_-(x,\omega)|^2\, |\omega_x|\, d\omega \le \int_{\mathcal{S}_+(x)} |f_+(x,\omega)|^2\, |\omega_x|\, d\omega\,, \quad x = 0,1. \tag{3.11}$$

(ii) $\mathcal{B}$, as an operator from $L^2(\mathcal{S}_+)$ to $L^2(\mathcal{S}_-)$, is of norm 1.

Proof: (i): follows straightforwardly from the Cauchy-Schwartz inequality and (3.10).
(ii): From (3.11), we deduce that $|\mathcal{B}f|^2_{L^2(\mathcal{S}_-)} \le |f|^2_{L^2(\mathcal{S}_+)}$, and therefore that $\|\mathcal{B}\| \le 1$. Now, from (3.10), any f in $\mathcal{C}^+$ is such that $\mathcal{B}f = Jf$. Then, $|\mathcal{B}f|^2_{L^2(\mathcal{S}_-)} = |f|_{L^2(\mathcal{S}_+)}$, showing that $\|\mathcal{B}\| = 1$. ■

We now remark that the dual operator $\mathcal{B}^*$ of $\mathcal{B}$ maps $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$ according to:

$$\mathcal{B}^*\phi = \beta J^*\phi + (1-\beta)\mathcal{K}^*(\phi) \tag{3.12}$$

with

$$\mathcal{K}^*(\phi)(x,\omega) = \int_{\mathcal{S}_-(x)} K(x,\omega\to\omega')\,|\omega'_x|\,\phi(x,\omega')\,d\omega',\ \omega\in\mathcal{S}_+(x)\,,\ x=0,1. \tag{3.13}$$

We now suppose:

Hypothesis 3.2 *The operator $\mathcal{K}$ is compact.*

We note that this implies that $\mathcal{K}^*$ is compact. We now prove:

Lemma 3.2 *Under Hypothesis 3.1, 3.2, we have:*

$$N(I - J\mathcal{B}^*) = \mathcal{C}^-\,,\quad N(I - J^*\mathcal{B}) = \mathcal{C}^+,$$

where N denotes the Null-Space.

Proof: We first remark from (3.10) that a function φ of $\mathcal{C}^+$ satisfies $(I - J^*\mathcal{B})\varphi = 0$. Conversely, let $\varphi \in L^2(\mathcal{S}_+)$ be a solution of $(I - J^*\mathcal{B})\varphi = 0$, then:

$$\varphi - \beta J^*J\varphi - (1-\beta)J^*\mathcal{K}(\varphi) = 0,$$

which implies (because $\beta < 1$) that $(I - J^*\mathcal{K})\varphi = 0$. This equation can be decomposed in:

$$(I - J^*\mathcal{K}(0))\varphi_0 = 0,\quad (I - J^*\mathcal{K}(1))\varphi_1 = 0,$$

where $\varphi_x = \varphi|_{\mathcal{S}_+(x)}$ and $\mathcal{K}(x) = \mathcal{K}|_{L^2(\mathcal{S}_+(x))}$, $x = 0,1$.

The operators $J^*\mathcal{K}(x)$, $x = 0,1$ satisfy the following properties which are deduced from hypotheses 3.1 and 3.2:
(i) $J^*\mathcal{K}(x)$ is a compact operator on $L^2(\mathcal{S}_+(x))$.
(ii) $J^*\mathcal{K}(x)$ is positive: $\varphi \ge 0$ implies $J^*\mathcal{K}(x)(\varphi) > 0$.
(iii) The constant function 1 is an eigenfunction of $J^*\mathcal{K}(x)$ associated with the eigenvalue 1, i.e. $J^*\mathcal{K}(x)(1) = 1$.

Thanks to the Krein-Rutman theorem, it follows that 1 is an eigenvalue of multiplicity 1. Therefore, $N(I - J^*\mathcal{K}(x)) = \text{Span}\{1\}$, $x = 0,1$ and thus $N(I - J^*\mathcal{K}) = \mathcal{C}^+$, which proves the result. The proof is clearly similar for $I - J\mathcal{B}^*$. ■

Lemma 3.3 *The following equalities hold:*

$$\mathcal{B}Q_+ = Q_-\mathcal{B} = JQ_+ = Q_-J\,, \quad \mathcal{B}P_+ = P_-\mathcal{B}. \tag{3.14}$$

and similarly (mutatis mutandis) for $\mathcal{B}^*$.

Proof: First, it is clear that $JQ_+ = Q_-J$. Let $\varphi = P_+\varphi + Q_+\varphi$ be the decomposition of $\varphi \in L^2(\mathcal{S}_+)$. Then:

$$\mathcal{B}\varphi = \mathcal{B}P_+\varphi + \mathcal{B}Q_+\varphi. \tag{3.15}$$

But $Q_+\varphi \in \mathcal{C}^+$ and, by Lemma 3.2, $\mathcal{B}Q_+\varphi = JQ_+\varphi = Q_-J\varphi$. Therefore,

$$\mathcal{B}\varphi = \mathcal{B}P_+\varphi + Q_-J\varphi. \tag{3.16}$$

We shall prove that (3.16) is the decomposition of $\mathcal{B}\varphi$ on P_- and Q_-. Indeed, let $q \in \mathcal{C}^-$, then, by duality, $(\mathcal{B}P_+\varphi, q)_{\mathcal{S}_-} = (P_+\varphi, \mathcal{B}^*q)_{\mathcal{S}_+}$, and $\mathcal{B}^*q = J^*q \in \mathcal{C}^+$. Hence, from the definition $P_+\varphi$, $(P_+\varphi, J^*q)_{\mathcal{S}_+} = 0$. Therefore, $\mathcal{B}P_+$ is orthogonal to $\mathcal{C}^-$ which proves the desired property. We deduce $\mathcal{B}P_+\varphi = P_-\mathcal{B}\varphi$ and $\mathcal{B}Q_+\varphi = Q_-\mathcal{B}\varphi$, which ends the proof of the Lemma. ∎

By elementary operator theory, we remark that, for every $\underline{\xi} \in \mathbb{R}^2$ and for every $|v| \in \mathbb{R}^+$, there exists $k(\underline{\xi}, |v|)$ such that $||\mathcal{K}P_+|| \le k(\underline{\xi}, |v|) < 1$. In the remainder, we shall assume that this constant is bounded away from 1, as $\underline{\xi}$ and $|v|$ vary. More precisely:

Hypothesis 3.3 *(i) There exists a constant* $k < 1$ *such that:*

$$||\mathcal{K}P_+||_{\mathcal{L}(L^2(\mathcal{S}_+),L^2(\mathcal{S}_-))} \le k < 1\,, |v| \in \mathbb{R}^+\,, \underline{\xi} \in \mathbb{R}^2. \tag{3.17}$$

(ii) There exists $\beta_0 < 1$ *such that* $0 \le \beta \le \beta_0 < 1\,, \underline{\xi} \in \mathbb{R}^2\,, |v| > 0\,, x = 0, 1.$

It follows that:

$$||\mathcal{B}P_+||_{\mathcal{L}(L^2(\mathcal{S}_+),L^2(\mathcal{S}_-))} \le \sqrt{\beta_0 + (1-\beta_0)k^2} = k_0. \tag{3.18}$$

We note that, if the kernel K is constant, then $\mathcal{K} = JQ_+$, and so $\mathcal{K}P_+ = JQ_+P_+ = 0$, and $\mathcal{K}$ satisfies the assumption (3.17). More generally, we have:

Lemma 3.4 *If* $K(\omega' \to \omega) \ge C > 0$*, where* C *is a constant, then assumption (3.17) holds.*

Proof: It is enough to prove the result for $x = 1$, the proof being analogous for $x = 0$ (with the sign of ω_x reversed). We prove equivalently that there exists k, $0 < k < 1$, only depending on C such that:

$$|\mathcal{K}(1)\varphi|^2_{L^2(S_-(1))} \le k^2|\varphi|^2_{L^2(S_+(1))}, \tag{3.19}$$

for all $\varphi \in L^2(S_+(1))$ such that:

$$\int_{\omega_x>0} \varphi\, |\omega_x|\, d\omega = 0. \tag{3.20}$$

Omitting the dependence of the kernel $K(\omega' \to \omega)$ upon $\underline{\xi}$ and $|v|$, we have:

$$\mathcal{K}\varphi(\omega) = \int_{\omega'_x>0} K(\omega' \to \omega)\, \varphi(\omega')\, |\omega'_x|\, d\omega' = \int_{\omega'_x>0} (K(\omega' \to \omega) - C/2)\, \varphi(\omega')\, |\omega'_x|\, d\omega'$$

because of (3.20). But $K(\omega' \to \omega) - C/2 \geq C/2 > 0$ and

$$\int_{\omega'_x>0} (K(\omega' \to \omega) - C/2)\, |\omega'_x|\, d\omega' = 1 - C\pi/2 \geq C\pi/2 > 0.$$

Therefore, noting $k^2 = 1 - C\pi/2$, we have $0 < k^2 < 1$ and by the Cauchy-Schwartz inequality:

$$\begin{aligned} |\mathcal{K}\varphi(\omega)|^2 &\leq k^2 \int_{\omega'_x>0} (K(\omega' \to \omega) - C/2)\, |\varphi(\omega')|^2\, |\omega'_x|\, d\omega' \\ &\leq k^2 \int_{\omega'_x>0} K(\omega' \to \omega) |\varphi(\omega')|^2\, |\omega'_x|\, d\omega'. \end{aligned}$$

Then, using the normalization property, we deduce that:

$$|\mathcal{K}\varphi(\omega)|^2_{L^2(\mathcal{S}_-(1))} = \int_{\omega_x<0} |K\varphi(\omega)|^2\, |\omega_x|\, d\omega \leq k^2 \int_{\omega'_x>0} |\varphi(\omega')|^2\, |\omega'_x|\, d\omega' = k^2 |\varphi|^2_{L^2(\mathcal{S}_+(1))}$$

Thus proving the result. ∎

As a final remark,we note the obvious fact:

Lemma 3.5 *Let $\phi = \phi(|v|^2)$ bounded and $f \in L^2(\mathcal{S}_+)$. Then $\mathcal{B}(\phi f) = \phi \mathcal{B}(f)$. Similarly, for $f \in L^2(\mathcal{S}_-)$, $\mathcal{B}^*(\phi f) = \phi \mathcal{B}^*(f)$.*

4 The transport operator

We define the following operator on $L^2(\Theta)$:

$$\mathcal{A}^\alpha f = \underline{v} \cdot \nabla_{\underline{\xi}} f - \underline{E} \cdot \nabla_{\underline{v}} f + \frac{1}{\alpha} \left(v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} \right), \tag{4.1}$$

with domain $D(\mathcal{A}^\alpha)$ defined by:

$$\begin{aligned} D(\mathcal{A}^\alpha) = \{ f \in L^2(\Theta),\, \mathcal{A}^\alpha f \in L^2(\Theta), P_+ f_+ \in L^2(\Gamma_+), \\ Q_+ f_+ \in L^2_{loc}(\Gamma_+), f_- = \mathcal{B} f_+ \}. \end{aligned} \tag{4.2}$$

where

$$L^2_{loc}(\Gamma_\pm) = \{f \text{ such that } \phi(|v|)f \in L^2(\Gamma_\pm), \forall \phi \in C^\infty, \phi \text{ bounded}\}. \quad (4.3)$$

$L^2_{loc}(\Gamma_\pm)$ is equipped with the family of semi-norms $|\cdot|_{\Gamma_\pm,R}$:

$$|f|^2_{\Gamma_\pm,R} = \int_{\Gamma_\pm,|v|\le R} |v_x|\,|f|^2\,d\Gamma. \quad (4.4)$$

$L^2_{loc}(\Gamma)$ is defined in a similar way with the family of semi-norms $|\cdot|_{\Gamma,R}$.

The goal of the present section is to establish trace estimates for functions of $D(\mathcal{A}^\alpha)$ which will show that $D(\mathcal{A}^\alpha)$ is closed for the graph norm

$$|f|^2_{\mathcal{A}^\alpha} = |f|^2_{L^2(\Theta)} + |\mathcal{A}^\alpha f|^2_{L^2(\Theta)}.$$

We then prove that $\mathcal{A}^\alpha$ generates a strongly continuous semi-group of contractions, thus providing an existence setting for the kinetic problem (2.1). We suppose:

Hypothesis 4.1 *(i)* $\underline{E} = \underline{E}(\underline{\xi}) \in (W^{1,\infty}(\mathbb{R}^2))^2$.
(ii) $B = B(\underline{\xi}) \in C^1 \cap W^{1,\infty}(\mathbb{R}^2)$.
(iii) There exists a constant $B_0 > 0$ *such that* $|B(\underline{\xi})| \ge B_0 > 0$*, for every* $\underline{\xi} \in \mathbb{R}^2$.

We first establish a Green's Formula for functions in $D(\mathcal{A}^\alpha)$. Let us notice that for two functions $f, g \in C^1_0(\Theta)$ we have :

$$\begin{aligned}(\mathcal{A}^\alpha f, g)_\Theta &= -(f, \mathcal{A}^\alpha g)_\Theta + \frac{1}{\alpha}\left(\int_{\mathbb{R}^2\times\mathbb{R}^3} v_x\,(fg)|_{x=1}\,d\Gamma - \int_{\mathbb{R}^2\times\mathbb{R}^3} v_x\,(fg)|_{x=0}\,d\Gamma\right)\\ &= -(f, \mathcal{A}^\alpha g)_\Theta + \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,f_+g_+\,d\Gamma - \int_{\Gamma_-} |v_x|\,f_-g_-\,d\Gamma\right). \quad (4.5)\end{aligned}$$

Define the space :

$$H(\mathcal{A}^\alpha) = \{f \in L^2(\Theta), \mathcal{A}^\alpha f \in L^2(\Theta)\} \quad (4.6)$$

By making $f = g$ in (4.5), we easily convince ourselves that the regularity $f \in H(\mathcal{A}^\alpha)$ is not sufficient to guarantee the integrability of $|f_+|^2$ over the boundary because of the minus sign at the right-hand-side of (4.5). Following [3], [24], we define:

$$H_0(\mathcal{A}^\alpha) = \{f \in H(\mathcal{A}^\alpha),\ f_- \in L^2(\Gamma_-)\} = \{f \in H(\mathcal{A}^\alpha),\ f_+ \in L^2(\Gamma_+)\}. \quad (4.7)$$

Then, from [3], [24], we deduce:

Lemma 4.1 (Green's Formula) *Under Hypothesis 4.1, for* f *and* g *in* $H_0(\mathcal{A}^\alpha)$ *with compact support with respect to* v*, we have:*

$$(\mathcal{A}^\alpha f, g)_\Theta + (f, \mathcal{A}^\alpha g)_\Theta = \frac{1}{\alpha}\left((f_+, g_+)_{\Gamma_+} - (f_-, g_-)_{\Gamma_-}\right). \quad (4.8)$$

We first prove that the "non constant" part of the trace at the boundary of a function of $D(\mathcal{A}^\alpha)$ is controlled by the graph norm.

Lemma 4.2 *If $f \in D(\mathcal{A}^\alpha)$, then there exists a constant $C > 0$ such that:*

$$|P_- f_-|^2_{L^2(\Gamma_-)} \leq |P_+ f_+|^2_{L^2(\Gamma_+)} \leq \frac{2\alpha}{1-k_0} (\mathcal{A}^\alpha f, f)_\Theta \leq C\alpha |f|^2_{\mathcal{A}^\alpha}. \tag{4.9}$$

Proof: We apply Green's Formula (4.8) with a cut off function $\chi_R(|v|^2)$ such that $\chi_R(|v|^2) = \chi(|v|^2/R^2)$, $\chi \in C^\infty(\mathbb{R}^+)$, $0 \leq \chi \leq 1$, $\chi(u) = 1$ for $u < 1$ and $\chi(u) = 0$ for $u > 2$ and obtain, thanks to Lemma 3.3 and Hypothesis 3.3:

$$\begin{aligned}
2(\mathcal{A}^\alpha \chi_R f, \chi_R f)_\Theta &= \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|\chi_R f_+|^2\, d\Gamma - \int_{\Gamma_-} |v_x|\,|\chi_R f_-|^2\, d\Gamma\right) \\
&= \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|\chi_R|^2\,|f_+|^2\, d\Gamma - \int_{\Gamma_-} |v_x|\,|\mathcal{B} f_+|^2\,|\chi_R|^2\, d\Gamma\right) \\
&= \frac{1}{\alpha}\int_{\Gamma_+} |v_x|\,\left(|P_+ f_+|^2 + |Q_+ f_+|^2\right)|\chi_R|^2\, d\Gamma \\
&\qquad -\frac{1}{\alpha}\int_{\Gamma_-} |v_x|\,\left(|P_- \mathcal{B} f_+|^2 + |Q_- \mathcal{B} f_+|^2\right)|\chi_R|^2\, d\Gamma \\
&= \frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|P_+ f_+|^2\,|\chi_R|^2\, d\Gamma - \int_{\Gamma_-} |v_x|\,|\mathcal{B} P_+ f_+|^2\,|\chi_R|^2\, d\Gamma\right) \\
&\geq \frac{1-k_0^2}{\alpha}\int_{\Gamma_+} |v_x|\,|P_+ f_+|^2 |\chi_R|^2\, d\Gamma,
\end{aligned}$$

where we have used that $P_- f_- = P_- \mathcal{B} f_+ = \mathcal{B} P_+ f_+$. But, we have:

$$\mathcal{A}^\alpha(\chi_R f) = \chi_R \mathcal{A}^\alpha f + \frac{2\underline{E}\cdot\underline{v}}{R^2} f \chi'\left(\frac{|v|^2}{R^2}\right). \tag{4.10}$$

Therefore,

$$2(\mathcal{A}^\alpha \chi_R f, \chi_R f)_\Theta \leq 2|\mathcal{A}^\alpha f|_{L^2(\Theta)}\,|f|_{L^2(\Theta)} + (C/R)|f|^2_{L^2(\Theta)}.$$

The result follows by letting $R \to \infty$. ■

We now notice that, if $f \in D(\mathcal{A}^\alpha)$, then $Q_- f_- = J Q_+ f_+$ (thanks to Lemma 3.3). Thus, there exists a single function $q(f) = q(x, \xi, |v|)$, $x = 0, 1, \xi \in \mathbb{R}^2, |v| > 0$, such that

$$q = Q_- f_- \text{ , on } \Gamma_-, \quad q = Q_+ f_+ \text{ , on } \Gamma_+ \tag{4.11}$$

We have:

Lemma 4.3 *Let $f \in D(\mathcal{A}^\alpha)$, then:*

$$|q(f)|^2_{\Gamma,R} \leq C\left(\alpha |f|^2_{\mathcal{A}^\alpha} + R|f|^2\right) \tag{4.12}$$

Proof: We multiply $\mathcal{A}^\alpha f$ by $\text{sgn}(v_x)\phi(x)f$, where $\phi(x) = 2x - 1$ and $\text{sgn}(v_x)$ is the sign of v_x. Note that $\phi(1) = 1$, $\phi(0) = -1$. Hence, applying Green's Formula (4.8) with the cut off function χ_R, we obtain:

$$
\begin{aligned}
&(\mathcal{A}^\alpha \chi_R f, \text{sgn}(v_x)\phi(x)\chi_R f)_{L^2(\Theta)} + \frac{2}{\alpha}|\sqrt{v_x}\chi_R f|^2_\Theta \\
&= \frac{1}{\alpha}\left(\int_{\mathbb{R}^2\times\mathbb{R}^3} \frac{|v_x|}{2} |\chi_R f(1)|^2 \, d\Gamma + \int_{\mathbb{R}^2\times\mathbb{R}^3} \frac{|v_x|}{2} |\chi_R f(0)|^2 \, d\Gamma \right) \\
&= \frac{1}{\alpha}\int_{\Gamma_+} \frac{|v_x|}{2} \left(|P_+ f_+|^2 + |Q_+ f_+|^2\right) |\chi_R|^2 \, d\Gamma \\
&\quad + \frac{1}{\alpha}\int_{\Gamma_-} \frac{|v_x|}{2} \left(|P_- f_-|^2 + |Q_- f_-|^2\right) |\chi_R|^2 \, d\gamma \\
&= \frac{1}{\alpha}\left(\int_{\Gamma_+} \frac{|v_x|}{2} |P_+ f_+|^2 \, |\chi_R|^2 \, d\Gamma + \int_{\Gamma_-} \frac{|v_x|}{2} |P_- f_-|^2 \, |\chi_R|^2 \, d\Gamma \right) \\
&\quad + \frac{1}{\alpha}\left(\int_{\Gamma_+} \frac{|v_x|}{2} |Q_+ f_+|^2 \, |\chi_R|^2 \, d\Gamma + \int_{\Gamma_-} \frac{|v_x|}{2} |Q_- f_-|^2 \, |\chi_R|^2 \, d\Gamma \right).
\end{aligned}
$$

The second term on the right hand side is estimated from below by:

$$
\frac{1}{\alpha}\left(\int_{\Gamma_+} \frac{|v_x|}{2} |q(f)|^2 |\chi_R|^2 \, d\Gamma + \int_{\Gamma_-} \frac{|v_x|}{2} |q(f)|^2 |\chi_R|^2 \, d\Gamma \right) > \frac{1}{\alpha}|q(f)|^2_{\Gamma,R},
$$

whereas, the first term on the left hand side is smaller than $|f|^2_{A^\alpha}$, and the second term on the left hand side is bounded by $(C/\alpha)|\sqrt{v_x}\chi_R f|^2_{L^2(\Theta)} = C(R/\alpha)|f|^2_{L^2(\Theta)}$. Hence, (4.3) follows. ■

Thanks to lemmas 4.2, 4.3, we can prove:

Lemma 4.4 *The operator $\mathcal{A}^\alpha$ given by (4.1) is a closed operator.*

Proof: Assume that a sequence of function $f_n \in D(\mathcal{A}^\alpha)$ converges to a function f, and that $\mathcal{A}^\alpha f_n$ converges to g. It is clear that $g = \mathcal{A}^\alpha f$ in the distributional sense. There is still to prove that $P_+ f_+ \in L^2(\Gamma_+)$, $Q_+ f_+ \in L^2_{loc}(\Gamma_+)$ and $f_- = \mathcal{B} f_+$. From Lemmas 4.2 and 4.3, we have that $P_+ f_{n+}$ and $Q_+ f_{n+}$ are Cauchy sequence in $L^2(\Gamma_+)$ and $L^2_{loc}(\Gamma_+)$ respectively, thus convergent to $P_+ f_+$ and $Q_+ f_+$. (This follows from the fact that the traces converge in a weak sense, like e.g. $H^{-1/2}$). Then, the continuity of $\mathcal{B}$ allows to find $f_- = \mathcal{B} f_+$. ■

In order to prove that the operator $-\mathcal{A}^\alpha$ given by (4.1) generates a strongly continuous semigroup of contractions it is sufficient to prove that its dual operator, $\mathcal{A}^{\alpha*}$, is accretive.

Lemma 4.5 *(i) We have $\mathcal{A}^{\alpha*} f = -\mathcal{A}^\alpha f$, with domain:*

$$
\begin{aligned}
D(\mathcal{A}^{\alpha*}) \;=\; &\{f \in L^2(\Theta)\,, \mathcal{A}^{\alpha*} f \in L^2(\Theta),\, P_- f_- \in L^2(\Gamma_-), \\
&\quad Q_- f_- \in L^2_{loc}(\Gamma_-), f_+ = \mathcal{B}^* f_-\}, \qquad (4.13)
\end{aligned}
$$

where $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ given by (3.12).
(ii) $\mathcal{A}^{\alpha *}$ is accretive, i.e. $(\mathcal{A}^{\alpha *} f, f)_\Theta \geq 0$, $\forall f \in D(\mathcal{A}^{\alpha *})$.

Proof: First, define $\widehat{\mathcal{A}^{\alpha *}}$ by $\widehat{\mathcal{A}^{\alpha *}} f = -\mathcal{A}^\alpha f$ with the domain defined by (4.13). Let $f \in D(\mathcal{A}^\alpha)$ and $f^* \in D(\widehat{\mathcal{A}^{\alpha *}})$. By Green's Formula (4.8), we have:

$$(\mathcal{A}^\alpha f\,, f^*)_\Theta = (f\,, -\mathcal{A}^\alpha f^*)_\Theta + (f_+, f_+^*)_{\Gamma_+} - (f_-, f_-^*)_{\Gamma_-}$$

But,

$$\begin{aligned} (f_+, f_+^*)_{\Gamma_+} - (f_-, f_-^*)_{\Gamma_-} &= (f_+, f_+^*)_{\Gamma_+} - (\mathcal{B} f_+, f_-^*)_{\Gamma_-} \\ &= (f_+, f_+^*)_{\Gamma_+} - (f_+, \mathcal{B}^* f_-^*)_{\Gamma_-} = 0. \end{aligned}$$

Therefore, $(\mathcal{A}^\alpha f, f^*)_\Theta \leq C(f^*)\,|f|_{L^2(\Theta)}$ proving that $f^* \in D(\mathcal{A}^{\alpha *})$. (The truncation argument, which must be used as above, is omitted for brevity), and that $D(\widehat{\mathcal{A}^{\alpha *}}) \subseteq D(\mathcal{A}^{\alpha *})$. Now, $D(\widehat{\mathcal{A}^{\alpha *}})$ is closed for the graph norm from the same arguments as for $\mathcal{A}^\alpha$. Therefore, $D(\widehat{\mathcal{A}^{\alpha *}})$ and $D(\mathcal{A}^{\alpha *})$ are two closed spaces for the graph norm, which contain the same dense subspace (for instance $\mathcal{D}(\Theta)$). Thus they are equal.
(ii) Let $f \in D(\mathcal{A}^{\alpha *})$. We have, thanks to Green's formula (via a truncation argument which is omitted):

$$\begin{aligned} 2(\mathcal{A}^{\alpha *} f, f)_\Theta &= -\frac{1}{\alpha}\left(\int_{\Gamma_+} |v_x|\,|f_+|^2\, d\Gamma - \int_{\Gamma_-} |v_x|\,|f_-|^2\, d\Gamma\right) \\ &= \frac{1}{\alpha}\left(\int_{\Gamma_-} |v_x|\,|f_-|^2\, d\Gamma - \int_{\Gamma_+} |v_x|\,|\mathcal{B}^* f_-|^2\, d\Gamma\right) \geq 0. \end{aligned}$$

The inequality follows from the Darrozes-Guiraud formula for $\mathcal{B}^*$, the proof of which is omitted. ■

Collecting all the previous Lemmas and using the Lumer-Phillips theorem (see [25]) gives the following:

Theorem 4.1 *The operator $-\mathcal{A}^\alpha$ given by (4.1) generates a strongly continuous semigroup of contractions on $L^2(\Theta)$.*

5 Convergence towards the macroscopic model.

Now, we consider the limit $\alpha \to 0$ of the following problem:

$$\alpha \frac{\partial}{\partial t} f^\alpha + \mathcal{A}^\alpha f^\alpha = 0 \quad \text{in } \Theta\,, \quad f_-^\alpha = \mathcal{B} f_+^\alpha \quad \text{on } \Gamma\,, \quad f^\alpha|_{t=0} = f_I\,. \tag{5.1}$$

To avoid the treatment of initial layers, we assume that the initial data are well prepared:

Hypothesis 5.1 *We suppose that there exists a function F_I such that $f_I(x, \underline{\xi}, v) = F_I(\underline{\xi}, |v|^2/2)$ and that f_I satisfies: $f_I \in L^2(\Theta)$, $(\underline{v} \cdot \nabla_{\underline{\xi}} - \underline{E} \cdot \nabla_{\underline{v}}) f_I \in L^2(\Theta)$.*

We note that hypothesis 5.1 implies that $f_I \in D(\mathcal{A}^\alpha)$ for all $\alpha > 0$. By Theorem 4.1, for any $T > 0$, problem (5.1) has a unique solution f^α belonging to $C^0([0,T], D(\mathcal{A}^\alpha)) \cap C^1([0,T], L^2(\Theta))$. In this section we are concerned with proving the main theorem 2.1. The proof will be divided in the following steps. First, we establish estimates on f^α showing the existence of a weak limit f^0 which does not depend on x and ω. After studying the auxilliary problem (2.8), (2.9), we show that the current converges weakly and we establish equation (2.5). Finally, we derive the continuity equation (2.4), which concludes the proof.

5.1 Weak limit of f^α

Lemma 5.1 *There exists a constant C, only depending on the data, such that:*

$$|f^\alpha|_{L^\infty(0,T;L^2(\Theta))} \le C\,, \tag{5.2}$$

$$\int_0^T |P_+ f_+^\alpha|^2_{L^2(\Gamma_+)}\, dt \le \alpha^2 C\,. \tag{5.3}$$

Moreover, for any $R > 0$, there exists C_R only depending on R and on the data, such that:

$$\int_0^T |q(f^\alpha)|^2_{\Gamma,R}\, dt \le C_R. \tag{5.4}$$

where the definition of $q(f^\alpha)$ is given by (4.11).

Proof: Multiplying the first equation of (5.1) by f^α, we have:

$$\alpha \left(|f^\alpha(t)|^2_{L^2(\Theta)} - |f_0|^2_{L^2(\Theta)} \right) + \int_0^t (\mathcal{A}^\alpha f^\alpha, f^\alpha)_\Theta\, ds = 0, \tag{5.5}$$

and using (4.9):

$$|f^\alpha(t)|^2_{L^2(\Theta)} + \frac{1-k_0^2}{2\alpha^2} \int_0^t |P_+ f_+^\alpha|^2_{L^2(\Gamma_+)}\, ds \le |f_I|^2_{L^2(\Theta)}, \tag{5.6}$$

which immediately gives (5.2) and (5.3). Moreover, from the proof of Lemma 4.3, we deduce:

$$|q(f^\alpha)|^2_{\Gamma,R} \le C \left(\alpha (\mathcal{A}^\alpha \chi_R f^\alpha, f^\alpha \chi_R \mathrm{sgn}(v_x)\phi)_\Theta + CR |f^\alpha|^2_{L^2(\Theta)} \right).$$

But, using (4.10) to evaluate $\mathcal{A}^\alpha(\chi_R f^\alpha)$ and the fact that $\mathcal{A}^\alpha f^\alpha = -\alpha \frac{\partial}{\partial t} f^\alpha$, we obtain:

$$\begin{aligned}
\int_0^T |q(f^\alpha)|^2_{\Gamma,R}\, ds &\le -\alpha^2 \int_0^T \left(\chi_R \frac{\partial}{\partial t} f^\alpha, \chi_R f^\alpha \mathrm{sgn}(v_x)\phi \right)_\Theta ds + CR \int_0^T |f^\alpha|^2_{L^2(\Theta)}\, ds \\
&\le -\alpha^2 \left[\int_\Theta |f^\alpha|^2 \mathrm{sgn}(v_x)\phi \chi_R^2\, dx\, d\underline{\xi}\, dv \right]_0^T + C \left(RT |f_0|^2_{L^2(\Theta)} \right) \\
&\le C_R.
\end{aligned}$$

Thus, proving (5.4). ■

As a consequence of Lemma 5.1, as α tends to 0, there exists a subsequence, still denoted by f^α, of solutions of problem (5.1), which converges in $L^\infty(0,T;L^2(\Theta))$ weak star to a function f^0. Furthermore, using the diagonal extraction process, the subsequence of $q(f^\alpha)$ converges to a function $q(x,\underline{\xi},|v|,t)$ with $x = 0,1$ in $L^2(0,T,L^2(\gamma\times B_R))$ weak star for any R, where B_R is the ball centered at 0 and of radius R in the velocity space. Also, from (5.3), the traces $P_+f_+^\alpha$ and $P_-f_-^\alpha$ converge in $L^2(0,T;L^2(\Gamma_+))$ and $L^2(0,T;L^2(\Gamma_-))$ (respectively) strongly towards zero.

Finally, we note that, since f^α is bounded in $L^\infty(0,T;L^2(\Theta))$, then, by equation (5.1), $\mathcal{A}^\alpha f^\alpha$ is bounded (and even tends to zero) in $H^{-1}(0,T;L^2(\Theta))$. This implies that $(v_x f^\alpha)|_\Gamma$ is bounded in $H^{-1}(0,T;H^{-1/2}(\gamma\times B_R))$ for any ball B_R in the velocity space, by standard properties of $H(div)$ spaces (see [24], [17]). Therefore, the traces of f^α on Γ have limits in the distributional sense that are the traces of f^0 on Γ. By the preceding considerations, we deduce that the traces $f_\pm^0$ of f^0 on $\Gamma_\pm$ satisfy:

$$P_-f_-^0 = P_+f_+^0 = 0\,,\quad Q_-f_-^0 = Q_+f_+^0 = q,$$

and so:

$$f^0|_\Gamma = q, \tag{5.7}$$

where $q = q(x,\underline{\xi},|v|,t)$, with $x = 0,1$, is independent of ω.

We now introduce the weak formulation:

Lemma 5.2 *Let f^α be the solution of problem (5.1). Then, f^α is a weak solution, i.e. for any test function $\phi\in C^1([0,T]\times\Theta)$, compactly supported in Θ and such that $\phi(\cdot,\cdot,T)=0$, we have:*

$$\begin{aligned}\int_0^T\int_\Theta f^\alpha\left(\alpha\frac{\partial}{\partial t}\phi + (\underline{v}\cdot\nabla_{\underline{\xi}}\phi - \underline{E}\cdot\nabla_{\underline{v}}\phi)\right)dt\,d\theta + \alpha\int_\Theta f_I\phi|_{t=0}\,d\theta \\ +\frac{1}{\alpha}\int_0^T\int_\Theta f^\alpha\left(v_x\frac{\partial}{\partial x}\phi - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\phi\right)dt\,d\theta \\ = \frac{1}{\alpha}\left(\int_0^T\int_{\Gamma_+}|v_x|\,f_+^\alpha(\phi_+ - \mathcal{B}^*\phi_-)\,dt\,d\Gamma\right).\end{aligned} \tag{5.8}$$

Proof: Multiply equation (5.1) by ϕ, use Green's Formula (4.8) and the boundary conditions. ■

Lemma 5.3 *The limit function f^0 is a function of $(\underline{\xi},|v|,t)$ only, i.e. $f^0 = f^0(\underline{\xi},|v|,t)$.*

Proof: Using (5.8) with ϕ compactly supported in Θ, we get:

$$\begin{aligned}\alpha^2\int_0^T\int_\Theta f^\alpha\frac{\partial}{\partial t}\phi\,dt\,d\theta + \alpha\int_0^T\int_\Theta f^\alpha(\underline{v}\cdot\nabla_{\underline{\xi}}\phi - \underline{E}\cdot\nabla_{\underline{v}}\phi)\,dt\,d\theta \\ +\int_0^T\int_\Theta f^\alpha\left(v_x\frac{\partial}{\partial x}\phi - (\underline{v}\times B)\cdot\nabla_{\underline{v}}\phi\right)dt\,d\theta + \alpha^2\int_\Theta f_I\phi|_{t=0}\,d\theta = 0.\end{aligned} \tag{5.9}$$

Hence, when α goes to 0 in (5.9), using the fact that f^α is bounded in $L^\infty(0,T,L^2(\Theta))$, we get:

$$\int_0^T \int_\Theta f^0 \left(v_x \frac{\partial}{\partial x}\phi - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\phi \right) dt\, d\theta = 0. \tag{5.10}$$

This is equivalent to saying that f^0 is a distributional solution of the problem

$$\mathcal{A}^0 f^0 = 0\,, \quad f^0|_\Gamma = q, \tag{5.11}$$

where $\mathcal{A}^0$ is defined by:

$$\mathcal{A}^0 f = v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f, \tag{5.12}$$

and $q = q(x, \xi, |v|, t)$ is independent of ω, as a consequence of (5.7).

To solve problem $\mathcal{A}^0 f = 0$, we first note that $\mathcal{A}^0$ operates only on the variables $(x, \omega) \in [0,1] \times \mathbb{S}^2$, and that $\underline{\xi} \in \mathbb{R}^2$ and $|v| \geq 0$ are mere parameters. Indeed, we can write:

$$\mathcal{A}^0 f = |v|\omega_x \frac{\partial f}{\partial x} + B(\underline{\xi}) \frac{\partial f}{\partial \omega}(e_x \times \omega) \tag{5.13}$$

where $\dfrac{\partial f}{\partial \omega}(e_x \times \omega)$ is the differential of f with respect to $\omega \in \mathbb{S}^2$ acting on the tangent vector to $\mathbb{S}^2$, $e_x \times \omega$.

We therefore only consider the dependence of f on $(x, \omega) \in [0,1] \times \mathbb{S}^2$. Note that $\mathbb{S}^2$ can be parameterized by the map $\omega(\sigma, \underline{\omega})$, where $\sigma = \omega_x/|\omega_x| \in \{-1, 1\}$ and $\underline{\omega} = (\omega_y, \omega_z)$. The fact that σ is equal to ± 1 recalls that we need two maps to parameterize the sphere in this way.

Next, we note $R^+_{(x,\sigma)}(\underline{\omega})$, the rotation of $\underline{\omega}$ about the x-axis of an angle bx, where

$$b = \frac{B(\underline{\xi})}{|v|\omega_x}\,, \quad \omega_x = \sigma\sqrt{1 - \omega_y^2 - \omega_z^2}.$$

In other words, $\underline{\omega}^\dagger = R^+_{(x,\sigma)}(\underline{\omega}) = (\omega_y^\dagger, \omega_z^\dagger)$ is given by:

$$\omega_y^\dagger = \omega_y \cos bx - \omega_z \sin bx\,, \quad \omega_z^\dagger = \omega_y \sin bx + \omega_z \cos bx\,. \tag{5.14}$$

We note that $\underline{\omega}^\dagger$ also depends on $|v|$ and $\underline{\xi}$, but we shall not stress this dependence otherwise needed.

Similarly, $R^-_{(x,\sigma)}(\underline{\omega})$ is the rotation of an angle $-bx$. Obviously, $\underline{\omega}^\dagger = R^+_{(x,\sigma)}(\underline{\omega})$ if and only if $\underline{\omega} = R^-_{(x,\sigma)}(\underline{\omega}^\dagger)$. Also, $R^-_{(x',\sigma)} R^+_{(x,\sigma)} = R^+_{(x-x',\sigma)} = R^-_{(x'-x,\sigma)}$. Then, with the change of unknowns:

$$f^\dagger(x, \sigma, \underline{\omega}) = f(x, \sigma, R^+_{(x,\sigma)}(\underline{\omega})), \tag{5.15}$$

a simple application of the chain rule gives:

$$|v|\omega_x \frac{\partial f^\dagger}{\partial x} = v_x \frac{\partial f}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}} f = \mathcal{A}^0 f = 0, \tag{5.16}$$

which means that $f^\dagger$ is constant with respect to x. Note that in obtaining (5.16), we have used that $\omega_x = \sigma|\omega_x|$ and $|\omega_x| = \sqrt{1-|\underline{\omega}|^2} = \sqrt{1-|\underline{\omega}^\dagger|^2}$ with $|\underline{\omega}|^2 = |\omega_y|^2 + |\omega_z|^2$. Therefore,

$$f^\dagger(x, \sigma, \underline{\omega}) = f^\dagger(0, \sigma, \underline{\omega}) = f^\dagger(1, \sigma, \underline{\omega}) \tag{5.17}$$

which, back to f, gives:

$$f(x, \sigma, \underline{\omega}) = f(0, \sigma, R^-_{(x,\sigma)}(\underline{\omega})) = f(1, \sigma, R^+_{(1-x,\sigma)}(\underline{\omega})). \tag{5.18}$$

Now, let f^0 be a solution of (5.11). Then, there exists $q(x)$, $x = 0, 1$, such that,

$$f^0(0, \sigma, \underline{\omega}) = q(0) \ , \quad f^0(1, \sigma, \underline{\omega}) = q(1).$$

From (5.18), this implies that

$$f^0(x, \sigma, \underline{\omega}) = q(0) = q(1) = q.$$

Writing again the full set of variables, we get:

$$f^0(x, v, t) = q(\underline{\xi}, |v|, t) \tag{5.19}$$

which was the result to be proved. ∎

Remark 5.1 The trace estimates (5.3) and (5.4) were essential to establish that the traces of the limit function f^0 converge in an L^2 sense. In [2], the convergence of the traces happen in a weaker sense and it is not clear why the limit trace should satisfy the boundary operator.

Formula (5.18) implies that:

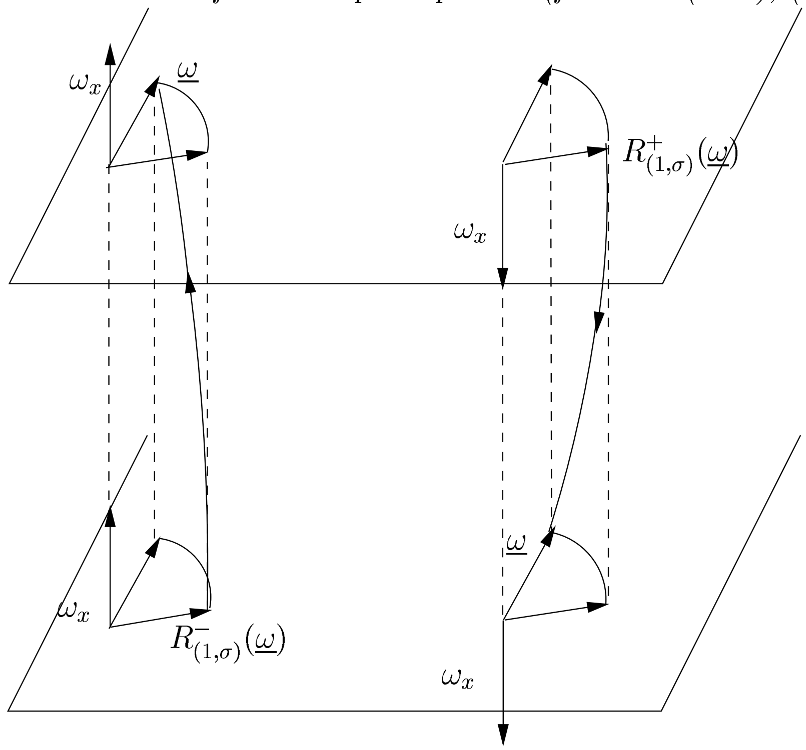
$$\begin{cases} f_+(1, \sigma, \underline{\omega}) = f_-(0, \sigma, R^-_{(1,\sigma)}(\underline{\omega})), & \sigma = 1, \\ f_+(0, \sigma, \underline{\omega}) = f_-(1, \sigma, R^+_{(1,\sigma)}(\underline{\omega})), & \sigma = -1, \end{cases} \tag{5.20}$$

which can be written compactly $f_+ = M_- f_-$, thus defining a map M_- from $L^2(\mathcal{S}_-)$ to $L^2(\mathcal{S}_+)$. The inverse map M_+ from $L^2(\mathcal{S}_+)$ to $L^2(\mathcal{S}_-)$ is defined by $f_- = M_+ f_+$ with:

$$\begin{cases} f_-(0, \sigma, \underline{\omega}) = f_+(1, \sigma, R^+_{(1,\sigma)}(\underline{\omega})), & \sigma = 1, \\ f_-(1, \sigma, \underline{\omega}) = f_+(0, \sigma, R^-_{(1,\sigma)}(\underline{\omega})), & \sigma = -1, \end{cases} \tag{5.21}$$

Remark that M_- and M_+ are isometries between the spaces $L^2(\mathcal{S}_-)$ and $L^2(\mathcal{S}_+)$. Formulas (5.20) and (5.21) are illustrated on Figure 5.1. The mappings M_- and M_+ will be used below. We also define the kinetic energy $\varepsilon = |v|^2/2$ and the energy distribution function $F(\underline{\xi}, \varepsilon, t) = f^0(\underline{\xi}, |v|, t)$.

Figure 5.1 *Action of the transport operator (formulae (5.20), (5.21))*



5.2 Auxiliary equation

We now study the following auxiliary problem, which will be useful for finding the limit of the current: to find $f(x,\omega)$ such that:

$$\mathcal{A}^{0*} f(x,\omega) = g(x,\omega)\,,\ (x,\omega) \in [0,1]\times\mathbb{S}^2 \qquad (5.22)$$

$$f_+(x,\omega) = \mathcal{B}^* f_-(x,\omega)\,,\ (x,\omega)\in\mathcal{S}_+, \qquad (5.23)$$

where $\mathcal{A}^{0*}$ is the formal adjoint of $\mathcal{A}^0$:

$$\mathcal{A}^{0*} f = -v_x \frac{\partial f}{\partial x} + (\underline{v}\times B)\nabla_{\underline{v}} f = -\left(|v|\omega_x \frac{\partial f}{\partial x} + B(\underline{\xi})\frac{\partial f}{\partial \omega}(e_x\times\omega)\right)\,.$$

We recall that $\mathcal{A}^{0*}$, like $\mathcal{A}^0$, operates only on $(x,\omega)\in[0,1]\times\mathbb{S}^2$, and not on $\underline{\xi}\in\mathbb{R}^2$, $|v|>0$ (see (5.13)), which are therefore omitted in the following discussion. Similarly, $\mathcal{B}^*$ only operates on $\omega\in\mathbb{S}^2$. We now prove:

Lemma 5.4 *Let $g\in L^2([0,1]\times\mathbb{S}^2)$ and define $G = G(x,\sigma,\underline{\omega})$ by:*

$$G(x,\sigma,\underline{\omega}) = \begin{cases} \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_x^1 g(x',\sigma,R^+_{(x'-x,\sigma)}(\underline{\omega}))\,dx'\,, & \sigma=+1,\\[2ex] \dfrac{1}{|v|\,|\omega_x|}\displaystyle\int_0^x g(x',\sigma,R^-_{(x-x',\sigma)}(\underline{\omega}))\,dx'\,, & \sigma=-1.\end{cases} \qquad (5.24)$$

Note that G also depends on $|v|$ and $\underline{\xi}$. Suppose that $\sqrt{|\omega_x|}G$ belongs to $L^2([0,1]\times\mathbb{S}^2)$ and that its trace G_- on Γ_- belongs to $L^2(\mathcal{S}_-)$ for almost every $(|v|,\underline{\xi}) \in \mathbb{R}_+ \times \mathbb{R}^2_{\underline{\xi}}$. Then, problem (5.22), (5.23) has a solution f such that $\sqrt{|\omega_x|}f \in L^2([0,1]\times\mathbb{S}^2)$ for almost every $(|v|,\underline{\xi}) \in \mathbb{R}_+ \times \mathbb{R}^2_{\underline{\xi}}$ if and only if the condition:

$$\int_0^1 \int_{\mathbb{S}^2} g(x,\omega)\, dx\, d\omega = 0 \tag{5.25}$$

holds. Furthermore, all solutions in this space are equal to f, up to an additive function of $\underline{\xi}$ and $|v|$.

Proof: Using the change of variables (5.14), we are led to:

$$|v|\omega_x \frac{\partial f^\dagger}{\partial x} = -g^\dagger. \tag{5.26}$$

Integrating (5.26) with respect to x, we obtain:

$$f^\dagger(x,\sigma,\underline{\omega}^\dagger) = \begin{cases} f^\dagger(1,\sigma,\underline{\omega}^\dagger) + \dfrac{1}{|v|\omega_x}\displaystyle\int_x^1 g^\dagger(x',\sigma,\underline{\omega}^\dagger)\, dx' , & \sigma = +1, \\ f^\dagger(0,\sigma,\underline{\omega}^\dagger) + \dfrac{-1}{|v|\omega_x}\displaystyle\int_0^x g^\dagger(x',\sigma,\underline{\omega}^\dagger)\, dx' , & \sigma = -1. \end{cases} \tag{5.27}$$

Back to the original variables, this gives:

$$f(x,\sigma,\underline{\omega}) = \begin{cases} f_+(1,\sigma,R^+_{(1-x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega}) , & \sigma = +1, \\ f_+(0,\sigma,R^-_{(x,\sigma)}(\underline{\omega})) + G(x,\sigma,\underline{\omega}) , & \sigma = -1. \end{cases} \tag{5.28}$$

Now it is clear that, if we find f_+ in $L^2(\mathcal{S}_+)$ such that f given by (5.28) satisfies the boundary conditions (5.23), then f is a solution of problem (5.22), (5.23) with the regularity $\sqrt{|\omega_x|}f \in L^2([0,1]\times\mathbb{S}^2)$. This follows from the assumption on G, and from the fact that $R^+(\underline{\omega})$ is a rotation of $\underline{\omega}$, and gives:

$$\int_{|\underline{\omega}|\leq 1} |f_+(1,\sigma,R^+_{(1-x,\sigma)}(\underline{\omega}))|^2\, d\underline{\omega} = \int_{|\underline{\omega}|\leq 1} |f_+(1,\sigma,\underline{\omega})|^2\, d\underline{\omega}$$

(Note that, in the parameterization $\omega = (\sigma,\underline{\omega})$, we have $|\omega_x|\, d\omega = d\underline{\omega}$).

Now, we show that there actually exists such an $f_+ \in L^2(\mathcal{S}_+)$. We also show that f_+ is unique up to an additive constant, which proves the last statement of the lemma, since $\underline{\xi}$ and $|v|$ are mere parameters throughout this proof. Evaluating (5.28) at $x = 0, 1$ we can write $f_- = M_+ f_+ + G_-$ where G_- is the trace of G on Γ_-. Thus, by means of the boundary condition (5.23) we have:

$$f_+ = \mathcal{B}^*(M_+ f_+ + G_-) \tag{5.29}$$

Using the same notations as in the proof of Lemma 3.2, (5.29) can be written:

$$\begin{cases} f_+(1) - \mathcal{B}^*(1) M_+(f_+(0)) = \mathcal{B}^*(1) G_-(1) \\ f_+(0) - \mathcal{B}^*(0) M_+(f_+(1)) = \mathcal{B}^*(0) G_-(0) \end{cases} , \tag{5.30}$$

and so:

$$\left\{ \begin{array}{l} f_+(1) - (\mathcal{B}^*(1)M_+\mathcal{B}^*(0)M_+)(f_+(1)) = h(1) \\ f_+(0) - (\mathcal{B}^*(0)M_+\mathcal{B}^*(1)M_+)(f_+(0)) = h(0) \end{array} \right. , \tag{5.31}$$

with:

$$\left\{ \begin{array}{l} h(1) = \mathcal{B}^*(1)M_+\mathcal{B}^*(0)G_-(0) + \mathcal{B}^*(1)G_-(1) \\ h(0) = \mathcal{B}^*(0)M_+\mathcal{B}^*(1)G_-(1) + \mathcal{B}^*(0)G_-(0) \end{array} \right. .$$

We now concentrate on the first equation of (5.31), the treatment of the second one being similar. Using the expression (3.12) of $\mathcal{B}^*$, we can write:

$$\mathcal{B}^*(1)M_+\mathcal{B}^*(0)M_+ = \beta(0)\beta(1)(J^*M_+)^2 + \mathcal{L}(1) \tag{5.32}$$

where $\mathcal{L}(1)$ is a compact operator on $L^2(\mathcal{S}_+(1))$. Moreover, equation (5.31) can be written:

$$(I - \mathcal{G}(1))(f_+(1)) = k(1), \tag{5.33}$$

where:

$$\mathcal{G}(1) = (I - \beta(0)\beta(1)(J^*M_+)^2)^{-1}\mathcal{L}(1), \tag{5.34}$$

$$k(1) = (I - \beta(0)\beta(1)(J^*M_+)^2)^{-1}h(1). \tag{5.35}$$

Now, it is an easy matter to check that $\mathcal{G}(1)$ is a compact operator on $L^2(\mathcal{S}_+(1))$, which is positive (i.e. if $\phi \in L^2(\mathcal{S}_+(1))$, $\phi \geq 0$, then $\mathcal{G}(1)\phi > 0$) and such that the constant functions on $\mathcal{S}_+(1)$ are eigenfunctions associated with the eigenvalue 1. Therefore, by the Krein-Rutman Theorem, the Null-Space $N(I - \mathcal{G}(1))$ is of dimension 1.

By the Fredholm Alternative, equation (5.33) has a solution f_+ in $L^2(\mathcal{S}_+(1))$ if and only if $k(1) \in N(I - \mathcal{G}^*(1))^\perp$. It is an easy matter to check that $N(I - \mathcal{G}^*(1))$ also consists of constant functions. Therefore, the solvability condition for equation (5.33) reads $(k(1), 1)_{L^2(\mathcal{S}_+(1))} = 0$, or:

$$(h(1), (I - \beta(0)\beta(1)(M_-J)^2)^{-1}1)_{L^2(\mathcal{S}_+(1))} = 0. \tag{5.36}$$

But, since $(I - \beta(0)\beta(1)(M_-J)^2)1 = 1 - \beta(0)\beta(1)$, condition (5.36) is also written $(h(1), 1)_{L^2(\mathcal{S}_+(1))} = 0$, or, by duality,

$$(G_-(0), \mathcal{B}(0)M_-\mathcal{B}(1)1)_{L^2(\mathcal{S}_-(0))} + (G_-(1), \mathcal{B}(1)1)_{L^2(\mathcal{S}_-(1))} = 0. \tag{5.37}$$

But, $\mathcal{B}(0)M_-\mathcal{B}(1)1 = 1$ and $\mathcal{B}(1)1 = 1$. So (5.37) gives:

$$(G_-(0), 1)_{L^2(\mathcal{S}_-(0))} + (G_-(1), 1)_{L^2(\mathcal{S}_-(1))} = 0, \tag{5.38}$$

which is written explicitely:

$$\int_0^1 \int_{|\underline{\omega}| \leq 1} \left(g(x, 1, R^+_{(x,1)}(\underline{\omega})) + g(x, -1, R^-_{(1-x,-1)}(\underline{\omega})) \right) \frac{1}{|\omega_x|} \, d\underline{\omega} \, dx = 0. \tag{5.39}$$

But, using the change of variables $\underline{\omega}' = R^+_{(x,1)}(\underline{\omega})$ for the first term and $\underline{\omega}' = R^-_{(1-x,-1)}(\underline{\omega})$ for the second one, condition (5.39) reads:

$$\int_0^1 \int_{|\omega|\leq 1} (g(x,1,\underline{\omega}) + g(x,-1,\underline{\omega}))\, \frac{1}{|\omega_x|}\, d\underline{\omega}\, dx = 0 \tag{5.40}$$

which is exactly the expression of the solvability condition (5.25) in the parametrization $\omega(\sigma,\underline{\omega})$.

The existence of $f_+(1)$ and its uniqueness up to an additive constant (i.e. belonging to the Null-Space of $(I - \mathcal{G}(1))$) are thus proved, under the solvabilty condition (5.25). Similarly, the existence of $f_+(0)$ and its uniqueness, up to an additive constant, are proved under the same condition. That the two arbitrary constants for $f_+(1)$ and $f_+(0)$ are equal follows easily from considering equations (5.30). ■

Remark 5.2 The present technique, based on the reduction to an integral equation at the boundary is mainly due to [2]. However, we note that the regularity $L^2(\mathcal{S}_-)$ of the trace does not imply the regularity $L^2([0,1]\times\mathbb{S}^2)$ for the function inside the domain because of the weight $|\omega_x|$. This point seems to have been overlooked in [2] and is connected to the question of characterizing the range of the trace operator for solutions of first order problems, which is still mainly open up to now.

Lemma 5.5 *The functions $g = \omega_i$, $(i = y, z)$ satisfy the assumptions of Lemma 5.4.*

Proof:

Let G_y be the function associated to $g = \omega_y$ by (5.24). We need to show that $\sqrt{|\omega_x|}G_y \in L^2([0,1]\times\mathbb{S}^2)$ and $(G_y)_- \in L^2(\mathcal{S}_-)$. The proof is obviously similar for $g = \omega_z$. After straightforward computations, we obtain

$$G_y(x,\sigma,\underline{\omega}) = \begin{cases} \dfrac{1}{B(\underline{\xi})}[\omega_y \sin b(1-x) + \omega_z(\cos b(1-x) - 1)]\, dx'\ ,\ \sigma = +1, \\ -\dfrac{1}{B(\underline{\xi})}[\omega_y \sin bx + \omega_z(1-\cos bx)]\, dx'\ ,\ \sigma = -1. \end{cases},$$

With the Hypothesis 4.1 (iii) on B, it is an easy matter to check that G_y satisfies the required hypothesis. ■

We note that the magnetic field operator has removed the singularity $|\omega_x|^{-1}$ that would otherwise be expected. The magnetic field thus contributes to maintain a finite diffusivity (see introduction).

By Lemma 5.4, there exist functions $D_i(x,\omega;\underline{\xi},\varepsilon)$, $(i = y, z)$, solutions of problem (5.22), (5.23) with right-hand-side $g = \omega_i$, unique up to additive functions of $\underline{\xi}$ and ε. We suppose that D_i satisfy the following regularity requirements:

Hypothesis 5.2 *(i) D_i, $(i = y, z)$, belong to $L^2([0,1]\times\mathbb{S}^2)$ for almost every $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$ and are C^1 bounded functions on Θ away from the set $\{v_x = 0\}$.*
(ii) The functions $\omega_i D_j(x, \omega; \underline{\xi}, \varepsilon)$ belongs to $L^1([0,1] \times \mathbb{S}^2)$ and $\int_0^1 \int_{\mathbb{S}^2} \omega_i D_j \, dx \, d\omega$ is a C^1 function of $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times (0, \infty)_\varepsilon$, uniformly bounded on $\mathbb{R}^2_{\underline{\xi}} \times [0, \infty)_\varepsilon$, and tending to 0 as $\varepsilon \to 0$.

Hypothesis 5.2 can be viewed as a regularity assumption on the data i.e. on the magnetic field B, the boundary scattering kernel K and the accommodation coefficient β. We do not look for explicit conditions on these data because the developments would be technical and of rather limited interest. We confine ourselves to noting that hypothesis 5.2 is not empty, because it is satisfied at least in the case of isotropic scattering. In this case [12], the scattering kernel K is equal to the constant π^{-1}, and we have for $\omega_x > 0$:

$$\begin{aligned} D_z &= \frac{1}{4\pi B}\omega_y^\dagger\left(\frac{1-\beta_0}{1-\beta_0\beta_1}(1-\cos b) + \cos b - \cos bx\right) \\ &\quad + \frac{1}{4\pi B}\omega_z^\dagger\left(\frac{1+\beta_0}{1-\beta_0\beta_1}\sin b - \sin b + \sin bx\right), \end{aligned}$$

with $\underline{\omega}^\dagger = R^-_{(x,\sigma)}(\underline{\omega})$ and similarly for $\omega_x < 0$, and for D_y. From hypothesis 4.1 (ii), (iii) and hypothesis 3.3 (ii), $\underline{D}$ is bounded and thus , $\underline{\omega}\,\underline{D}$ is clearly integrable in (x, ω). Furthermore, formula (4.34) and (4.35) of [12] shows that $\int_0^1 \int_{\mathbb{S}^2} \underline{\omega}\,\underline{D} \, dx \, d\omega$ is a C^1 function of $\underline{\xi}$, ε for $\underline{\xi} \in \mathbb{R}^2$ and $\varepsilon > 0$, as soon as B and the accommodation coefficient β are C^1 and that it is bounded and tends to 0 as $\varepsilon \to 0$. Finally, away from the plane $\{v_x = 0\}$, all derivative of D_z are smooth, showing that Hypothesis 5.2 is satisfied.

From Hypothesis 5.2 (ii), we deduce that the diffusivity tensor (2.7) is defined and is a C^1 function of $(\underline{\xi}, \varepsilon) \in \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$. We also note that the definition of D_{ij} does not depend on the arbitrary additive function of $\underline{\xi}$ and ε which enters in the definition of D_j.

5.3 The current equation

Let us define the current $\underline{J}^\alpha(\underline{\xi}, \varepsilon, t) = (J^\alpha_y, J^\alpha_z)$ as follows:

$$\begin{aligned} \underline{J}^\alpha(\underline{\xi}, \varepsilon, t) &= \frac{|v|}{\alpha}\int_0^1 \int_{\mathbb{S}^2} \underline{v} f^\alpha(x, \underline{\xi}, |v|, \omega, t) \, dx \, d\omega \\ &= \frac{2\varepsilon}{\alpha}\int_0^1 \int_{\mathbb{S}^2} \underline{\omega} f^\alpha(x, \underline{\xi}, \varepsilon, \omega, t) \, dx \, d\omega. \end{aligned} \tag{5.41}$$

We first prove the following technical lemma:

Lemma 5.6 *Let $\varphi(x, v)$ be a C^1 function. We have:*

$$\sqrt{2\varepsilon}\int_0^1 \int_{\mathbb{S}^2} (\nabla_{\underline{v}}\varphi)(x, \sqrt{2\varepsilon}\omega) \, dx \, d\omega = \frac{\partial}{\partial\varepsilon} J_\varphi, \quad J_\varphi(\varepsilon) = 2\varepsilon \int_0^1 \int_{\mathbb{S}^2} \underline{\omega}\, \varphi(x, \sqrt{2\varepsilon}\omega) \, dx \, d\omega \tag{5.42}$$

Proof: We only compute the y component. We note that

$$\frac{\partial \varphi}{\partial v_y} = \frac{\partial \varphi}{\partial |v|}\omega_y + \frac{1}{|v|}\frac{\partial \varphi}{\partial \omega}(e_y - \omega_y \omega), \tag{5.43}$$

where $(\partial\varphi/\partial\omega)(e_y - \omega_y\omega)$ denotes the derivative of φ with respect to ω acting on the tangent vector $e_y - \omega_y\omega$ Then, integrating (5.43) with respect to x and ω and using that

$$\int_{\mathbb{S}^2} \frac{\partial \varphi}{\partial \omega}(e_y - \omega_y\omega)\, d\omega = -\int_{\mathbb{S}^2} \varphi(\omega)\, \mathrm{div}_{\mathbb{S}^2}(e_y - \omega_y\omega)\, d\omega = -\int_{\mathbb{S}^2} \varphi(\omega)(-2\omega_y)\, d\omega,$$

which is deduced from Stoke's Theorem on the sphere, we easily find (5.42). ■

We denote by Θ' the position-energy space $\Theta' = \mathbb{R}^2_{\underline{\xi}} \times \mathbb{R}^+_\varepsilon$ and by $d\theta'$ its volume element $d\theta' = d\underline{\xi}\, d\varepsilon$. We note that $dv = |v|^2 d|v|\, d\omega = \sqrt{2\varepsilon}\, d\varepsilon\, d\omega$.

Lemma 5.7 *As α goes to 0, the current $\underline{J}^\alpha(\underline{\xi}, \varepsilon, t)$ converges in the sense of distribution to $\underline{J}(\underline{\xi}, \varepsilon, t)$ given by (2.5). More precisely, for every $\underline{\psi} = (\psi_y, \psi_z) \in C^1(\Theta' \times [0,T], \mathbb{R}^2)$ with compact support in $\mathbb{R}^2_{\underline{\xi}} \times (0,\infty)_\varepsilon \times [0,T[$, we have:*

$$\lim_{\alpha\to 0} \int_0^T \int_{\Theta'} \underline{J}^\alpha \cdot \underline{\psi}\, d\theta'\, dt = \int_0^T \int_{\Theta'} F\left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right) \cdot (\mathbb{D}^T \underline{\psi})\, dt\, d\theta', \tag{5.44}$$

where $\mathbb{D}^T$ denotes the transpose of $\mathbb{D}$.

We note that the right-hand side of equ. (5.44) is the weak form of that of equ. (2.5).

Proof: We use the weak formulation (5.8) with $\phi = \sqrt{2\varepsilon}\underline{\psi}(\underline{\xi}, \varepsilon, t) \cdot \underline{D}(x, \omega; \underline{\xi}, \varepsilon)$ $\chi_\rho(v_x)$ for test function. Since $\underline{D}$ is not smooth at $v_x = 0$, we use the truncation function χ_ρ which is smooth, vanishes identically for $|v_x| \leq \rho$ and is equal to 1 for $|v_x| \geq 2\rho$. Hypothesis 5.2 provides all the necessary assumptions to allow the passage $\rho \to 0$ in (5.8). Therefore, this passage will be omitted in the following proof and we just use $\underline{\psi}\, \underline{D}$ as a test function as if it were smooth. Because of (5.22), (5.23), we have for $i \in \{y, z\}$:

$$\begin{aligned}\left(v_x\frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\right)(\sqrt{2\varepsilon}\psi_i\, D_i) &= \sqrt{2\varepsilon}\psi_i \left(v_x\frac{\partial}{\partial x} - (\underline{v} \times B) \cdot \nabla_{\underline{v}}\right)(D_i) \\ &= -\sqrt{2\varepsilon}\psi_i\, \omega_i\,, \quad \mathrm{in}\Theta,\end{aligned}$$

and

$$(\sqrt{\varepsilon}\psi_i\, D_i)_+ - \mathcal{B}^*(\sqrt{\varepsilon}\psi_i\, D_i)_- = \sqrt{\varepsilon}\psi_i[(D_i)_+ - \mathcal{B}^*(D_i)_-] = 0\,, \quad \mathrm{on}\Gamma\,.$$

Therefore:

$$
\begin{aligned}
&\frac{1}{\alpha}\int_0^T\int_\Theta f^\alpha\left(v_x\frac{\partial}{\partial x}-(\underline{v}\times B)\cdot\nabla_{\underline{v}}\right)(\sqrt{2\varepsilon}\underline{\psi}\cdot\underline{D})\,dt\,d\theta\\
&\qquad-\frac{1}{\alpha}\int_0^T\int_{\Gamma_+}|v_x|\,f^\alpha(\,(\sqrt{2\varepsilon}\underline{\psi}\cdot\underline{D})_+-\mathcal{B}^*(\sqrt{2\varepsilon}\underline{\psi}\cdot\underline{D})_-\,dt\,d\Gamma\\
&=-\frac{1}{\alpha}\int_0^T\int_\Theta f^\alpha\,\underline{\omega}\cdot\underline{\psi}\sqrt{2\varepsilon}\,dt\,d\theta=-\int_0^T\int_{\Theta'}\underline{J}^\alpha\cdot\underline{\psi}\,dt\,d\theta'\,,
\end{aligned}
$$

which a posteriori justifies the introduction of the auxilliary function $\underline{D}$. Thus, the weak formulation (5.8) yields:

$$
\begin{aligned}
\int_0^T\int_{\Theta'}\underline{J}^\alpha\cdot\underline{\psi}\,dt\,d\theta'&=\alpha\int_0^T\int_\Theta\sqrt{2\varepsilon}f^\alpha\underline{D}\cdot\frac{\partial}{\partial t}\underline{\psi}\,dt\,d\theta+\alpha\int_\Theta\sqrt{2\varepsilon}f_I\underline{D}\cdot\underline{\psi}|_{t=0}\,d\theta\\
&\quad+\int_0^T\int_\Theta f^\alpha(\underline{v}\cdot\nabla_{\underline{\xi}}-\underline{E}\cdot\nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{D}\cdot\underline{\psi})\,dt\,d\theta\,. \qquad (5.45)
\end{aligned}
$$

Now, we let α tend to 0. Because $\underline{D}\in L^2([0,1]\times\mathbb{S}^2)$, for almost every $(\underline{\xi},\varepsilon)\in\mathbb{R}^2_{\underline{\xi}}\times\mathbb{R}^+_\varepsilon$, the first and second terms on the right hand side of (5.45) converge to 0. Now, the limit of the last term exists and we have, because of Lemma 5.3:

$$
\begin{aligned}
\lim_{\alpha\to0}\int_0^T\int_\Theta f^\alpha\underline{v}\cdot\nabla_{\underline{\xi}}(\sqrt{2\varepsilon}\underline{D}\cdot\underline{\psi})\,dt\,d\theta&=\int_0^T\int_\Theta f^0\nabla_{\underline{\xi}}\cdot[2\varepsilon\underline{\omega}(\underline{D}\cdot\underline{\psi})]\,dt\,d\theta\\
&=\int_0^T\int_{\Theta'}F\nabla_{\underline{\xi}}\cdot\left((2\varepsilon)^{3/2}\int_0^1\int_{\mathbb{S}^2}\underline{\omega}(\underline{D}\cdot\underline{\psi})\,dx\,d\omega\right)dt\,d\theta'\\
&=\int_0^T\int_{\Theta'}F\nabla_{\underline{\xi}}\cdot[\mathbb{D}^T\underline{\psi}]\,dt\,d\theta', \qquad (5.46)
\end{aligned}
$$

Similarly, using (5.42), we have:

$$
\begin{aligned}
\lim_{\alpha\to0}\int_0^T\int_\Theta f^\alpha(\underline{E}\cdot\nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{D}\cdot\underline{\psi})\,dt\,d\theta&=\int_0^T\int_\Theta f^0(\underline{E}\cdot\nabla_{\underline{v}})(\sqrt{2\varepsilon}\underline{D}\cdot\underline{\psi})\,dt\,d\theta\\
&=\int_0^T\int_{\Theta'}F\underline{E}\cdot\left(\sqrt{2\varepsilon}\int_0^1\int_{\mathbb{S}^2}\nabla_{\underline{v}}(\sqrt{2\varepsilon}\underline{D}\cdot\underline{\psi})\,dx\,d\omega\right)dt\,d\theta'\\
&=\int_0^T\int_{\Theta'}F\underline{E}\cdot\frac{\partial}{\partial\varepsilon}\left((2\varepsilon)^{3/2}\int_0^1\int_{\mathbb{S}^2}\underline{\omega}(\underline{D}\cdot\underline{\psi})\,dx\,d\omega\right)dt\,d\theta'\\
&=\int_0^T\int_{\Theta'}F\underline{E}\cdot\frac{\partial}{\partial\varepsilon}(\mathbb{D}^T\underline{\psi})\,dt\,d\theta'\,. \qquad (5.47)
\end{aligned}
$$

Lemma 5.7 follows by collecting (5.46) and (5.47). ■

5.4 The continuity equation

To end the proof of Theorem 2.1, it remains to prove that equations (2.4) and (2.6) hold true in the weak sense. This is the object of the following:

Lemma 5.8 *For any test function $\psi(\underline{\xi}, \varepsilon, t)$ belonging to $C^2(\Theta' \times [0,T])$, with compact support in $\mathbb{R}^2_{\underline{\xi}} \times (0,\infty)_\varepsilon \times [0,T[$, we have:*

$$\int_0^T \int_{\Theta'} \left(4\pi\sqrt{2\varepsilon} F \frac{\partial \psi}{\partial t} + \underline{J} \cdot \left(\nabla_{\underline{\xi}} - \underline{E}\frac{\partial}{\partial \varepsilon}\right)\psi\right) dt\, d\theta' + \int_{\Theta'} 4\pi\sqrt{2\varepsilon} F_I \psi|_{t=0}\, d\theta' = 0. \tag{5.48}$$

Note that equ. (5.48) is the weak form of equs. (2.4) and (2.6).

Proof: We first define the function:

$$F^\alpha(\underline{\xi}, \varepsilon, t) = \frac{1}{4\pi}\int_0^1 \int_{\mathbb{S}^2} f^\alpha(x, \underline{\xi}, |v|, \omega, t) dx\, d\omega, \tag{5.49}$$

which weakly converges to F as α tends to 0. Dividing equation (5.8) by α, and using $\phi = \psi(\underline{\xi}, \varepsilon, t)$ as a test function, we obtain the weak form of the continuity equation, which looks exactly similar to (5.48) except that F and $\underline{J}$ are replaced by F^α and $\underline{J}^\alpha$. Using the weak convergence of F^α and $\underline{J}^\alpha$ to F and $\underline{J}$ respectively (see Lemma 5.7), allows to pass to the limit in the continuity equation for F^α and to obtain equation (5.48). ■

6 Properties of the diffusivity

In the next two propositions, we prove that the diffusion tensor $\mathbb{D}$ defined in (2.7) satisfies the Onsager relation $\mathbb{D}(B)^T = \mathbb{D}(-B)$ and that it is positive definite, i.e. $(\mathbb{D}Y, Y) > 0$.

Proposition 6.1 *The diffusion tensor $\mathbb{D}$ satisfies $\mathbb{D}(B)^T = \mathbb{D}(-B)$.*

Proof: For $f(x,\omega)$, we define the transformation $\mathcal{J}f$: $\mathcal{J}f(x,\omega) = f(x,-\omega)$. We make the dependence of $\mathcal{A}^0$ and $\mathbb{D}$ upon B explicit by writing $\mathcal{A}^0(B)$ and $\mathbb{D}(B)$. We begin by noting that $\mathcal{J}f$ is a solution of $\mathcal{A}^{0^*}(-B)\mathcal{J}f = \mathcal{J}g$, $\mathcal{J}f_+ = \mathcal{B}^*\mathcal{J}f_-$ if and only if the function f is solution of $\mathcal{A}^0(B)f = g$, $f_- = \mathcal{B}f_+$. This follows from the reciprocity relation (3.9), and from:

$$\mathcal{A}^{0^*}(-B)\mathcal{J}f = \mathcal{J}\mathcal{A}^0(B)f \quad , \quad \mathcal{A}^0(B)\mathcal{J}f = \mathcal{J}\mathcal{A}^{0^*}(-B)f. \tag{6.1}$$

Now, we have (see (3.2) for the definition of $(\cdot,\cdot)_{\mathcal{S}}$):

$$(\mathcal{A}^{0^*}(B)D_i(B), \mathcal{J}D_j(-B))_{\mathcal{S}} = (D_i(B), \mathcal{A}^0(B)\mathcal{J}D_j(-B))_{\mathcal{S}}.$$

But on the one hand:

$$\left(\mathcal{A}^{0*}(B)D_i(B), \mathcal{J}D_j(-B)\right)_{\mathcal{S}} = (\omega_i, \mathcal{J}D_j(-B))_{\mathcal{S}} = -(\omega_i, D_j(-B))_{\mathcal{S}}$$

and on the other hand, with (6.1):

$$\begin{aligned}\left(D_i(B), \mathcal{A}^0(B)\mathcal{J}D_j(-B)\right)_{\mathcal{S}} &= \left(D_i(B), \mathcal{J}\mathcal{A}^{0*}(-B)D_j(-B)\right)_{\mathcal{S}} \\ &= (D_i(B), \mathcal{J}\omega_j)_{\mathcal{S}} = -(D_i(B), \omega_j)_{\mathcal{S}}\end{aligned}$$

Therefore, $(D_i(B), \omega_j)_{\mathcal{S}} = (\omega_i, D_j(-B))_{\mathcal{S}}$, which is, up to the multiplication by $(2\varepsilon)^{3/2}$, the result to be proved. ∎

Proposition 6.2 *The diffusion tensor* $\mathbb{D}$ *is positive definite: more precisely, for all* $\underline{\xi} \in \mathbb{R}^2$ *and all* $\varepsilon_0 > 0$*, there exists* $C = C(\varepsilon_0) > 0$ *such that:*

$$(\mathbb{D}Y, Y) = \sum_{i,j=1}^{2} \mathbb{D}_{ij} Y_i Y_j \geq C|Y|^2 = C\sum_{i=1}^{2} Y_i^2\,, \quad \forall Y, \underline{\xi} \in \mathbb{R}^2\,, \quad \forall \varepsilon \geq \varepsilon_0\,. \tag{6.2}$$

Proof: Let $Y = (y_1, y_2)$ such that $|Y| > 0$. From the definition of $\mathbb{D}$ we obtain that:

$$(\mathbb{D}Y, Y) = (2\varepsilon)^{3/2} \int_0^1 \int_{\mathbb{S}^2} \left(\sum_{i=1}^{2} \omega_i y_i\right) \left(\sum_{i=1}^{2} D_i y_i\right) dx\, d\omega. \tag{6.3}$$

Define $\Phi(x,\omega)$ as follows:

$$\Phi(x,\omega) = \sum_{i=1}^{2} y_i D_i(x,\omega). \tag{6.4}$$

Then:

$$\sum_{i=1}^{2} \omega_i y_i = \mathcal{A}^{0*}\Phi(x,\omega),$$

and equation (6.3) reads:

$$\begin{aligned}(\mathbb{D}Y, Y) &= (2\varepsilon)^{3/2}(\mathcal{A}^{0*}\Phi, \Phi)_{\mathcal{S}} = (2\varepsilon)^{3/2}\left(\int_{\mathcal{S}_-} |\omega_x|\,|\Phi_-|^2\, d\omega - \int_{\mathcal{S}_+} |\omega_x|\,|\Phi_+|^2\, d\omega\right) \\ &\geq (2\varepsilon_0)^{3/2}\left(|\Phi_-|^2_{L^2(\mathcal{S}_-)} - |\mathcal{B}^*\Phi_-|^2_{L^2(\mathcal{S}_+)}\right) \geq 0,\end{aligned} \tag{6.5}$$

Now, if there exists Y such that $(\mathbb{D}Y, Y) = 0$, the corresponding Φ satisfies:

$$|\mathcal{B}^*\Phi_-|^2_{L^2(\mathcal{S}_+)} = |\Phi_-|^2_{L^2(\mathcal{S}_-)}. \tag{6.6}$$

From the transposition of equation (3.18) to the adjoint operator $\mathcal{B}^*$, equation (6.6) is possible only if Φ_- is a constant function of ω, on each connected component of

$\mathcal{S}_-$ (i.e. an element of $\mathcal{C}^-$, see section 3). Then, $\Phi_- = J\Phi_+$ is the same constant. Denoting by $\Phi_0 = \Phi|_{x=0}$, $\Phi_1 = \Phi|_{x=1}$and using (5.28), we have, for $\omega_x > 0$:

$$\begin{aligned}\Phi_0 &= \Phi_1 + \frac{1}{|v||\omega_x|}\int_0^1 \underline{y}\cdot R^+_{(x',\sigma)}(\underline{\omega})\,dx' \\ &= \Phi_1 + \frac{1}{B}[y_1(\omega_1 \sin b + \omega_2(\cos b - 1)) + y_2(-\omega_1(\cos b - 1) + \omega_2 \sin b)].\end{aligned}$$

Obviously, for Φ_0 and Φ_1 to be independent of $\underline{\omega}$, the only possibility is $y_1 = y_2 = 0$, which contradicts the fact that $|Y| > 0$. This ends the proof of (6.2). ■

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