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On the Cohomology of the Lie Superalgebra of Contact Vector Fields on $S^{1|2}$

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Abstract

We investigate the first cohomology space associated with the embedding of the Lie superalgebra $K(2)$ of contact vector fields on the (1,2)-dimensional supercircle $S^{1|2}$ in the Lie superalgebra $SΨDO(S^{1|2})$ of superpseudodifferential operators with smooth coefficients. Following Ovsienko and Roger, we show that this space is ten-dimensional with only even cocycles and we give explicit expressions of the basis cocycles.

1 Introduction

V. Ovsienko and C. Roger [5] calculated the space $H^1(\text{Vect}(S^1), \Psi DO(S^1))$, where $\text{Vect}(S^1)$ is the Lie algebra of smooth vector fields on the circle $S^1$ and $\Psi DO(S^1)$ is the space of pseudodifferential operators with smooth coefficients. The action is given by the natural embedding of $\text{Vect}(S^1)$ in $\Psi DO(S^1)$. They used the results of D. B. Fuchs [3] on the cohomology of $\text{Vect}(S^1)$ with coefficients in weighted densities to determine the cohomology with coefficients in the graded module $\text{Gr}(\Psi DO(S^1))$, namely $H^1(\text{Vect}(S^1), \text{Gr}^p(\Psi DO(S^1)))$; here $\text{Gr}^p(\Psi DO(S^1))$ is isomorphic, as $\text{Vect}(S^1)$-module, to the space of weighted densities $F_p$ of weight $-p$ on $S^1$. To compute $H^1(\text{Vect}(S^1), \Psi DO(S^1))$, V. Ovsienko and C. Roger applied the theory of spectral sequences to a filtered module over a Lie algebra.

In a recent paper [2], using the same methods as in the paper [5], two of the authors computed $H^1(K(1), SΨDO(S^{1|1}))$, where $K(1)$ is the Lie superalgebra $K(1)$ of contact vector fields on the supercircle $S^{1|1}$ and $SΨDO(S^{1|1})$ is the space of superpseudodifferential operators on $S^{1|1}$.

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Here, we follow again the same methods by V. Ovsienko and C. Roger [5] to calculate $H^1(\mathcal{K}(2), \mathcal{S}\mathcal{D}\mathcal{O}(S^{1|2}))$.

The paper ([5]) contains also the classification of polynomial deformations of the natural embedding of $\text{Vect}(S^1)$ in $\mathcal{S}\mathcal{D}\mathcal{O}(S^1)$. The multi-parameter deformations of the embedding of $\mathcal{K}(1)$ into $\mathcal{S}\mathcal{D}\mathcal{O}(S^{1|1})$ are classified in ([4]). Our aim is this classification for the case $S^{1|2}$.

2 Definitions and Notations

Let $S^{1|n}$ be the supercircle with local coordinates $(\varphi; \theta_1, \ldots, \theta_n)$, where $\theta = (\theta_1, \ldots, \theta_n)$ are the odd variables. More precisely, let $x = e^{i\varphi}$, in what follows by $S^{1|n}$ we mean the supermanifold $(\mathbb{C}^*)^{1|n}$, whose underlying is $\mathbb{C} \setminus \{0\}$. Any contact structure on $S^{1|n}$ can be given by the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i.$$ 

Let $\mathcal{K}(n)$ be the Lie superalgebra of vector fields on $S^{1|n}$ whose Lie action on $\alpha_n$ amounts to a multiplication by a function. Any element of $\mathcal{K}(n)$ is of the form (see [1])

$$v_F = F \partial_x + \frac{(-1)^{p(F)+1}}{2} \sum_{i=1}^{n} \eta_i(F) \eta_i,$$

where $F \in C^\infty(S^{1|n})$, $p(F)$ is the parity of $F$ and $\eta_i = \partial_\theta_i - \theta_i \partial_x$. The bracket is given by

$$[v_F, v_G] = v_{\{F,G\}},$$

where

$$\{F,G\} = FG' - F'G + \frac{(-1)^{p(F)+1}}{2} \sum_{i=1}^{n} \eta_i(F) \eta_i(G).$$

The Lie superalgebra $\mathcal{K}(n)$ is called the Lie superalgebra of contact vector fields.

The superspace of the supercommutative algebra of superpseudodifferential symbols on $S^{1|n}$ with its natural multiplication is spanned by the series

$$\mathcal{S}\mathcal{P}(n) = \left\{ A = \sum_{k=-M}^{\infty} \sum_{\epsilon = (\epsilon_1, \ldots, \epsilon_n)} a_{k,\epsilon}(x, \theta) \xi^k \bar{\theta}_1^{\epsilon_1} \cdots \bar{\theta}_n^{\epsilon_n} | a_{k,\epsilon} \in C^\infty(S^{1|n}); \; \epsilon_i = 0, 1; \; M \in \mathbb{N} \right\},$$

where $\xi$ corresponds to $\partial_x$ and $\bar{\theta}_i$ corresponds to $\partial_\theta_i$ ($p(\bar{\theta}_i) = 1$). The space $\mathcal{S}\mathcal{P}(n)$ has a structure of the Poisson Lie superalgebra given by the following bracket:

$$\{A, B\} = \frac{\partial(A)}{\partial \xi} \frac{\partial(B)}{\partial x} - \frac{\partial(A)}{\partial x} \frac{\partial(B)}{\partial \xi} - (-1)^{p(A)} \sum_{i=1}^{n} \left( \frac{\partial(A)}{\partial \theta_i} \frac{\partial(B)}{\partial \theta_i} + \frac{\partial(A)}{\partial \bar{\theta}_i} \frac{\partial(B)}{\partial \bar{\theta}_i} \right).$$
The associative superalgebra of superpseudodifferential operators \( SΨDO(S^{1|n}) \) on \( S^{1|n} \) has the same underlying vector space as \( SP(n) \), but the multiplication is now defined by the following rule:

\[
A \circ B = \sum_{\alpha \geq 0, \nu \equiv 0, 1} \frac{(-1)^{p(A)+1}}{\alpha!} (\partial_x^\alpha \partial_\theta^\nu A)(\partial_x^\alpha \partial_\theta^\nu B).
\]

This composition rule induces the supercommutator defined by:

\[
[A, B] = A \circ B - (-1)^{p(A)p(B)} B \circ A.
\]

### 3 The space of weighted densities on \( S^{1|2} \)

Recall the definition of the \( Vect(S^1) \)-module of weighted densities on \( S^1 \). Consider the 1-parameter action of \( Vect(S^1) \) on \( C^\infty(S^1) \) given by

\[
L^\lambda_{X(x)}(f(x)) = X(x)f'(x) + \lambda X'(x)f(x),
\]

where \( f \in C^\infty(S^1) \) and \( \lambda \in \mathbb{R} \). Denote \( F_{\lambda} \) the \( Vect(S^1) \)-module structure on \( C^\infty(S^1) \) defined by this action. Note that the adjoint \( Vect(S^1) \)-module is isomorphic to \( F_{-1} \). Geometrically, \( F_{\lambda} \) is the space of weighted densities of weight \( \lambda \) on \( S^1 \), i.e., the set of all expressions: \( f(x)(dx)^\lambda \), where \( f \in C^\infty(S^1) \). We have analogous definition of weighted densities in the supercase (see [2]) with \( dx \) replaced by \( \alpha_n \).

Consider the 1-parameter action of \( K(n) \) on \( C^\infty(S^{1|n}) \) given by the rule:

\[
L^\lambda_{v_F}(G) = v_F(G) + \lambda F' \cdot G,
\]

where \( F, G \in C^\infty(S^{1|n}) \), \( F' = \partial_x F \). We denote this \( K(n) \)-module by \( \mathfrak{F}_{\lambda} \) and the \( K(2) \)-module by \( \mathfrak{H}_{\lambda} \). Geometrically, the space \( \mathfrak{H}_{\lambda} \) is the space of all weighted densities on \( S^{1|2} \) of weight \( \lambda \):

\[
\phi = f(x, \theta)\alpha^\lambda_{\theta}, \ f(x, \theta) \in C^\infty(S^{1|2}).
\]

**Remarks 3.1.** 1) The adjoint \( K(2) \)-module is isomorphic to \( \mathfrak{H}_{-1} \). This isomorphism induces a contact bracket on \( C^\infty(S^{1|2}) \) given by:

\[
\{ F, G \} = L^{-1}_{v_F}(G) = FG' - F'G + (-1)^{p(F)+1} 2 \sum_{i=1}^2 (\eta_i F)(\eta_i G).
\]

2) As a \( Vect(S^1) \)-module, the space of weighted densities \( \mathfrak{H}_{\lambda} \) is isomorphic to

\[
\mathfrak{F}_{\lambda} \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}} \oplus \mathfrak{F}_{\lambda+\frac{1}{2}}) \oplus \mathfrak{F}_{\lambda+1}.
\]
4 The structure of $\mathcal{SP}(2)$ as a $\mathcal{K}(2)$-module

The natural embedding of $\mathcal{K}(2)$ into $\mathcal{SP}(2)$ defined by

$$\pi(v_F) = F\xi + \frac{(-1)^{p(F)+1}}{2} \sum_{i=1}^{2} \eta_i(F)\zeta_i,$$

where $\zeta_i = \bar{\theta}_i - \theta_i\xi$, (4.1)

induces a $\mathcal{K}(2)$-module structure on $\mathcal{SP}(2)$.

Setting $\deg x = \deg \bar{\theta}_i = 0$, $\deg \xi = \deg \bar{\theta}_i = 1$ for all $i$, we endow the Poisson superalgebra $\mathcal{SP}(2)$ with a $\mathbb{Z}$-grading:

$$\mathcal{SP}(2) = \bigoplus_{n \in \mathbb{Z}} \mathcal{SP}_n,$$

where $\bigoplus_{n \in \mathbb{Z}} = (\bigoplus_{n < 0}) \oplus \prod_{n \geq 0}$ and

$$\mathcal{SP}_n = \left\{ F\xi^{-n} + G\xi^{-n-1}\bar{\theta}_1 + H\xi^{-n-1}\bar{\theta}_2 + T\xi^{-n-2}\bar{\theta}_1\bar{\theta}_2 \mid F, G, H, T \in C^\infty(S^{1|2}) \right\}$$

is the homogeneous subspace of degree $-n$. Each element of $\mathcal{SP}(S^{1|2})$ can be expressed as

$$A = \sum_{k \in \mathbb{Z}} (F_k + G_k\xi^{-1}\bar{\theta}_1 + H_k\xi^{-1}\bar{\theta}_2 + T_k\xi^{-2}\bar{\theta}_1\bar{\theta}_2)\xi^{-n},$$

where $F_k, G_k, H_k, T_k \in C^\infty(S^{1|2})$. We define the order of $A$ to be

$$\text{ord}(A) = \sup\{k \mid F_k \neq 0 \text{ or } G_k \neq 0 \text{ or } H_k \neq 0 \text{ or } T_k \neq 0\}.$$  

This definition of order equips $\mathcal{SP}(S^{1|2})$ with a decreasing filtration as follows: set

$$\mathcal{F}_n = \{ A \in \mathcal{SP}(S^{1|2}), \text{ord}(A) \leq -n\},$$

where $n \in \mathbb{Z}$. So one has

$$\ldots \subset \mathcal{F}_{n+1} \subset \mathcal{F}_n \subset \ldots$$  

This filtration is compatible with the multiplication and the Poisson bracket, that is, for $A \in \mathcal{F}_n$ and $B \in \mathcal{F}_m$, one has $A \circ B \in \mathcal{F}_{n+m}$ and $\{A, B\} \in \mathcal{F}_{n+m-1}$. This filtration makes $\mathcal{SP}(S^{1|2})$ an associative filtered superalgebra. Consider the associated graded space

$$\text{Gr}(\mathcal{SP}(S^{1|2})) = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n/\mathcal{F}_{n+1}.$$ 

The filtration (4.3) is also compatible with the natural action of $\mathcal{K}(2)$ on $\mathcal{SP}(S^{1|2})$. Indeed, if $v_F \in \mathcal{K}(2)$ and $A \in \mathcal{F}_n$, then

$$v_F(A) = [v_F, A] \in \mathcal{F}_n.$$
The induced $K(2)$-module on the quotient $F_n/F_{n+1}$ is isomorphic to the $K(2)$-module $SP_n$. Therefore, the $K(2)$-module $Gr(SΨDO(S^{1|2}))$, is isomorphic to the graded $K(2)$-module $SP(2)$, that is

$$SP(2) \simeq \bigoplus_{n \in \mathbb{Z}} F_n/F_{n+1}.$$ 

Recall that a $C^\infty$ function on $S^{1|2}$ has the form $F = f_0 + f_1 \theta + f_2 \theta + f_{12} \theta_1 \theta_2$ with $f_0, f_1, f_2, f_{12} \in C^\infty(S^1)$ and a $C^\infty$ function on $S_i^{1|1} (i = 1, 2)$, where $S_i^{1|1}$ is the supercircle with local coordinates $(\varphi, \theta_i)$, has the form $F = f_0 + f_i \theta_i$ ($f_{12} = f_{3-i} = 0$) with $f_0, f_i \in C^\infty(S^1)$. Then the Lie superalgebra $K(2)$ has two subsuperalgebras $K(1)_i$ for $i = 1, 2$ isomorphic to $K(1)$ defined by

$$K(1)_i = \left\{ v_F = F \partial_x + \frac{(-1)^p(F)+1}{2} \sum_{i=1}^{2} \eta_i(F) \eta_i | F \in C^\infty(S_i^{1|1}) \right\},$$

Therefore, $SP(2)$ and $SP_n$ are $K(1)_i$-modules.

For $i = 1, 2$, let $S^i_\lambda$ be the $K(1)_i$-module of weighted densities of weight $\lambda$ on $S_i^{1|1}$.

**Proposition 4.1.**  
1) As a $K(1)_i$-module, $i = 1, 2$, we have

$$SP_n \simeq S_n \oplus \Pi(S^{n+\frac{1}{2}}_n \oplus S^{n+\frac{1}{2}}_n) \oplus S_{n+1} \text{ for } n = 0, -1.$$ 

2) For $n \neq 0, -1$:

a) The following subspace of $SP_n$ :

$$SP_{n, i} = \left\{ B_F^{(n, i)} = F \partial_x + \frac{(-1)^p(F)+1}{2} \sum_{i=1}^{2} \eta_i(F) \eta_i | F \in C^\infty(S^{1|2}) \right\} (4.4)$$

is a $K(1)_i$- module, $i = 1, 2$, isomorphic to $S_{n+1}$.

b) As a $K(1)_i$-module we have

$$SP_n/SP_{n, i} \simeq S_n \oplus \Pi(S^{n+\frac{1}{2}}_n \oplus S^{n+\frac{1}{2}}_n), \ i = 1, 2.$$ 

Proof: First, note that for $n = 0, -1$, the $K(1)_i$-module $SP_n$ with the grading (4.2) is the direct sum of four $K(1)_i$-modules, $i = 1, 2$. 


For \( n = 0 \), the four \( \mathcal{K}(1)_i \)-modules are

\[
\mathcal{S}\mathcal{P}(0, 0) = \left\{ A_F^{(0, 0)} = F \mid F \in C^\infty(S^{1|2}) \right\},
\]

\[
\mathcal{S}\mathcal{P}(0, \frac{1}{2}, i) = \left\{ A_F^{(0, \frac{1}{2}, i)} = \theta_i F - \frac{1}{2}(1 - 2\theta_{3-i}\partial_{\theta_{3-i}})(F)\bar{\theta}_i \xi^{-1} - \theta_{3-i}\partial_{\bar{\theta}_i}(F)\bar{\theta}_i \xi^{-1} + F^\prime \theta_{3-i}\partial_{\bar{\theta}_i}(F)\bar{\theta}_i \xi^{-2} \mid F \in C^\infty(S^{1|2}) \right\},
\]

\[
\overline{\mathcal{S}\mathcal{P}}(0, \frac{1}{2}, i) = \left\{ \bar{A}_F^{(0, \frac{1}{2}, i)} = \frac{1}{2}(3F - (-1)^{p(F)}F)\bar{\theta}_i \xi^{-1} + (-1)^{p(F)}(\partial_{\theta_{3-i}} - \partial_{\bar{\theta}_i} + \theta_i \partial_\xi)(F)\bar{\theta}_i \xi^{-2} \mid F \in C^\infty(S^{1|2}) \right\},
\]

\[
\mathcal{S}\mathcal{P}(0, 1, i) = \left\{ A_F^{(0, 1, i)} = F\theta_{3-i}\bar{\theta}_i \xi^{-1} + \theta_{3-i}(\eta_{3-i} - \frac{1}{2}\bar{\eta}_i)(F)\xi_3 \xi_{3-i} \xi^{-2} \mid F \in C^\infty(S^{1|2}) \right\}.
\]

For \( n = -1 \), the four \( \mathcal{K}(1)_i \)-modules are

\[
\mathcal{S}\mathcal{P}(-1, 0) = \left\{ A_F^{(-1, 0)} = F\xi + \frac{(-1)^{p(F)+1}}{2} (\eta_1(F)\xi_1 + \eta_2(F)\xi_2) \mid F \in C^\infty(S^{1|2}) \right\},
\]

\[
\mathcal{S}\mathcal{P}(-1, \frac{1}{2}, i) = \left\{ A_F^{(-1, \frac{1}{2}, i)} = F\xi_i - (\theta_{3-i}\eta_i + \theta_i \bar{\theta}_{3-i})(F)\bar{\theta}_i \xi_i - (-1)^{p(F)}\partial_{\theta_{3-i}}(F)\bar{\theta}_i \xi_i \mid F \in C^\infty(S^{1|2}) \right\},
\]

\[
\overline{\mathcal{S}\mathcal{P}}(-1, \frac{1}{2}, i) = \left\{ \bar{A}_F^{(-1, \frac{1}{2}, i)} = F\xi_i + (1 - \theta_{3-i}\eta_i)(F)\bar{\theta}_i \xi_i \mid F \in C^\infty(S^{1|2}) \right\},
\]

\[
\mathcal{S}\mathcal{P}(-1, 1, i) = \left\{ A_F^{(-1, 1, i)} = (-1)(\theta_{3-i}\bar{\theta}_i + \theta_{3-i}(\eta_{3-i} - \frac{1}{2}\bar{\eta}_i)(F)\xi_3 \xi_{3-i} \xi^{-1} \mid F \in C^\infty(S^{1|2}) \right\}.
\]

The action of \( \mathcal{K}(1)_i \) on \( \mathcal{S}\mathcal{P}(n, 0) \) and on \( \mathcal{S}\mathcal{P}(n, 1, i) \) for \( n = 0, -1 \) is induced by the embedding (4.1) as follows

\[
v_G \cdot A_F^{(n, 0)} = \left\{ \pi(v_G), A_F^{(n, 0)} \right\} = A_{\Sigma_G^{n+1}}(F) \quad \text{and} \quad v_G \cdot A_F^{(n, 1, i)} = \left\{ \pi(v_G), A_F^{(n, 1, i)} \right\} = A_{\Sigma_G^{n+1}}(F),
\]

where \( F \in C^\infty(S^{1|2}) \) and \( G \in C^\infty(S_1^{1|1}) \). Therefore, the natural maps

\[
\psi_{n, 0} : \mathfrak{S}_n \longrightarrow \mathcal{S}\mathcal{P}(n, 0) \quad \text{and} \quad \psi_{n, 1} : \mathfrak{S}_{n+1} \longrightarrow \mathcal{S}\mathcal{P}(n, 1, i)
\]

\[
F\alpha_{2,n} \longrightarrow A_F^{(n, 0)} \quad \text{and} \quad F\alpha_{2,n+1} \longrightarrow A_F^{(n, 1, i)} \quad (4.5)
\]

provide us with isomorphisms of \( \mathcal{K}(1)_i \)-modules, \( i = 1, 2 \).
The induced where

\[ F \] 

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Second, for \( n \neq 0, -1 \), the action of \( K(1)_n \) on \( SP_{n, 0, i} \) is given by

\[
v_G \cdot A_F^{(n, \frac{1}{2}, i)} = \left\{ \begin{array}{l}
\pi(v_G), A_F^{(n, \frac{1}{2}, i)} \\
F \xi^{-n} + \frac{(-1)^{p(F)}}{2n+1} \theta_{3-i} \eta_{3-i} \eta_i (F) \xi^{-n-1} + \\
(\theta_{3-i} + \frac{n+1}{2n+1} \eta_{3-i}) (F) \bar{\theta}_{3-i} \xi^{-n-1} + \\
\frac{n+1}{2n+1} \eta_{3-i} \eta_i (F) \bar{\theta}_{3-i} \xi^{-n-2} \end{array} \right. \]

where \( F \in C^\infty(S^{1|2}) \) and \( G \in C^\infty(S^{1|1}) \). Therefore, the natural maps

\[
\psi^n_{i, \alpha, 2} : \Pi(\tilde{S}_{n+1}) \to SP_{n, \frac{1}{2}, i} \]

\[
\Pi(F_{\alpha_2}^{n+1}) \to A_F^{(n, \frac{1}{2}, i)} \]

provides us with isomorphisms of \( K(1)_n \)-modules.

The action of \( K(1)_n \) on \( SP_{n, j, i} \) and on \( \tilde{SP}_{n, \frac{1}{2}, i} \) for \( n = 0, -1 \) is given by

\[
v_G \cdot A_F^{(n, j, i)} = \left\{ \begin{array}{l}
\pi(v_G), \tilde{A}_F^{(n, j, i)} \\
\tilde{A}_F^{(n, j, i)} \end{array} \right. \]

where \( F \) is a natural map on \( SP_{n, j, i} \) and \( \tilde{SP}_{n, \frac{1}{2}, i} \).

The action of \( K(1)_n \) on \( \tilde{SP}_{n, j, i} \) and on \( \tilde{SP}_{n, \frac{1}{2}, i} \) is induced by the the action of \( K(1)_n \) on \( SP_{n, j, i} \) and \( \tilde{SP}_{n, \frac{1}{2}, i} \) and a direct computation shows that one has:

\[
v_G \cdot A_F^{(n, j, i)} = \tilde{A}_F^{(n, j, i)} \quad \text{ for } j = 0, \frac{1}{2} \]

and

\[
v_G \cdot \tilde{A}_F^{(n, \frac{1}{2}, i)} = \tilde{A}_F^{(n, \frac{1}{2}, i)} \]

where \( F \in C^\infty(S^{1|2}) \) and \( G \in C^\infty(S^{1|1}) \). Therefore, the natural maps

\[
\psi^n_{i, \alpha, 2} : \Pi(\tilde{S}_{n+1}) \to SP_{n, \frac{1}{2}, i} \]

\[
\Pi(F_{\alpha_2}^{n+1}) \to \tilde{A}_F^{(n, \frac{1}{2}, i)} \]

provides us with isomorphisms of \( K(1)_n \)-modules.
where $F \in C^\infty(S^{1|2})$ and $G \in C^\infty(S^{1|1})$, $i = 1, 2$. Therefore, the natural maps

$$
\psi_{n, 0}^i : \mathfrak{g}_n \rightarrow SP(n, 0, i), \quad \psi_{n, \frac{1}{2}}^i : \Pi(\mathfrak{g}_n + \frac{1}{2}) \rightarrow SP(n, \frac{1}{2}, i)
$$

and

$$
\tilde{\psi}_{n, \frac{1}{2}}^i : \Pi(\mathfrak{g}_n + \frac{1}{2}) \rightarrow \overline{SP}(n, \frac{1}{2}, i)
$$

provide us with isomorphisms of $K(1)_i$-modules. This completes the proof.

5 The first cohomology space $H^1(K(2), SP(2))$

Let us first recall some fundamental concepts from cohomology theory ([3]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a super vector space $V = V_0 \oplus V_1$. The space $\text{Hom}(\mathfrak{g}, V)$ is $\mathbb{Z}_2$-graded via

$$
\text{Hom}(\mathfrak{g}, V)_b = \oplus_{a \in \mathbb{Z}_2} \text{Hom}(\mathfrak{g}_a, V_{a+b}); \quad b \in \mathbb{Z}_2.
$$

(5.1)

According to the $\mathbb{Z}_2$-grading (5.1), each $c \in Z^1(\mathfrak{g}, V)$, is broken to $(c', c'') \in \text{Hom}(\mathfrak{g}_0, V) \oplus \text{Hom}(\mathfrak{g}_1, V)$ subject to the following three equations:

$$
\begin{align*}
(E_1) \quad & c'([g_1, g_2]) - g_1.c'(g_2) + g_2.c'(g_1) = 0 \quad \text{for any } g_1, g_2 \in \mathfrak{g}_0, \\
(E_2) \quad & c''([g, h]) - g.c''(h) + h.c'(g) = 0 \quad \text{for any } g \in \mathfrak{g}_0, h \in \mathfrak{g}_1, \\
(E_3) \quad & c'([h_1, h_2]) - h_1.c''(h_2) - h_2.c''(h_1) = 0 \quad \text{for any } h_1, h_2 \in \mathfrak{g}_1.
\end{align*}
$$

Proposition 5.1. 1)

$$
H^1(K(1)_i, \mathfrak{g}_\lambda) = \begin{cases} 
\mathbb{R}^3 & \text{if } \lambda = 0, \\
\mathbb{R}^2 & \text{if } \lambda = 1, \\
0 & \text{otherwise}.
\end{cases}
$$

The respective nontrivial 1-cocycles are

$$
C^0(v_F) = \frac{1}{4}(3F + (-1)^p(F)), \quad C^1(v_F) = F', \quad C^2(v_F) = \tilde{\eta}_i(F')\theta_{3-i} \quad \text{if } \lambda = 0,
$$

$$
C^3(v_F) = \tilde{\eta}_i(F'')\theta_{3-i} \quad \text{if } \lambda = 1,
$$

where $\tilde{\eta}_i = \partial_{\theta_i} + \theta_i\partial_x$, $v_F \in K(1)_i$ and $F = f_0 + f_i\theta_i$.

2)

$$
H^1(K(1)_i, \mathfrak{g}_\lambda) = \begin{cases} 
\mathbb{R}^3 & \text{if } \lambda = \frac{1}{2}, \\
\mathbb{R}^2 & \text{if } \lambda = -\frac{1}{2}, \\
0 & \text{otherwise}.
\end{cases}
$$
Theorem 5.3. It is spanned by the following 1-cocycles:

\[
\begin{align*}
C_4(v_F) &= \frac{1}{4}(3F + (-1)^p(F)\theta_3) \quad \text{if } \lambda = -\frac{1}{2}, \\
C_6(v_F) &= \tilde{\eta}_i(F') \quad \text{if } \lambda = \frac{1}{2}, \\
C_7(v_F) &= \tilde{\eta}_i(F'') \quad \text{if } \lambda = \frac{3}{2}.
\end{align*}
\]

(5.4)

To prove Proposition 5.1, we need the following result (see [2]).

Proposition 5.2. [2]

1) The space \(H^1(K(1)_i, \mathfrak{g}^i_0)\), \(i = 1, 2\), has the following structure:

\[
H^1(K(1)_i, \mathfrak{g}^i_0) \simeq \begin{cases} 
\mathbb{R}^2 & \text{if } \lambda = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

The space \(H^1(K(1)_i, \mathfrak{g}^i_0)\) is generated by the cohomology classes of the 1-cocycles

\[
c_0(v_F) = \frac{1}{4}(3F + (-1)^p(F)F) \quad \text{and} \quad c_1(v_F) = F'.
\]

(5.5)

2) \(H^1(K(1)_i, \mathfrak{g}^i_1) \simeq \begin{cases} 
\mathbb{R} & \text{if } \lambda = \frac{1}{2}, \frac{3}{2}, \\
0 & \text{otherwise}.
\end{cases}
\]

It is spanned by the nontrivial 1-cocycles

\[
\begin{align*}
c_2(v_F) &= \tilde{\eta}_i(F') \quad \text{if } \lambda = \frac{1}{2}, \\
c_3(v_F) &= \tilde{\eta}_i(F'') \quad \text{if } \lambda = \frac{3}{2}.
\end{align*}
\]

(5.6)

Proof of Proposition 5.1: Let \(F\alpha_2^1 = (f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2)\alpha_2^1 \in \mathfrak{g}_\lambda\). The map

\[
\Phi : \mathfrak{g}_\lambda \rightarrow \mathfrak{g}_\lambda^i \oplus \mathfrak{g}_{\lambda + \frac{1}{2}}
\]

\[
F\alpha_2^1 \rightarrow ((1 - \theta_3\partial\theta_3)(F)\alpha_1^{i,i}, (-1)^p(F) + 1\theta_3\partial\theta_3(F)\alpha_{1,i}^{i,i,\frac{1}{2}}),
\]

where \(\alpha_{1,i} = dx + \theta_i d\theta_i, i = 1, 2\), provides us with an isomorphism of \(K(1)_i\)-modules. This map induces the following isomorphism between cohomology spaces:

\[
H^1(K(1)_i, \mathfrak{g}_\lambda) \simeq H^1(K(1)_i, \mathfrak{g}_\lambda^i) \oplus H^1(K(1)_i, \mathfrak{g}_{\lambda + \frac{1}{2}}).
\]

We deduce from this isomorphism and Proposition 5.2, the 1-cocycles (5.3–5.4).

The space \(H^1(K(2), S^2P(2))\) inherits the grading (4.2) of \(S^2P(2)\), so it suffices to compute it in each degree. The main result of this section is the following.

Theorem 5.3. The space \(H^1(K(2), S^2P_n)\) is purely even. It has the following structure:

\[
H^1(K(2), S^2P_n) \simeq \begin{cases} 
\mathbb{R}^3 & \text{if } n = -1 \\
\mathbb{R}^6 & \text{if } n = 0 \\
\mathbb{R} & \text{if } n = 1 \\
0 & \text{otherwise}.
\end{cases}
\]
For $n = -1$, the nontrivial 1-cocycles are:

\[
\begin{align*}
\text{Υ}_1(v_F) &= η_1 η_2(F) χ^{-1}_1 χ_2, \\
\text{Υ}_2(v_F) &= F' χ^{-1}_1 χ_2, \\
\text{Υ}_3(v_F) &= \left(\frac{1}{4}(F + (-1)^{p(F)} + F) + η_2 η_1(F_1 θ_2)\right) χ^{-1}_1 χ_2,
\end{align*}
\]

For $n = 0$, the nontrivial 1-cocycles are:

\[
\begin{align*}
\text{Υ}_4(v_F) &= \frac{1}{4}(F + (-1)^{p(F)} + F) + η_2 η_1(F_1 θ_2), \\
\text{Υ}_5(v_F) &= F', \\
\text{Υ}_6(v_F) &= η_1 η_2(F), \\
\text{Υ}_7(v_F) &= (-1)^{p(F)} \left(η_1(F') χ_1 + η_2(F') χ_2\right) χ^{-1}, \\
\text{Υ}_8(v_F) &= F''' χ^{-2}_1 χ_2 + (-1)^{p(F)} \left(η_2(F') χ_1 - η_1(F') χ_2\right) χ^{-1}, \\
\text{Υ}_9(v_F) &= η_1 η_2(F') χ^{-2}_1 χ_2.
\end{align*}
\]

For $n = 1$, the nontrivial 1-cocycle is:

\[
\text{Υ}_{10}(v_F) = \frac{2}{3} F^\prime \prime′ χ^{-3}_1 χ_2 + (-1)^{p(F)} \left(η_2(F') χ_1 - η_1(F') χ_2\right) χ^{-2} + 2η_1 η_2(F') χ^{-1}.
\]

To prove Theorem 5.3, we need first to prove the following lemma:

**Lemma 5.4.** Let $C$ be a even (resp. odd) 1-cocycle from $K(2)$ to $SP_n$, $n ∈ Z$. If its restriction to $K(1)_1$ and to $K(1)_2$ is a coboundary, then $C$ is a coboundary.

**Proof.** Let $C$ be a even (resp. odd) 1-cocycle of $K(2)$ with coefficients in $SP_n$ such that its restriction to $K(1)_1$ and to $K(1)_2$ is a coboundary. Using the condition of a 1-cocycle, we prove that there exists $G ∈ SP_n$ such that

\[
C(v_{f_0 + f_i θ_i}) = \{v_{f_0 + f_i θ_i} , G\} \text{ for any } f_0, f_i ∈ C^∞(S^1) \text{ and } i = 1, 2
\]

and

\[
C(v_{f_i θ_i θ_2}) = \{v_{f_i θ_i θ_2} , G\} \text{ for any } f_{12} ∈ C^∞(S^1).
\]

We deduce that $C(v_F) = \{v_F , G\}$, for any $F ∈ C^∞(S^1|2)$, and therefore $C$ is a coboundary of $K(2)$. \[ \square \]

**Proof of Theorem 5.3:** According to Lemma 5.4, the restriction of any nontrivial 1-cocycle of $K(2)$ with coefficients in $SP_n$ to $K(1)_1$ or to $K(1)_2$ is a nontrivial 1-cocycle.

Using Proposition 4.1 and Proposition 5.1, we obtain:

\[
H^1(K(1)_1, SP_n) \simeq \begin{cases} 
\mathbb{R}^7 & \text{if } n = -1 \\
\mathbb{R}^6 & \text{if } n = 0.
\end{cases}
\]
In the case \( n = -1 \), the space \( H^1(K(1), SP_{-1}) \) is spanned by the following 1-cocyles:

\[
\beta_i^l(v_F) = \psi^l_{-1, i}(C_l(v_F)), \quad l = 0, 1, 2,
\]
\[
\beta_i^l(v_F) = \psi^l_{-1, 4}(\Pi(C_l(v_F))),
\]
\[
\tilde{\beta}_i^l(v_F) = \tilde{\psi}^l_{-1, 4}(\Pi(C_l(v_F)));
\]
\[
\beta_i^l(v_F) = \psi^l_{0, 1}(C_5(v_F)),
\]
\[
\tilde{\beta}_i^l(v_F) = \tilde{\psi}^l_{0, 1}(\Pi(C_5(v_F))).
\]

In the case \( n = 0 \), the space \( H^1(K(1), SP_0) \) is spanned by the following 1-cocycle:

\[
\beta_{i+6}^l(v_F) = \psi_i^l_{0, 0}(C_l(v_F)), \quad l = 0, 1, 2,
\]
\[
\beta_{i}^l(v_F) = \psi^l_{0, 1}(C_5(v_F)),
\]
\[
\beta_{i}^l(v_F) = \psi^l_{0, 1}(\Pi(C_5(v_F))),
\]
\[
\tilde{\beta}_i^l(v_F) = \tilde{\psi}^l_{0, 1}(\Pi(C_5(v_F)));
\]

where the cocycles \( C_0, \ldots, C_6 \) are defined by the formulæ (5.3)–(5.4) and \( \psi^l_{n, j}, \tilde{\psi}^l_{n, j} \) are as in (4.5)–(4.6).

According to the same propositions, we obtain \( H^1(K(1), SP_n / SP_{n, i}) \) and \( H^1(K(1), SP_{n, i}) \) for \( n \neq 0, -1 \) and \( i = 1, 2 \). By direct computations, one can now deduce \( H^1(K(1), SP_n) \).

Second, note that any nontrivial 1-cocycle of \( K(2) \) with coefficients in \( SP_n \) should retain the following general form: \( \Upsilon = \Upsilon^0 + \Upsilon^1 + \Upsilon^2 + \Upsilon^3 \) where \( \Upsilon^0 : \text{Vect}(S^1) \to SP_n, \Upsilon^1, \Upsilon^2 : \mathcal{F}_{-\frac{1}{2}} \to SP_n \) and \( \Upsilon^3 : \mathcal{F}_0 \to SP_n \) are linear maps. The space \( H^1(K(1), SP_n, i) \), \( i = 1, 2 \), determines the linear maps \( \Upsilon^0, \Upsilon^1 \) and \( \Upsilon^2 \). The 1-cocycle conditions determines \( \Upsilon^3 \). More precisely, we get:

For \( n = -1 \), the space \( H^1(K(2), SP_{-1}) \) is generated by the nontrivial cocycles \( \Upsilon_1, \Upsilon_2 \) and \( \Upsilon_3 \) corresponding to the cocycles \( \beta_2^l, \beta_5^l \) and \( \beta_4^l \), respectively, via their restrictions to \( K(1) \).

For \( n = 0 \), the space \( H^1(K(2), SP_0) \) is spanned by the nontrivial cocycles \( \Upsilon_4, \Upsilon_5, \Upsilon_6, \Upsilon_7, \Upsilon_8 \) and \( \Upsilon_9 \) corresponding to the cocycles \( \beta_6^l, \beta_7^l, \beta_8^l, \beta_{10}^l, \beta_{11}^l, \beta_9^l \), respectively, via their restrictions to \( K(1) \), where \( \Upsilon_7 = \Upsilon_7 + \Upsilon_9 \) and \( \Upsilon_8 = \Upsilon_8 + \Upsilon_6 \).

Finally, for \( n = 1 \), the space \( H^1(K(2), SP_1) \) is generated by the nontrivial cocycle \( \Upsilon_{10} \) corresponding to the cocycle \( \psi_{1, 0}(C_3(v_F)) \) with \( \psi_{1, 0}^l \) as in (4.7) via its restriction to \( K(1) \).

Theorem 5.3 is proved. \( \square \)

6 The space \( H^1(K(2), SPDO(S^{1/2})) \)

6.1 The spectral sequence for a filtered module over a Lie (super)algebra

The reader should refer to [6], for the details of the homological algebra used to construct spectral sequences. We will merely quote the results for a filtered module \( M \) with decreasing filtration \( \{M_n\}_{n \in \mathbb{Z}} \) over a Lie (super)algebra \( g \) so that \( M_{n+1} \subset M_n \), \( \bigcup_{n \in \mathbb{Z}} M_n = M \) and \( gM_n \subset M_n \).
Consider the natural filtration induced on the space of cochains by setting:

\[ F^n(C^*(g, M)) = C^*(g, M_n), \]

then we have:

- \( dF^n(C^*(g, M)) \subset F^n(C^*(g, M)) \) (i.e., the filtration is preserved by \( d \));
- \( F^{n+1}(C^*(g, M)) \subset F^n(C^*(g, M)) \) (i.e. the filtration is decreasing).

Then there is a spectral sequence \((E_r^{p,q}, d_r)\) for \( r \in \mathbb{N} \) with \( d_r \) of degree \((r, 1-r)\) and

\[
E_0^{p,q} = F^p(C^{p+q}(g, M))/F^{p+1}(C^{p+q}(g, M)) \quad \text{and} \quad E_1^{p,q} = H^{p+q}(g, \text{Grad}^p(M)).
\]

To simplify the notations, we have to replace \( F^n(C^*(g, M)) \) by \( F^nC^* \). We define

\[
Z_r^{p,q} = F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}),
\]

\[
B_r^{p,q} = F^pC^{p+q} \cap d(F^{p-r}C^{p+q-1}),
\]

\[
E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_r^{p,q}).
\]

The differential \( d_r \) maps \( Z_r^{p,q} \) into \( Z_r^{p+r,q-r+1} \), and hence includes a homomorphism

\[
d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}
\]

The spectral sequence converges to \( H^*(C, d) \), that is

\[
E_\infty^{p,q} \simeq F^pH^{p+q}(C, d)/F^{p+1}H^{p+q}(C, d),
\]

where \( F^pH^*(C, d) \) is the image of the map \( H^*(F^pC, d) \rightarrow H^*(C, d) \) induced by the inclusion \( F^pC \rightarrow C \).

### 6.2 Computing \( H^1(K(2), S\Psi DO(S^{1|2})) \)

Now we can check the behavior of the cocycles \( \Upsilon_1, \ldots, \Upsilon_{10} \) under the successive differentials of the spectral sequence. Cocycles \( \Upsilon_1, \Upsilon_2 \) and \( \Upsilon_3 \) belong to \( E_1^{-1,2} \), cocycles \( \Upsilon_4, \ldots, \Upsilon_9 \) belong to \( E_1^{0,1} \) and cocycle \( \Upsilon_{10} \) belongs to \( E_1^{1,0} \). Consider a cocycle in \( \mathcal{SP}(2) \), but compute its differential as if it were with values in \( S\Psi DO(S^{1|2}) \) and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image under \( d_1 \). The higher order differentials \( d_r \) can be calculated by iteration of this procedure, the space \( E_r^{p+r,q-r+1} \) contains the subspace coming from \( H^{p+q+1}(K(2); \text{Grad}^{p+1}(S\Psi DO(S^{1|2}))) \).

It is now easy to see that the cocycles \( \Upsilon_1, \ldots, \Upsilon_6 \) will survive in the same form. Computing supplementary higher order terms for the other cocycles, we obtain
Theorem 6.1. The space $H^1(\mathcal{K}(2), \mathfrak{SDO}(S^{1/2}))$ is purely even. It is spanned by the classes of the following nontrivial 1-cocycles

\[
\begin{align*}
\Theta_1(v_F) &= \eta_1\eta_2(F)\xi^{-1}\zeta_1\zeta_2, \\
\Theta_2(v_F) &= F\xi^{-1}\zeta_1\zeta_2, \\
\Theta_3(v_F) &= \left(\frac{1}{4}(F + (-1)^{p(F)} + 1) + \eta_1(F\theta_1\theta_2)\right)\xi^{-1}\zeta_1\zeta_2, \\
\Theta_4(v_F) &= \frac{1}{4}(F + (-1)^{p(F)} + 1) + \eta_2(F\theta_1\theta_2), \\
\Theta_5(v_F) &= F', \\
\Theta_6(v_F) &= \eta_1\eta_2(F), \\
\Theta_7(v_F) &= \sum_{n=0}^{\infty} \frac{(-1)^{p(F)+n}}{n+1} \left(\eta_1(F^{(n+1)})\zeta_1 + \eta_2(F^{(n+1)})\zeta_2\right)\xi^{-n-1} + \\
&\sum_{n=0}^{\infty} \frac{2(-1)^{n}}{n+2} F^{(n+2)}\xi^{-n-1}, \\
\Theta_8(v_F) &= \sum_{n=0}^{\infty} (-1)^{n} F^{(n+2)}\xi^{-n-2}\zeta_2 + \\
&\sum_{n=0}^{\infty} (-1)^{n} \eta_1\eta_2(F^{(n)})\xi^{-n}, \\
\Theta_9(v_F) &= \sum_{n=0}^{\infty} (-1)^{n} \eta_1 F^{(n+1)}\xi^{-n-2}\zeta_1\zeta_2 + \\
&\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n+2} F^{(n+2)}\xi^{-n-1}, \\
\Theta_{10}(v_F) &= \sum_{n=0}^{\infty} (-1)^{n} \frac{2n}{n+2} F^{(n+2)}\xi^{-n-2}\zeta_1\zeta_2 + \\
&\sum_{n=1}^{\infty} (-1)^{n} \eta_1 F^{(n+1)}\xi^{-n-1}\zeta_2 + \\
&\sum_{n=1}^{\infty} (-1)^{n} \frac{2n}{n+1} \eta_2 F^{(n+1)}\xi^{-n-1}\zeta_1 + \\
&\sum_{n=0}^{\infty} 2(-1)^{n+1}\eta_1\eta_2(F^{(n)})\xi^{-n}.
\end{align*}
\]

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References


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