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Abstract

The paper is devoted to the mathematical analysis of an optimal control problem arising in a generalized principal-agent model with limited liability constraint. We present the economic model and the mathematical formulation.

Though the problem seems "simple", it presents many difficulties that are hard to overcome. After a formal resolution, we give an existence and uniqueness result for this problem.

Keywords : Optimal control, Bottleneck constraints, Incentives, Delegation, Personnel Economics

Classification AMS : 49J20, 49M25

1 Introduction

Contracts which incorporate limits on the maximum loss that an agent can be forced to bear as a consequence of contracting with a principal are referred to as limited liability contracts. It is well known, for example, that, when a project owner requires the unobservable effort of a risk neutral agent, limited liability constraint precludes the agent from paying the owner the full value of her project. In short, under moral hazard, limited liability constraint makes it no longer possible for a principal to "sell the firm" to a risk neutral agent. Similarily, imagine a principal hiring a risk neutral agent to choose a project among several alternatives, the probability of success of the project being a function of the agent's talent to choose a good project. Assume that agent's talent is his private information. The introduction of limited liability constraint

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within this adverse selection framework precludes the principal from punishing the agent when he turns out to be inefficient.

To simplify the resolution of this kind of model, the modeler usually first guesses the way the limited liability constraint is binding and checks ex post that this constraint is indeed stricly satisfied. Under moral hazard, the economic intuition is that only the limited liability constraint in case of failure may be binding. Under adverse selection, the modeler guesses that only the limited liability constraint associated with the least efficient agent may be binding. The purpose of this research is to investigate how the modeler selects a limited liability constraint on utility within a generalized principal-agent problem. A generalized principal-agent problem arises when the principal's payoff function depends on both the private information and the unobservable action of the agent (see. Myerson (1982)).

Consider the following standard principal-agent framework. First a principal follows an agent's recommendation to choose a project among several alternatives. For example, a chief executive officer follows a product manager's recommendation to choose on how and when a new product is launched. The agent is endowed with a talent parameter that we interpret as his match to the project. We assume that the agent's talent is his private information. Then the agent spends an unobservable effort to realize the project. This effort may have different interpretations, e.g. time or attention devoted to project realization. For example, selling effort would include developing and updating customers address lists or contracting customers about upcoming sales and new merchandise. Hence, the project's success probability depends both on the agent's talent and effort. Therefore a model in which adverse selection and moral hazard are jointly present is analyzed. The resolution is adapted from Faynzilberg and Kumar (2000) but we assume that the agent is risk neutral and protected by limited liability constraint on utility.

Section 2 briefly describes the generalized principal-agent model and the principal's optimization problem following Faynzilberg and Kumar's procedure. Faynzilberg and Kumar analyze the existence of optimal contracts within a generalized principal-agent problem. The agent being risk averse within their decomposition procedure, we adapt their framework to risk neutral agent protected by limited liability on utility. The introduction of limited liability constraint on utility is analyzed by an economic intuition. Then, we perform a formal and numerical resolution of the optimal control problem that we derive from this model. The last section is devoted to the rigorous proof of an existence result: we may give the analytical form of the solution.

2 The model

There are two risk-neutral actors in the model: a principal and an agent. The principal (the owner of the firm) is the residual claimant of profits generated by a project that requires the agent's unobservable effort in order to be realized. The level of output resulting from the relationship is determined by a production function, $y = f(t, e, \varepsilon)$, with three inputs: the agent's talent $t \in T = [0,1]$ that we interpret as how well the agent matches to the project, the agent's effort $e \in E = [0,1]$ and a random shock ε (Nature). To simplify matters as much as possible, take the case where the space is binary: $y \in \{\underline{y}, \overline{y}\}$. Success generates gross value $\overline{y}$ while failure provides $\underline{y}$ with $\underline{y} < \overline{y}$. $p(e,t)$ denotes the probability that the project succeeds given the inputs (e,t). Talent and effort are essential for success, so $p(e,0) = 0$ and $p(0,t) = 0$ respectively for all e and t. Higher levels of effort and higher realizations of t decrease the conditional probability that the smaller level of performance will be realized, i.e., $p_e(e,t) \geq 0$ and $p_t(e,t) \geq 0$, where subscripts denote partial derivatives. The expected marginal impact of the agent's effort is also assumed to be greater in more productive environments, that is $p_{et}(e,t) \geq 0$. The agent associates a monetary cost $g(e,t)$ with undertaking an effort equal to e. We make the following standard assumptions: $g(e,t)$ is strictly increasing and convex in e, $g_e > 0$, $g_{ee} > 0$ and higher talents have lower effort cost and lower marginal effort cost, that is, $g_t < 0$ and $g_{et} < 0$. Throughout the paper, the reservation utility of the agent is normalized to zero. We assume that the agent is protected by limited liability on utility. This assumption implies that the optimal transfer to the agent cannot be so low that the agent's utility can be negative. In case of employment contracts, many legal restrictions limit the worker's liability. Among these restrictions one can point out laws exonerating the worker from liability for damages caused during the execution of the contract. The resulting utility function of the agent is: $u(w,e,t) = w(y,t) - g(e,t)$. The principal is a risk neutral individual with a profit function: $\pi(y,w,t) = y - w(y,t)$.

The output is determined both by the agent's talent (private information) and the agent's unobservable effort. The principal faces adverse selection followed by moral hazard. The game begins with the agent privately observing his talent t. Next, the principal offers to the agent an opportunity to participate in the production of the outcome y, determined by the agent's talent, the agent's effort and Nature. None of these inputs is observable by the principal. At the time the principal is contracting with the agent, she knows that the agent's talent is drawn from a distribution function $F(t)$ with density function $f(t)$ for every $t \in T$. The agent's acceptance (respectively refusal) of the principal's offer leads to the agent's choice of a type contingent transfer from a menu offered by the principal, and an effort (respectively ends the game with the agent realizing his reservation utility). Finally, Nature chooses a random productivity shock. The game ends with the principal making to the agent a transfer $w(y,t)$ from the set $\{w(\underline{y},t), w(\overline{y},t)\}$. The principal's problem is the following

$$\underset{e(.),w(.,.)}{\text{Maximize}} \int_0^1 \left\{ p(e,t)(\overline{y} - w(\overline{y},t)) + (1-p(e,t))(\underline{y} - w(\underline{y},t)) \right\} f(t)dt \tag{1a}$$

subject to, for all $e \in E$, $t \in T$ and $t' \in T$:

$$p(e,t)w(\overline{y},t) + (1-p(e,t))w(\underline{y},t) - g(e,t) \geq 0 \tag{1b}$$

$$w(y,t) - g(e,t) \geq 0 \tag{1c}$$

$$e \in \arg\max \; p(e,t)w(\overline{y},t) + (1-p(e,t))w(\underline{y},t) - g(e,t) \tag{1d}$$

$$U(w(y,t),e(t),t) \geq U(w(y,t'),e(t'),t). \tag{1e}$$

where

$$U(w(y,t'),e(t'),t) = p(e(t'),t)w(\overline{y},t') + (1-p(e(t'),t))w(\underline{y},t') - g(e(t'),t).$$

The principal maximizes (1a) her expected utility which depends on the probability of success. The agent's talent being his private information, $p(e,t)$ is conditional to the density function of the agent's talent. Inequalities (1b) and (1c) ensure respectively the individual rationality of agent's participation and the agent's limited liability on utility. Expressions (1d) and (1e) capture the generalized incentive compatible constraints. To simplify matters as much as possible we make the following assumptions

$$t \sim U[0,1] \;,\; g(e,t) = \frac{e^2}{t} \;\forall t \in]0,1] \text{ and } p(e,t) = te.$$

First, the principal performs a "conditional" optimization: she chooses an optimal incentive-compatible contract for every level of the agent's indirect expected utility. In short, the principal chooses an optimal contract ex-ante as a function of the agent's indirect expected utility. Then the principal selects an optimal level of the indirect expected utility itself, subject to the participation constraint and to the limited liability constraint on utility. The optimal contract ex-ante as a function of the agent's indirect expected utility, $\forall t \in]0,1]$ is (see Faynzilberg and Kumar (2000)).

$$e(t) = t\sqrt{\frac{h(t)}{3}},\; w(\underline{y},t) = V(t) - \frac{th(t)}{3},\; w(\overline{y},t) = V(t) - \frac{th(t)}{3} + \tfrac{2}{t}\sqrt{\frac{h(t)}{3}},$$

with $V(t_a)$ the agent's indirect expected utility and $h(t_a)$ the variation of the agent's indirect expected utility when the agent's talent increases ($\frac{dV(t)}{dt} = h(t)$). Knowing the optimal contract ex-ante as a function of the agent's indirect expected utility, the principal selects an optimal level of the indirect expected utility itself, subject to the participation constraint and to the limited liability constraint on utility.

While much of the limited liability literature has treated the adverse selection and moral hazard separately, we investigate limited liability constraint on utility within a generalized principal-agent framework. Under models that contain only moral hazard, the economic intuition is that only the limited liability constraint in case of failure may be binding because agent's utility in case of success is greater than agent's utility in case of failure. Under models that contain only adverse selection, the modeler guesses that only the limited liability constraint associated with the least efficient agent may be binding because the agent's utility when he is efficient is greater than when he is inefficient. Given the joined presence of adverse selection and moral hazard and the particular form of our problem (two outputs and a continuum of inputs) it may be not possible to apply usual economic intuition. First since the optimal contract ex-ante implies that $w(\overline{y}, t) \geq w(\underline{y}, t)$, the relevant constraint (1c) is $w(\underline{y}, t) - g(e, t) \geq 0$. The principal's program can be rewritten as

$$\underset{V(t_a)}{\text{Maximize}} \int_0^1 \left\{ t_a^2 (\overline{y} - \underline{y}) \sqrt{\frac{h(t_a)}{3}} - V(t_a) - \frac{t_a h(t_a)}{3} + \underline{y} \right\} dt_a \tag{2}$$

subject to, for $t \in]0, 1]$

$$\frac{dV(t_a)}{dt_a} = h(t_a) \ , \ V(0) = 0 \ , \tag{3}$$

$$V(t_a) - \frac{2t_a h(t_a)}{3} \geq 0. \tag{4}$$

The principal maximizes her expected profit as a function of the optimal contract ex-ante (2) subject to the constraint (3) and to the limited liability constraint on utility rewritten thanks to the optimal contract ex-ante (4) .

Second, we should take into account adverse selection. We do not know whether agent's utility is an increasing function of talent, even if we know that it is true in expectation ($h(t) \geq 0$). A first intuitive way to solve this problem is to find the solution of the unconstrained program. It can be shown that without limited liability constraint on the agent's utility, the agent's utility in case of failure is negative, the principal selling the firm to the risk neutral agent and the agent's utility in case of success is positive. The introduction of limited liability constraint on utility precludes the agent from bearing all the risks of the relationship in case of failure. So within our framework, the modeler guesses that only the limited liability constraint on utility in case of failure associated with any talent is binding. The economic intuition allows to set $w(\underline{y}, t) - g(e, t) = 0$.

We are now going to focus directely on the above optimal control problem without using economic intuition to bind the limited liability constraint on utility. It has a very "simple" form: we have to minimize a strictly convex cost functional with a linear differential state equations and bottleneck constraints. This kind of contraints appear frequently in economics models and can be treated via dynamical programming methods as in Bellman (2003) or Miriča (1985). More recently, Bergounioux &Tiba (1996) and Bergounioux & Tröltzsch (1998) have studied such problems. The state equation was a partial differential one but these problems were (in our opinion) much simpler. The techniques that have been used in the quoted papers are not useful here.

3 The optimal control problem

3.1 Setting the problem

From now, we consider the optimal control problem that we write as follows :

$$\begin{cases} \max \tilde{J}(V,h) \stackrel{def}{=} \displaystyle\int_0^1 (-V(t)f(t) - \varphi(h(t)))\,dt \\ V'(t) = h(t) \text{ on }]0,1[,\ V(0) = 0\ , \\ 0 \le t\,h(t) \le \alpha\,V(t) \qquad \forall t \in]0,1]\ , \end{cases} \tag{5}$$

where

$$\varphi(h)(t) = \begin{cases} \left[\dfrac{t\,h(t)}{3} - t^2(\overline{y} - \underline{y})\sqrt{\dfrac{h}{3}}\right] f(t) & \text{if } h \ge 0 \\ +\infty & \text{else} \end{cases} \quad \text{and } \alpha \ge 1.$$

Here "$h \ge 0$" stands for : "$h(t) \ge 0$ for all $t \in [0,1]$". Note that $\alpha = 3/2$ in the model that we presented in the previous section. Moreover, in the sequel we set $\overline{y} - \underline{y} = \delta$.
The above function is convex and Gâteaux -differentiable at $h > 0$ and

$$\frac{\partial \varphi}{\partial h}(h)(t) = \frac{t}{3} - \frac{t^2\delta}{2\sqrt{3\,h(t)}}\ .$$

We note that the feasible domain is not empty since it involves 0. This implies that the infimum is non positive.

As the state equation is easy to solve we may give an equivalent form for this problem, namely :

$$(\mathcal{P}_1) \qquad \begin{cases} \min J(h) \stackrel{def}{=} \displaystyle\int_0^1 (\varphi(h(t)) - h(t)\,F(t))\,dt \\ 0 \le t\,h(t) \le \alpha \displaystyle\int_0^t h(s)\,ds \qquad \forall t \in]0,1]\ , \\ h \in L^1(0,1)\ . \end{cases}$$

where F is the primitive function of f that vanishes at 1, i.e

$$F(t) = \int_1^t f(s)\,ds\ .$$

Indeed

$$\tilde{J}(V,h) = \int_0^1 (-V(t)F'(t) - \varphi(h(t)))\,dt = \int_0^1 (V'(t)F(t) - \varphi(h(t)))\,dt$$

with $V(0) = 0$ and $F(1) = 0$. With the state equation we get

$$J(h) = -\tilde{J}(V,h) = \int_0^1 (\varphi(h(t)) - h(t)\,F(t))\,dt\ .$$

In the sequel we set, $\forall h \geq 0$ and $\forall t \in [0,1]$ $\psi(h)(t) = \varphi(h(t)) - h(t)\,F(t))$ i.e.

$$\psi(h)(t) = \left[\frac{t\,h(t)}{3} - t^2\delta\sqrt{\frac{h}{3}}\right] f(t) - h(t)\int_1^t f(s)\,ds\ . \tag{6}$$

Remark 3.1 *The "natural" space for this optimal control problem is $L^1(0,1)$ since we only need the functions to be integrable. This implies that an eventual Lagrange multiplier must belong to $L^\infty(0,1)$. In fact, we shall see in the forthcoming sections that we are not able to prove the existence of solutions for any $\alpha \geq 1$. We shall have to add regularity assumptions on the feasible functions that have to belong to $L^\infty(0,1)$ in some cases.*

The functional J is strictly convex with respect to h and the constraints are linear. Therefore the solution of this problem (if it exists) is unique. First, we should prove that problem $(\mathcal{P})_1$ has a solution h^*. Though this problem seems quite simple, we are not able to prove any existence result with classical minimization techniques since the function J is not coercive and the feasible set is not bounded. Moreover, usual qualification conditions are not easy to verify (even the quite weak condition of Zowe and Kurcyusz (1979) and we cannot prove easily the existence of Lagrange multipliers a priori.

In order to find the solution of this problem, we first perform a formal and numerical analysis that allows to guess the solution. So we exhibit what could be a suitable Lagrange multiplier in $L^1(0,1)$. Then, we prove that a Karush-Kuhn-Tucker type optimality system is satisfied. As the problem is convex, this proves that we have found the unique solution of problem $(\mathcal{P}_1)$.

3.2 Formal resolution of problem $(\mathcal{P}_1)$

Assume h^* is the unique solution of $(\mathcal{P})$. Let us set

$$L(h)(t) = t\,h(t) - \alpha\int_0^t h(s)\,ds\ .$$

L is a linear continuous operator from $L^\infty(0,1)$ to $L^\infty(0,1)$. Assume we are able to find a Lagrange multiplier $\lambda \in L^2(0,1)$, so that we may define the Lagrangian function of problem $(\mathcal{P})$ as

$$\mathcal{L}(h,\lambda) = J(h) + \int_0^1 \lambda(t)L(h)(t)\,dt\ . \tag{7}$$

We keep the constraint $h \geq 0$ without using a Lagrange multiplier. The optimality system is :

$$\left(\frac{\partial\mathcal{L}(h^*,\lambda^*)}{\partial h}, h - h^*\right) \geq 0 \quad \text{forall } h \geq 0, \tag{8a}$$

$$\lambda^* \geq 0 \text{ and } \lambda^*(t)L(h^*)(t) = 0 \text{ a.e. } t \in [0,1]\ , \tag{8b}$$

$$L(h^*)(t) \leq 0 \text{ and } h^*(t) \geq 0 \quad \text{a.e. } t \in [0,1] \ , \tag{8c}$$

Here $(\cdot,\cdot)$ demotes the $L^2(0,1)$ inner product. We propose to use the Uzawa algorithm to solve this optimality system. We recall it for convenience:

Uzawa Algorithm

1. Choose $\lambda_o \in L^2(0,1)$ and set $n = 1$;
2. Iteration n: λ_{n-1} is known.
   (a) Compute h_n solution of
   $$\min\{\mathcal{L}(h, \lambda_{n-1}) \mid h \geq 0 \ \} \ , \tag{9}$$
   (b) Set $\lambda_n(t) = \max\left(0, \lambda_{n-1}(t) + \rho\, L(h_n)(t)\,\right)$
   where $\rho > 0$.
3. Stopping criterion : stop or set $n = n + 1$ and go to 2.

We focus on equation (9) (λ_{n-1} is known). First, we compute the unconstrained minimum :

$$\min\{\mathcal{L}(h, \lambda_{n-1}) \mid h \in L^2(0,1) \ \} \ ,$$

and we shall realize a posteriori that the constraint $h \geq 0$ is satisfied. Therefore, it is the solution to (9) as well. So we have to solve

$$\forall v \in L^2(0,1) \qquad \left(\frac{\partial \mathcal{L}(h, \lambda_{n-1})}{\partial h}, v\right) = 0 \ , \tag{10}$$

$$\left(\frac{\partial \mathcal{L}(h, \lambda_{n-1})}{\partial h}, v\right) =$$
$$\int_0^1 \left(\frac{\partial \varphi(h)}{\partial h} - F(t)\right) v(t)\, dt + \int_0^1 \lambda_{n-1}(t) \left(\frac{\partial L(h)}{\partial h}\right) v(t)\, dt \ .$$

Let us compute

$$\int_0^1 \lambda_{n-1}(t) \left(\frac{\partial L(h)}{\partial h}\right) v(t)\, dt = \int_0^1 \lambda_{n-1}(t) \left(t v(t) - \alpha \int_0^t v(s) ds\right)\, dt \ .$$

Setting

$$U(t) = \int_0^t v(s)\, ds \ , \text{ for any } v \in L^2(0,1) \ ,$$

and

$$\Lambda_{n-1}(t) = \int_1^t \lambda_{n-1}(s)\, ds \ ,$$

(the primitive function of λ_{n-1} that vanishes at 1) with an integration by parts, gives

$$\int_0^1 \left(\lambda_{n-1}(t) \int_0^t v(s)ds \right) \, dt = \int_0^1 \Lambda'_{n-1}(t) U(t) \, dt = - \int_0^1 \Lambda_{n-1}(t) v(t) \, dt.$$

So

$$\int_0^1 \lambda_{n-1}(t) \left(\frac{\partial L(h)}{\partial h} \right) v(t) \, dt = \int_0^1 \left(t\lambda_{n-1}(t) + \alpha \Lambda_{n-1}(t) \ \right) v(t) \, dt \ .$$

Finally equation (10) turns to be

$$\forall v \in L^2(0,1) \quad \int_0^1 \left(\frac{\partial \varphi(h)}{\partial h} - F(t) + t\lambda_{n-1}(t) + \alpha \Lambda_{n-1}(t) \right) v(t) \, dt = 0 \ ,$$

that is

$$\frac{\partial \varphi(h)}{\partial h} - F(t) + t \, \lambda_{n-1}(t) + \alpha \Lambda_{n-1}(t) = 0 \text{ a.e. } t \in [0,1] \ .$$

This yields

$$f(t) \left(\frac{t}{3} - \frac{t^2 \delta}{2\sqrt{3\, h(t)}} \right) - F(t) + t\lambda_{n-1}(t) + \alpha \, \Lambda_{n-1}(t) = 0 \quad \text{a.e.} \tag{11}$$

This gives, for every $t \in]0,1]$,

$$\frac{t}{3} - \frac{t^2 \delta}{2\sqrt{3\, h(t)}} = \frac{F(t) - t\lambda_{n-1}(t) - \alpha \Lambda_{n-1}(t)}{f(t)} \ ,$$

$$\frac{t^2 \delta}{2\sqrt{3\, h(t)}}) = \frac{t}{3} - \frac{F(t) - t\lambda_{n-1}(t) - \alpha \Lambda_{n-1}(t)}{f(t)}$$

$$= \frac{t\, f(t) - 3\, [F(t) - t\lambda_{n-1}(t) - \alpha \Lambda_{n-1}(t)]}{3\, f(t)},$$

$$2\sqrt{3\, h(t)} = \frac{3\, t^2 \delta \, f(t)}{t\, f(t) - 3\, [F(t) - t\lambda_{n-1}(t) - \alpha \Lambda_{n-1}(t)]},$$

and finally

$$h_n(t) = \frac{3}{4} \left[\frac{t^2 \delta \, f(t)}{t\, f(t) - 3\, F(t) + 3\, t\lambda_{n-1}(t) + 3\, \alpha \, \Lambda_{n-1}(t)} \right]^2 . \tag{12}$$

We see that $h_n \geq 0$ and the constraint $h \geq 0$ is fulfilled.

3.3 Numerical results ($\alpha = 3/2$)

We perform the numerical tests in the very case where $\alpha = 3/2$. This corresponds to the model presented in Section 2
The discretization of the problem is done using a finite difference scheme on $[0, 1]$.
Let $N \in \mathbb{N}$ and $t_i = i\,\Delta t, i = 0, \cdots, N$ where $\Delta t = \dfrac{1}{N}$.
We approximate $\Lambda_{n-1}(t) = \displaystyle\int_1^t \lambda_{n-1}(s)\,ds = -\int_t^1 \lambda_{n-1}(s)\,ds$ with a trapezoidal rule, namely

$$\Lambda_{n-1}(t_i) \simeq -\Delta t \left(\sum_{k=i+1}^{N-2} \lambda_{n-1}(t_i) + \frac{\lambda_{n-1}(1) + \lambda_{n-1}(t_i)}{2} \right) .$$

The numerical tests have been peformed using MATLAB software and the following data :

$$f(t) \equiv 1 \text{ so that } F(t) = t - 1,\ d = 1, c = 0 \text{ so that } \delta = 1 .$$

The stopping criterion is the following

$$\max\{\|h_n - h_{n-1}\|, |(\lambda_n, L(h_n))|\} \le \varepsilon ,$$

where ε is a prescribed tolerance. For different values of N we have tested different values for ρ and the initial guess for λ. We present the solution that was obtained with $N = 2000,\ \varepsilon = 5.10^{-4},\ \rho = 3$ and $\lambda_o = 0.25$.

$$J(h^*) \simeq -1.9725\,10^{-2},\ \text{number of iterations } = 151 .$$

We may note that h^* is not derivable at 0. This explains the "bad" numerical behavior of the computed solution in the neighborhood of 0 (V is "better").

3.4 Computing the "exact" solution

From now, we assume that

$$f(t) \equiv 1 \text{ and } F(t) = t - 1 . \tag{13}$$

The numerical computation shows that the optimal solution should satisfy $L(h^*) = 0$. This allows us to compute (formally) the exact (analytical) solution. Assume that we have

$$\forall t \in]0, 1] \qquad t\,h^*(t) = \alpha\,V^*(t) .$$

Therefore V^* is solution to the differential equation :

$$\frac{dV^*}{dt} = \frac{\alpha}{t} V(t) \qquad \forall t \in]0, 1],\ V(0) = 0 .$$

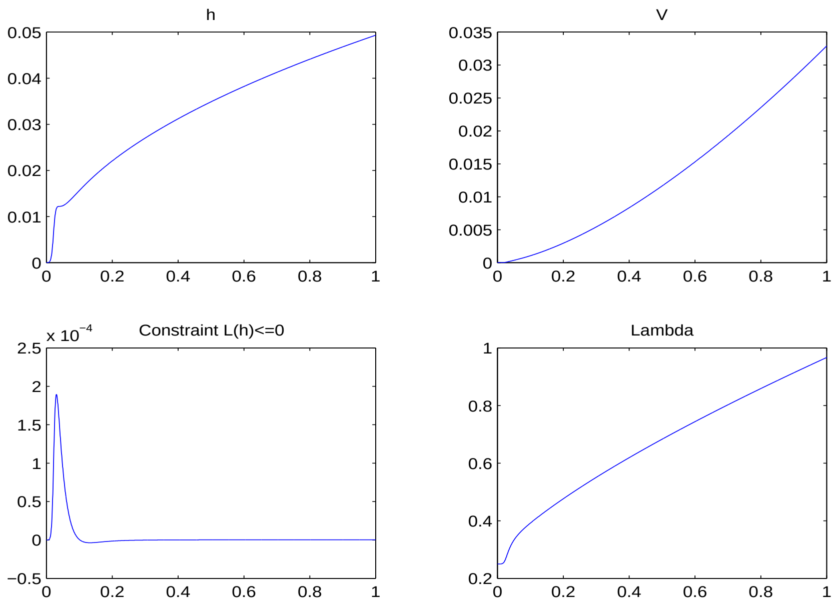

Figure 1: Figure 1. Solution h^* and V^* with λ^* and the constraint $L(h^*)$

This gives $V^*(t) = \nu t^\alpha$ and $h^*(t) = \nu\alpha t^{\alpha-1}$ where ν is a non negative number to determine.
Let us compute

$$J(h^*) = \int_0^1 \left[\frac{\nu\alpha}{3} t^\alpha - t^2 \delta \sqrt{\frac{\alpha\nu}{3}} t^{\frac{\alpha-1}{2}} - \nu\alpha t^{\alpha-1}(t-1) \right] dt \overset{def}{=} \Phi(\nu) \ .$$

A short computation shows that

$$\Phi(\nu) = \nu \frac{\alpha+3}{3(\alpha+1)} - \frac{2\delta\sqrt{\alpha}}{\sqrt{3}(\alpha+5)} \sqrt{\nu} \ .$$

The minimum of Φ is obtained for $\nu = 3\alpha \left(\frac{\delta(\alpha+1)}{(\alpha+5)(\alpha+3)} \right)^2$, that is in the case where $\delta = 1$, $\nu = 3\alpha \left(\frac{\alpha+1}{(\alpha+5)(\alpha+3)} \right)^2$.
Finally we may give the "exact" solution of the problem

$$h^*(t) = 3 \left(\frac{\alpha(\alpha+1)}{(\alpha+3)(\alpha+5)} \right)^2 t^{\alpha-1} \ , \tag{14}$$

and

$$J(h^*) = -\frac{\alpha(\alpha+1)}{(\alpha+3)(\alpha+5)^2} \ . \tag{15}$$

This gives for $\alpha = 3/2$:

$$h^*(t) = \frac{25}{507}\sqrt{t},\ V^*(t) = \frac{50}{1521}t^{\frac{3}{2}} \text{ and } J(h^*) = -\frac{10}{507} \simeq -1.9723\,10^{-2} .$$

Now that we have a good candidate to be the solution to our problem. We are going to prove it. In the sequel we shall set

$$C_\alpha \stackrel{def}{=} 3\left(\frac{\alpha(\alpha+1)}{(\alpha+3)(\alpha+5)}\right)^2 . \tag{16}$$

4 Existence of the solution

In order to prove that h^* is the exact solution that we are looking for we would like to exhibit a Lagrange multiplier such that an Karush-Kuhn-Tucker optimality system is satisfied. This Lagrange multiplier should be the ajoint state that we study in the forthcoming subsection.

4.1 The adjoint equation

We consider now the following equation:

$$t\,\lambda(t) + \alpha\int_1^t \lambda(s)\,ds + \frac{d\psi}{dh}(h^*)(t) = 0 \qquad \text{on }]0,1] , \tag{17}$$

where h^* is given by (14). As $F(t) \equiv (t-1)$ and $\delta = 1$ we have

$$\psi(h)(t) = \left(1 - \frac{2\,t}{3}\right)h(t) - t^2\sqrt{\frac{h}{3}} ,$$

so that

$$\frac{d\psi}{dh}(h^*)(t) = 1 - \frac{2\,t}{3} - \frac{t^{\frac{5-\alpha}{2}}}{2\sqrt{3\,C_\alpha}} = 1 - \frac{2\,t}{3} - \frac{(\alpha+3)(\alpha+5)}{6\,\alpha(\alpha+1)}t^{\frac{5-\alpha}{2}} . \tag{18}$$

Setting $\Lambda(t) = \int_1^t \lambda(s)\,ds$, we get the differential equation

$$\begin{cases} t\,\Lambda'(t) = -\alpha\Lambda(t) - \left(1 - \frac{2\,t}{3} - \frac{(\alpha+3)(\alpha+5)}{6\,\alpha(\alpha+1)}t^{\frac{5-\alpha}{2}}\right) & \text{on }]0,1[, \\ \Lambda(1) = 0 . \end{cases} \tag{19}$$

The solution λ^* of (17) is $\lambda^*(t) = \Lambda^{*'}(t)$ where Λ^* is the solution of (19). The resolution of (19) gives

$$\Lambda^*(t) = -\frac{1}{\alpha} + \frac{2\,t}{3\,(\alpha+1)} + \frac{(\alpha+3)}{3\alpha(\alpha+1)}t^{\frac{5-\alpha}{2}} ;$$

we obtain

$$\lambda^*(t) = \Lambda^{*'}(t) = \frac{2}{3\,(\alpha+1)} + \frac{(5-\alpha)(\alpha+3)}{6\alpha(\alpha+1)}t^{\frac{3-\alpha}{2}} . \tag{20}$$

Theorem 4.1 *Assume that* $\alpha \in [1, 5]$*. Then the function* λ^* *defined by (20) satisfies*

$$\lambda^*(t) \geq 0 \textit{ forall } t \in]0, 1] \quad ,$$

and $\lambda^* \in L^1(0, 1)$*.*
Moreover, if $\alpha \leq 3$*,* λ^* *belongs to* $\mathcal{C}^o([0, 1])$*.*

We may note that if $\alpha > 5$, then λ^* does not belong to $L^1(0, 1)$ and its sign is not constant. This method fails in this case: we cannot conclude for $\alpha > 5$. In addition, if $\alpha \in]3, 5]$, λ^* does not belong to $L^\infty(0, 1)$: therefore we cannot prove any existence result for $(\mathcal{P}_1)$ but for $(\mathcal{P}_\infty)$ (that is the same problem where the space function is no longer $L^1(0, 1)$ but $L^\infty(0, 1)$).

4.2 The optimality system when $\alpha \leq 5$

From now we assume that

$$1 \leq \alpha \leq 5 \ ; \tag{21}$$

this involves the case $\alpha = 3/2$ of the model described in Section 2.
Let recall the Lagrangian function defined by (7)

$$\mathcal{L}(h, \lambda) = J(h) + \int_0^1 \lambda(t) L(h)(t) \, dt \ ,$$

for $(h, \lambda) \in L^\infty(0, 1) \times L^1(0, 1)$. The function $\mathcal{L}$ is convex with respect to h and linear with respect to λ. Let us compute

$$\frac{\partial \mathcal{L}}{\partial h}(\tilde{h}, \lambda) h = \int_0^1 \frac{d\psi}{dh}(\tilde{h})(t) \, h(t) \, dt + \int_0^1 \lambda(t) \left(t h(t) - \alpha \int_0^t h(s) \, ds \right) \, dt \ ,$$

for any $h \in L^\infty(0, 1)$. As $\displaystyle\int_0^t h(s) \, ds = V(t)$,

$$\int_0^1 \lambda(t) \left(\int_0^t h(s) \, ds \right) \, dt = \int_0^1 \lambda(t) V(t) \, dt$$

$$= \left[\left(\int_1^t \lambda(s) \, ds \right) \, V(t) \right]_0^1 - \int_0^1 h(t) \left(\int_1^t \lambda(s) \, ds \ \right) \, dt$$

$$= - \int_0^1 \left(\int_1^t \lambda(s) \, ds \right) h(t) \, dt \ .$$

Therefore

$$\frac{\partial \mathcal{L}}{\partial h}(\tilde{h}, \lambda) h = \int_0^1 \left(\frac{d\psi}{dh}(\tilde{h})(t) + t\lambda(t) + \alpha \int_1^t \lambda(s) \, ds \right) h(t) \, dt \ .$$

Let us compute $\dfrac{\partial \mathcal{L}}{\partial h}(h^*, \lambda^*) \, h$, where h^* is given by (14) and λ^* is given by (17) (or equivalently (20)):

$$\frac{\partial \mathcal{L}}{\partial h}(h^*, \lambda^*) h = \int_0^1 \left(\frac{d\psi}{dh}(h^*)(t) + t\lambda^*(t) + \alpha \int_1^t \lambda(^* s) \, ds \right) h(t) \, dt \ = 0 \ .$$

So we obtain the desired optimality system

Theorem 4.2 *Assume $1 \leq \alpha \leq 5$. The pair $(h^*, \lambda^*) \in L^\infty(0,1) \times L^1(0,1)$ satisfies the following optimality conditions :*

$$\frac{\partial \mathcal{L}}{\partial h}(h^*, \lambda^*) = 0 \ , \tag{22a}$$

$$\lambda^* \geq 0 \ \text{on} \ [0,1] \ , \tag{22b}$$

$$h^* \geq 0, \quad L(h^*) = 0 \ \text{on} \ [0,1] \ , \tag{22c}$$

4.3 Existence of the solution of $(\mathcal{P}_\infty)$

As the Lagrange multiplier λ^* belongs to $L^1(0,1)$ we cannot conclude that $(\mathcal{P}_1)$ has an optimal solution. Indeed, the "duality" product $\int_0^1 \lambda^*(t)\, L(h)(t)\, dt$ does not make sense unless $h \in L^\infty(0,1)$. Therefore we consider the problem

$$(\mathcal{P}_\infty) \qquad \begin{cases} \min J(h) \overset{def}{=} \displaystyle\int_0^1 (\varphi(h(t)) - h(t)\, F(t))\, dt \\ 0 \leq t\, h(t) \leq \alpha \displaystyle\int_0^t h(s)\, ds \qquad \forall t \in]0,1] \ , \\ h \in L^\infty(0,1) \ . \end{cases}$$

As the lagrangian function $\mathcal{L}$ is convex with respect to h, the optimality system of Theorem 4.2 allows to conclude that h^* is the optimal solution of $(\mathcal{P}_\infty)$ and λ^* is a Lagrange multiplier associated to the constraint $L(h) \leq 0$. Therefore, we may enounce the main result of this paper:

Theorem 4.3 *If $1 \leq \alpha \leq 5$, problem $(\mathcal{P}_\infty)$ has a unique solution h^* given by (14).*

Proof - The proof is standard, but we recall it for convenience. Since $\mathcal{L}$ is convex with respect to h, equation (22a) yields

$$\forall h \in L^\infty(0,1) \qquad \mathcal{L}(h^*, \lambda^*) \leq \mathcal{L}(h, \lambda^*) \ . \tag{23}$$

For every $h \in L^\infty(0,1)$ such that $h \geq 0$ and $L(h) \leq 0$ this gives

$$J(h^*) + \int_0^1 \lambda^*(t) L(h^*)(t)\, dt \leq J(h) + \int_0^1 \lambda^*(t) L(h)(t)\, dt \leq 0 \ ,$$

since $\lambda^* \geq 0$.
As $L(h^*) = 0$, we finally have

$$\forall h \in L^\infty(0,1) \text{ such that } h \geq 0 \text{ and } L(h) \leq 0, \qquad J(h^*) \leq J(h) \ .$$

As h^* is feasible this means that h^* is a solution of $(\mathcal{P})$. As J is strictly convex, h^* is the unique solution. □
We may precise this result if $\alpha \in [1,3]$, since $\lambda^* \in L^\infty(0,1)$ in this case.

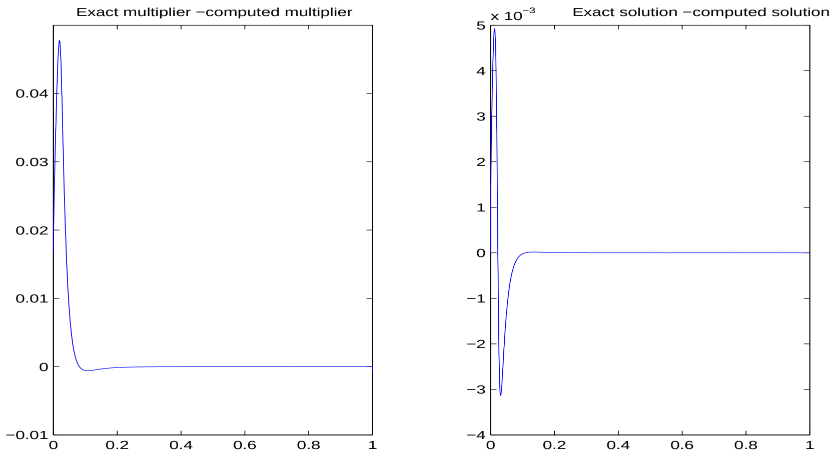

Figure 2: Comparison between exact and computed solution

4.4 Existence of the solution of $(\mathcal{P}_1)$ when $\alpha \in [1,3]$

We noticed that $\lambda^* \in L^\infty(0,1)$ if $\alpha \in [1,3]$ and (of course) $h^* \in L^1(0,1)$. Relations (22) and (23) remain valid. By density of $\mathcal{C}_c(]0,1[)$ in $L^1(0,1)$ and continuity of $\mathcal{L}$ with respect to the L^1-norm we conclude with (23) that

$$\forall h \in L^1(0,1) \qquad \mathcal{L}(h^*,\lambda^*) \le \mathcal{L}(h,\lambda^*) \ .$$

The end of the proof is the same as in Theorem 4.3. □

We may note that if $\alpha = 3/2$ then $\lambda^* = \dfrac{4}{15} + \dfrac{7}{15}\, t^{\frac{3}{4}}$. This corresponds to the computed multiplier we have found in Section 3 (Figure 1.) We present in Figure 2. the difference between the computed solutions and the analytical ones.

5 Conclusion

We prove existence results within a generalized principal-agent model with limited liability on the agent's utility. Due to the joined presence of adverse selection and moral hazard and the particular form of our problem, it is no longer possible to apply usual economic intuition used within models that contain only one type of private information to bind the limited liability constraint. A first way to solve the problem is to find the solution of the unconstrained model and then to guess the way the limited liability constraint is binding. We show within our framework that there is no need to take into account adverse selection, the usual economic intuition used within models that contains only moral hazard being sufficient to bind the limited liability constraint.

A formal way to solve this problem is to focus directly on the optimal control problem, the limited liability constraint becoming a mixed inequality constraint.

From the mathematical point of view the case where $\alpha > 5$ is still open : it seems (at least numerically) that the optimal solution (if it exists) no longer satisfies $L(h) = 0$. So we have to solve it via different methods. The work is in progress and if the problem is completely solved, we hope it will help to let the modelization process more general.