Approximating partial derivatives of first and second order by quadratic spline quasi-interpolants

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Abstract

Given a bivariate function f defined in a rectangular domain Ω, we approximate it by a C^1 quadratic spline quasi-interpolant (abbr. QI) and we take partial derivatives of this QI as approximations to those of f. We give error estimates and asymptotic expansions for these approximations. We also propose a simple algorithm for the determination of stationary points, illustrated by a numerical example.

Key words: Quadratic spline quasi-interpolant; Partial derivative approximation; Stationary points detection.

1 Introduction and notations

Let $\Omega = I \times J = [a, b] \times [c, d]$ be a rectangular domain. For a given steplength $h > 0$, we assume that $b = a+mh$ and $d = c+nh$ and we consider the uniform partitions $X_m = \{x_i = a + ih, 0 \leq i \leq m\}$ and $Y_n = \{y_j = c + jh, 0 \leq j \leq n\}$, respectively in I and J. We also need the midpoints $s_i = \frac{1}{2}(x_{i-1} + x_i)$ and $t_j = \frac{1}{2}(y_{j-1} + y_j)$ of subintervals defined by the two partitions. For the sake of simplicity, and in order to avoid the use of boundary B-splines and functionals, we add two points at each end of intervals (see Figure 1) , denoted respectively by $x_{-2}, x_{-1}, x_{m+1}, x_{m+2}$ and $y_{-2}, y_{-1}, y_{n+1}, y_{n+2}$. In the same way, we have two

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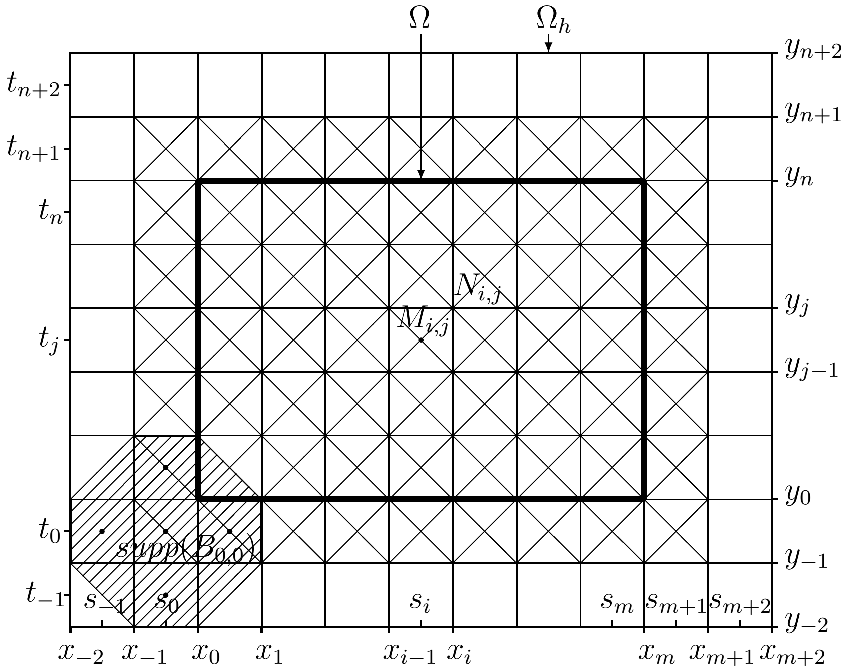

Figure 1. Domains Ω and Ω_h

more midpoints at each end: $s_{-1}, s_0, s_{m+1}, s_{m+2}$ and $t_{-1}, t_0, t_{n+1}, t_{n+2}$. Then we define $\Omega_h = [a-2h, b+2h] \times [c-2h, d+2h] = [x_{-2}, x_{m+2}] \times [y_{-2}, y_{n+2}]$. We assume in the following that f is defined in the bigger domain Ω_h. This domain is endowed with the so-called criss-cross (or type 2) triangulation $\mathcal{T}_{mn}$ consisting in drawing diagonals in each subsquare $\Omega_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ of Ω_h. The centres of these subsquares are the points $M_\alpha = (s_i, t_j)$, and their vertices are the points $N_\alpha = (x_i, y_j)$. Let $A = \{\alpha = (i,j),\ 0 \le i \le m+1,\ 0 \le j \le n+1\}$, $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$. Then the C^1 quadratic spline quasi-interpolant considered in this paper is defined by

$$Qf = \sum_{\alpha \in A} \mu_\alpha(f) B_\alpha$$

where the coefficient functionals are

$$\mu_\alpha(f) = \frac{3}{2} f(M_\alpha) - \frac{1}{8} \left(f(M_{\alpha-\varepsilon_1}) + f(M_{\alpha+\varepsilon_1}) + f(M_{\alpha-\varepsilon_2}) + f(M_{\alpha+\varepsilon_2}) \right).$$

and the B_α's are the classical C^1-quadratic B-splines (or box-splines). For their properties, see e.g. [2, 3]. Note that the family $\{B_\alpha, \alpha \in A\}$ of B-splines whose support intersect Ω is only a spanning system, not a basis, of the space $S_2(\Omega, \mathcal{T}_{mn})$ of quadratic splines defined in Ω endowed with the triangulation $\mathcal{T}_{mn}$. However, this has no effect on the definitions and results given below. Note also that for the construction of Qf, we need values of f at points of

$\{M_\gamma \in \Omega_h, \gamma \in \Gamma\}$ where $\Gamma = \{\gamma = (i,j),\ -1 \le i \le m+2, -1 \le j \le n+2\}$, except the four pairs $(-1,-1), (m+2,-1), (-1,n+2), (m+2,n+2)$.

By introducing the fundamental splines

$$\tilde{B}_\gamma = \frac{3}{2} B_\gamma - \frac{1}{8}\left(B_{\gamma-\varepsilon_1} + B_{\gamma+\varepsilon_1} + B_{\gamma-\varepsilon_2} + B_{\gamma+\varepsilon_2}\right),$$

it is also very convenient and useful to write Qf in the "quasi-Lagrange" form

$$Qf = \sum_{\gamma\in\Gamma} f(M_\gamma)\tilde{B}_\gamma.$$

In particular, the Lebesgue function Λ of Q, defined by

$$\Lambda = \sum_{\gamma\in\Gamma} |\tilde{B}_\gamma|$$

allows us to evaluate its infinity norm (see Section 3.1 below)

$$\|Q\|_\infty = |\Lambda|_\infty = \max\{\Lambda(x,y),\ (x,y) \in \Omega\} = 1.5.$$

The operator Q is constructed in order to be *exact* on the space Π_2 of bivariate *quadratic polynomials* (i.e. of total degree at most 2). In other words, it satisfies $Qp = p$ for all $p \in \Pi_2$.

There exist other types of quadratic spline approximants on the same triangulation (see e.g.[9] with applications to contour plotting) and on other types of triangulations, e.g. the Powell-Sabin one ([8]).

Here is an outline of the paper. In Section 2, for the sake of completeness, we give the full equations of fundamental splines $\tilde{B}_\gamma$ and of their first and second order partial derivatives, in each of the 68 subtriangles of its support. In Section 3, we give error estimates for infinity norms of $f - Qf$ and its p.d. In Section 4, we give a general algorithm for the exact computation of stationary points of Qf. Finally, in Section 5, we give a numerical example in order to illustrate this algorithm.

Partial derivatives are be denoted by one of the following forms :

$$\partial_1 = \frac{\partial}{\partial x} = D^{10}, \quad \partial_2 = \frac{\partial}{\partial y} = D^{01},$$
$$\partial_1^2 = \frac{\partial^2}{\partial x^2} = D^{20}, \quad \partial_1\partial_2 = \frac{\partial^2}{\partial x \partial y} = D^{11}, \quad \partial_2^2 = \frac{\partial^2}{\partial y^2} = D^{02}.$$

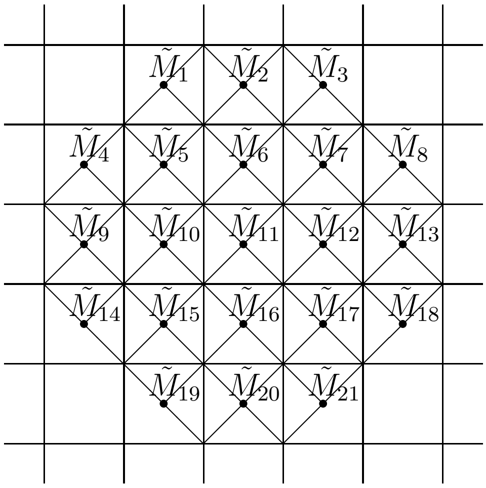

Figure 2. Centres $\tilde{M}_k$ of the 21 supports of fundamental splines $\tilde{B}_\gamma$ covering the subsquare Ω_{ij} centered at $M_{ij} = \tilde{M}_{11}$

In Taylor's formulas, we use Ciarlet's notations [5], with $C_k^j = \frac{k!}{j!(k-j)!}$:

$$D^k f(M).(M_0M)^k = \sum_{j=0}^{k} C_k^j (x - x_0)^j (y - y_0)^{k-j}.$$

For norms of derivatives, we set, for $k \geq 1$:

$$\|D^k f\|_{\infty,\Omega} = \max\{|D^\alpha f|_{\infty,\Omega},\ |\alpha| = k\},$$

where , for $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2 = k$:

$$|D^\alpha f|_{\infty,\Omega} = \max\{|D^\alpha f(M)|,\ M \in \Omega\}.$$

Often, we use the simpler notation $|.|_\infty$ instead of $|.|_{\infty,\Omega}$, without explicit reference to the domain Ω.

2 Equations of fundamental splines and of their partial derivatives

The local representation of B-splines and fundamental splines in the Bernstein basis of bivariate quadratic polynomials in each triangle of their support has been given in the technical reports [13b] and [13c]. For the sake of simplicity, we now give the equations of fundamental splines w.r.t. local cartesian cordinates $(u, v) \in [-\frac{1}{2}, \frac{1}{2}]^2$, with $(x, y) = (s_i + uh, t_j + vh)$, associated with each subsquare Ω_{ij} of the partition, the origin being taken at its centre M_{ij}. We consider only the 21 fundamental splines $\tilde{B}_\gamma$, for which the support cover (at least partly)

Table 1
Equations of $16\tilde{B}_\gamma$

k	N	S	E	W
1	$-\bar{p}_1^2$	0	0	$-\bar{p}_1^2$
2	$-\frac{1}{2}(1+4\bar{q}_2)$	$-\frac{1}{2}p_3^2$	$p_1^2-\frac{1}{2}p_3^2$	$\bar{p}_1^2-\frac{1}{2}p_3^2$
3	$-p_1^2$	0	$-p_1^2$	0
4	$-\bar{p}_1^2$	0	0	$-\bar{p}_1^2$
5	$13\bar{p}_1^2+2\bar{p}_1-1$	$-(1-\bar{p}_1)^2$	$-(1-\bar{p}_1)^2$	$13\bar{p}_1^2+2\bar{p}_1-1$
6	$4(1+6\bar{q}_2)$	$2q_3$	$13p_1^2+4(1+6\bar{q}_2)$	$13p_1^2+4(1+6\bar{q}_2)$
7	$13p_1^2-2p_1-1$	$-(1+p_1)^2$	$13p_1^2-2p_1-1$	$-(1+p_1)^2$
8	$-p_1^2$	0	$-p_1^2$	0
9	$\bar{p}_1^2-\frac{1}{2}\bar{p}_2^2$	$p_1^2-\frac{1}{2}\bar{p}_2^2$	$-\frac{1}{2}\bar{p}_2^2$	$-\frac{1}{2}(1-4q_1)$
10	$13p_1^2+4(1-6q_1)$	$13\bar{p}_1^2+4(1-6q_1)$	$2\bar{q}_4$	$4(1-6q_1)$
11	p_4	p_4	p_4	p_4
12	$13\bar{p}_1^2+4(1+6\bar{q}_1)$	$13p_1^2+4(1+6\bar{q}_1)$	$4(1+6\bar{q}_1)$	$2q_4$
13	$p_1^2-\frac{1}{2}p_2^2$	$\bar{p}_1^2-\frac{1}{2}p_2^2$	$-\frac{1}{2}(1+4\bar{q}_1)$	$-\frac{1}{2}p_2^2$
14	0	$-p_1^2$	0	$-p_1^2$
15	$-(1-p_1)^2$	$13p_1^2+2p_1-1$	$-(1-p_1)^2$	$13p_1^2+2p_1-1$
16	$2\bar{q}_3$	$4(1-6q_2)$	$13p_1^2+4(1-6q_2)$	$13\bar{p}_1^2+4(1-6q_2)$
17	$-(1+\bar{p}_1)^2$	$13\bar{p}_1^2-2\bar{p}_1-1$	$13\bar{p}_1^2-2\bar{p}_1-1$	$-(1+\bar{p}_1)^2$
18	0	$-\bar{p}_1^2$	$-\bar{p}_1^2$	0
19	0	$-p_1^2$	0	$-p_1^2$
20	$-\frac{1}{2}\bar{p}_3^2$	$-\frac{1}{2}(1-4q_2)$	$\bar{p}_1^2-\frac{1}{2}\bar{p}_3^2$	$p_1^2-\frac{1}{2}b p_3^2$
21	0	$-\bar{p}_1^2$	$-\bar{p}_1^2$	0

the subsquare Ω_{ij}. We number them by indices k from 1 to 21, associated to their support centres $\tilde{M}_k$, as given Figure 2 Table 1 gives the equations of fundamental splines $\tilde{B}_\gamma$. Tables 2 and 3 give the equations of their first partial derivatives. Finally, Table 4 gives the equations of their second partial derivatives. In Table 1, we use the following abbreviations:

$$p_1=u+v,\quad \bar{p}_1=u-v,\quad p_2=1+2u,\quad \bar{p}_2=1-2u,$$
$$p_3=1+2v,\quad \bar{p}_3=1-2v,\quad p_4=22-24(u^2+v^2),$$
$$q_1=u+v^2,\quad \bar{q}_1=u-v^2,\quad q_2=v+u^2,\quad \bar{q}_2=v-u^2,$$
$$q_3=u^2+13v^2+12v+2,\quad \bar{q}_3=u^2+13v^2-12v+2,$$

Table 2
Equations of $16h\partial_1 \tilde{B}_\gamma$

k	N	S	E	W
1	$-u+v$	0	0	$-u+v$
2	$2u$	0	$u+v$	$u-v$
3	$-(u+v)$	0	$-(u+v)$	0
4	$-u+v$	0	0	$-u+v$
5	$13(u-v)+1$	$1-u+v$	$1-u+v$	$13(u-v)+1$
6	$-24u$	$2u$	$-11u-13v$	$-11u+13v$
7	$13(u+v)-1$	$-(1+u+v)$	$13(u+v)-1$	$-(1+u+v)$
8	$-(u+v)$	0	$-(u+v)$	0
9	$1-u-v$	$1-u+v$	$1-2u$	1
10	$13(u+v)-12$	$13(u-v)-12$	$26u-12$	-12
11	$-24u$	$-24u$	$-24u$	$-24u$
12	$13(u-v)+12$	$13(u+v)+12$	12	$26u+12$
13	$-1-u+v$	$-(1+u+v)$	-1	$-(1+2u)$
14	0	$-(u+v)$	0	$-(u+v)$
15	$1-u-v$	$13(u+v)+1$	$1-u-v$	$13(u+v)+1$
16	$2u$	$-24u$	$-11u+13v$	$-11u-13v$
17	$-1-u+v$	$13(u-v)-1$	$13(u-v)-1$	$-1-u+v$
18	0	$-u+v$	$-u+v$	0
19	0	$-(u+v)$	0	$-(u+v)$
20	0	$2u$	$u-v$	$u+v$
21	0	$-u+v$	$-u+v$	0

$$q_4 = 13u^2 + v^2 + 12u + 2, \quad \bar{q}_4 = 13u^2 + v^2 - 12u + 2.$$

3 Global error estimates for smooth functions

In this section, we assume that $f \in C^3(\Omega_h)$: in [13a], error bounds have been already computed in the case of a non-uniform triangulation. In the specific case of uniform triangulations, the constants can be substantially reduced.

Table 3
Equations of $16h\partial_2\tilde{B}_\gamma$

k	N	S	E	W
1	$u-v$	0	0	$u-v$
2	-1	$-(1+2v)$	$-1+u-v$	$-(1+u+v)$
3	$-(u+v)$	0	$-(u+v)$	0
4	$u-v$	0	0	$u-v$
5	$-13(u-v)-1$	$-1+u-v$	$-1+u-v$	$-13(u-v)-1$
6	12	$26v+12$	$-13(u-v)+12$	$12+13(u+v)$
7	$13(u+v)-1$	$-(1+u+v)$	$13(u+v)-1$	$-(1+u+v)$
8	$-(u+v)$	0	$-(u+v)$	0
9	$-u+v$	$u+v$	0	$2v$
10	$13u-11v$	$-(13u+11v)$	$2v$	$-24v$
11	$-24v$	$-24v$	$-24v$	$-24v$
12	$-(13u+11v)$	$13u-11v$	$-24v$	$2v$
13	$u+v$	$u-v$	$2v$	0
14	0	$-(u+v)$	0	$-(u+v)$
15	$1-u-v$	$13(u+v)+1$	$1-u-v$	$13(u+v)+1$
16	$26v-12$	-12	$13(u+v)-12$	$-13(u-v)-12$
17	$1+u-v$	$-13(u-v)+1$	$-13(u-v)+1$	$1+u-v$
18	0	$u-v$	$u-v$	0
19	0	$-(u+v)$	0	$-(u+v)$
20	$1-2v$	1	$1-u-v$	$1+u-v$
21	0	$u-v$	$u-v$	0

Theorem 1 *The following error estimates are valid, for constants* $C_0 \leq \frac{5}{96}$, $C_1 \leq 2$ *and* $C_2 \leq \frac{41}{4}$ *respectively:*

$$
\begin{aligned}
|f-Qf|_\infty &\leq C_0 h^3 \|D^3 f\|_\infty, \\
|D^\beta(f-Qf)|_\infty &\leq C_1 h^2 \|D^3 f\|_\infty, \text{ for } |\beta|=1, \\
|D^\beta(f-Qf)|_{\infty,T} &\leq C_2 h \|D^3 f\|_\infty, \text{ for } |\beta|=2, \text{ and for all } T \in \mathcal{T}_{mn}.
\end{aligned}
$$

Table 4
Values of second derivatives of $16h^2\tilde{B}_\gamma$

	∂_1^2				$\partial_1\partial_2$				∂_2^2			
k	N	S	E	W	N	S	E	W	N	S	E	W
1	−1	0	0	−1	1	0	0	1	−1	0	0	−1
2	2	0	1	1	0	0	1	−1	0	−2	−1	−1
3	−1	0	−1	0	−1	0	−1	0	−1	0	−1	0
4	−1	0	0	−1	1	0	0	1	−1	0	0	−1
5	13	−1	−1	13	−13	1	1	−13	13	−1	−1	13
6	−24	2	−11	−11	0	0	−13	13	0	26	13	13
7	13	−1	13	−1	13	−1	13	−1	13	−1	13	−1
8	−1	0	−1	0	−1	0	−1	0	−1	0	−1	0
9	−1	−1	−2	0	−1	1	0	0	1	1	0	2
10	13	13	26	0	13	−13	0	0	−11	−11	2	−24
11	−24	−24	−24	−24	0	0	0	0	−24	−24	−24	−24
12	13	13	0	26	−13	13	0	0	−11	−11	−24	2
13	−1	−1	0	−2	1	−1	0	0	1	1	2	0
14	0	−1	0	−1	0	−1	0	−1	0	−1	0	−1
15	−1	13	−1	13	−1	13	−1	13	−1	13	−1	13
16	2	−24	−11	−11	0	0	13	−13	26	0	13	13
17	−1	13	13	−1	1	−13	−13	1	−1	13	13	−1
18	0	−1	−1	0	0	1	1	0	0	−1	−1	0
19	0	−1	0	−1	0	−1	0	−1	0	−1	0	−1
20	0	2	1	1	0	0	−1	1	−2	0	−1	−1
21	0	−1	−1	0	0	1	1	0	0	−1	−1	0

Proof Let us choose a triangle T of type N(North) and the midpoint $M_0 = (x_0, y_0)$ of its upper edge. By Taylor's formula, we obtain, for all $M = (x, y) \in T$,

$$f(M) = p_2(M) + r_2(M),$$

where p_2 and q_2 are respectively the quadratic Taylor polynomial and the associated remainder

$$p_2(M) = f(M_0) + Df(M_0).(M_0M) + \frac{1}{2}D^2f(M)(M_0M)^2,$$

$$r_2(M) = \frac{1}{6}\int_0^1 D^3f((1-t)M + tM_0).(M_0M)^3\,dt.$$

As $|M_0M|_1 = |x - x_0| + |y - y_0| \leq h/2$, we get

$$|r_2(M)| \leq \frac{1}{6}\|D^3f\|_\infty|M_0M|_1^3 \leq \frac{h^3}{48}\|D^3f\|_\infty.$$

On the other hand, $Qf = Qp_2 + Qr_2 = p_2 + Qr_2$ since Q is exact on Π_2. Therefore, using the above majoration

$$|f - Qf|_\infty = |r_2 - Qr_2| = |(I - Q)r_2| \leq (1 + \|Q\|_\infty)|r_2| = \frac{5}{2}|r_2| \leq \frac{5h^3}{96}\|D^3f\|_\infty.$$

For $\partial_1(f - Qf)$, we can use the following technique. We do not go into details and we only give a sketch of the computations. We start from the expression of the derivative $\partial_1 Qf = \sum_\gamma f(M_\gamma)\partial_1\tilde{B}_\gamma$ and from Taylor's formula of order 2:

$$f(M_\gamma) = f(M) + Df(M).(MM_\gamma) + \frac{1}{2}D^2f(M).(MM_\gamma)^2$$
$$+\frac{1}{6}\int_0^1 D^3f((1-t)M_\gamma + tM).(MM_\gamma)^3\,dt.$$

Replacing $f(M_\gamma)$ by its expansion in the formula of $\partial_1 Qf$, we obtain

$$\partial_1 Qf(M) = f(M) + \frac{1}{6}\sum_\gamma[\int_0^1 D^3f((1-t)M_\gamma + tM).(MM_\gamma)^3\,dt]\,\partial_1\tilde{B}_\gamma(M).$$

This is a consequence of the following facts. We use the notation $M_\gamma = (x_\gamma, y_\gamma)$ and the property of Q to be exact on Π_2. As $\sum_\gamma \tilde{B}_\gamma = 1, \sum_\gamma x_\gamma\tilde{B}_\gamma = x$, $\sum_\gamma y_\gamma\tilde{B}_\gamma = y$, then $\sum_\gamma \partial_1\tilde{B}_\gamma = 0, \sum_\gamma x_\gamma\partial_1\tilde{B}_\gamma = 1$, and $\sum_\gamma y_\gamma\partial_1\tilde{B}_\gamma = 0$. Therefore we get

$$\sum_\gamma \partial_1\tilde{B}_\gamma[Df(M).(MM_\gamma)] = \sum_\gamma \partial_1\tilde{B}_\gamma[(x_\gamma - x)\partial_1 f(M) + (y_\gamma - y)\partial_2 f(M) = \partial_1 f(M),$$

In a similar way, as $\sum_\gamma x_\gamma^2 \tilde{B}_\gamma = x^2$, $\sum_\gamma y_\gamma^2 \tilde{B}_\gamma = y^2, \sum_\gamma x_\gamma y_\gamma \tilde{B}_\gamma = xy$, then $\sum_\gamma x_\gamma^2 \partial_1 \tilde{B}_\gamma = 2x$, $\sum_\gamma x_\gamma y_\gamma \partial_1 \tilde{B}_\gamma = y$, $\sum_\gamma y_\gamma^2 \partial_1 \tilde{B}_\gamma = 0$. and we get

$$\sum_\gamma \partial_1 \tilde{B}_\gamma D^2 f(M).[(MM_\gamma)^2] = 0,$$

We deduce from the above expression that

$$|\partial_1 Qf(M) - f(M)| \leq \frac{1}{6} \|D^3 f\|_\infty \sum_\gamma |MM_\gamma|_1^3 |\partial_1 \tilde{B}_\gamma|$$

Now, setting $\varphi_\gamma(M) = |MM_\gamma|_1^3 |\partial_1 \tilde{B}_\gamma|$, we use Table 2 to compute the exact maximum of φ_γ in the triangle T. We obtain the following bounds: $\varphi_\gamma \leq 9$ for $k = 1, 3, 4, 8, 9, 13, 15, 17$; $\varphi_\gamma \leq 19$ for $k = 5, 7, 10, 12$; $\varphi_\gamma \leq 8$ for $k = 2, 16$; $\varphi_\gamma \leq 12$ for $k = 6, 11$, from which we deduce $\sum_\gamma \varphi_\gamma \leq 2$ and finally

$$|\partial_1 Qf - f|_\infty \leq 2h^2 \|D^3 f\|_\infty.$$

A similar method is used for second order derivatives. □

4 An algorithm for computing stationary points

In view of the results of the preceding sections, we propose the following algorithm for the detection of *stationary points* of Qf:

(i) compute π and χ at centres M_{ij} and vertices N_{ij} of subsquares of the partition.
(ii) select the subset $\mathcal{T}' \subset \mathcal{T}_{mn}$ of triangles in which the sum of signs of π at the three vertices is ± 1.
(iii) select the subset $\mathcal{T}'' \subset \mathcal{T}'$ of those triangles in which the sum of signs of χ at the three vertices is ± 1.
(iv) in each triangle of $\mathcal{T}''$, solve the system of equations $\pi = \chi = 0$, i.e. compute the exact intersection point of the two corresponding segments. We thus obtain the set of stationary points of Qf.
(v) study locally the sign of $H = \sigma^2 - \rho\tau$: if $H < 0$, then if $\rho > 0$ and $\tau > 0$, then there is a local minimum, else if $\rho < 0$ and $\tau < 0$, then there is a local maximum. Finally, if $H \geq 0$, we have a saddle point or a degenerate point (see e.g. [7]).

Remark 2 *Of course, one can exchange the roles of* π *and* χ *in the above algorithm.*

5 Numerical example

We test the previous algorithm with a Scilab ([12]) program and obtain the following results.

Example 3 *Franke's function (see e.g. [1], p. 144):*
$f_2(x,y) = .75exp(-\frac{1}{4}((9x-2)^2+(9y-2)^2)+.75exp(-\frac{1}{49}(9x+1)^2-\frac{1}{10}(9y+1))$
$+.5exp(-\frac{1}{4}((9x-7)^2+(9y-3)^2)-.2exp(-(9x-4)^2-(9y-7)^2).$

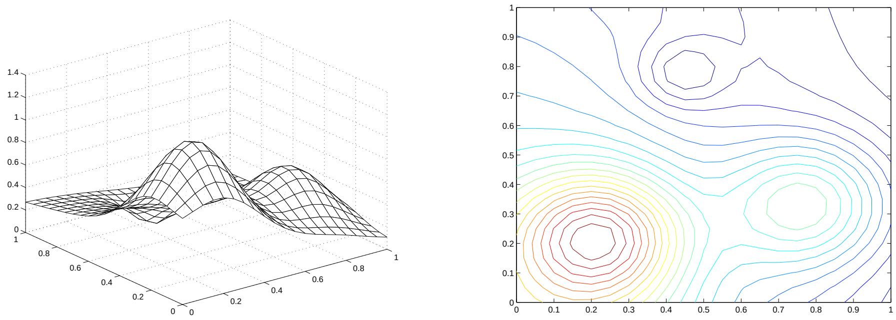

Figure 3. Surface $z = f(x, y)$ and its level lines

We compute five stationary points $(x_{i,h}, y_{i,h},\ i = 1, ..., 5$: one local minimum, two local maxima and two saddle points or degenerate points. For the sake of comparison, we have found with Matlab the following stationary points : $(x_1, y_1) = (0.456, 0.784)$, $(x_2, y_2) = (0.206, 0.208)$, $(x_3, y_3) = (0.755, 0.326)$, $(x_4, y_4) = (0.556, 0.277)$, $(x_5, y_5) = (0.616, 0.857)$. Table 5 gives the error $E_i(h) = ((x_{i,h} - x_i)^2 + (y_{i,h} - y_i)^2)^{1/2}$, $i = 1, ..., 5$, for a few values of h.

Table 5
Approximation errors for the computed stationary points

h	$E_1(h)$	$E_2(h)$	$E_3(h)$	$E_4(h)$	$E_5(h)$
1/5	1.03E-01	2.55E-03	4.20E-02	1.22E-02	3.82E-02
1/10	5.38E-03	1.48E-03	2.74E-03	5.39E-03	1.50E-02
1/20	8.26E-04	1.91E-04	2.69E-04	4.32E-04	1.14E-02
1/40	2.52E-04	6.62E-05	7.42E-05	6.53E-05	1.59E-03
1/80	5.51E-05	4.86E-05	1.46E-05	4.74E-05	2.97E-04

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