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Short Fibers Suspension in Steady Recirculating Flows

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As a consequence of the increasing use of composite materials, there has been much work on constitutive equations and computational mechanics for short fibers composites. Since these materials are generally made of a matrix and fibers reinforcement, the mechanical properties of the conformed pieces depend greatly on the fibers' orientation in the solid material. However, it turns out that this orientation is determined by the forming process, so that it is interesting to develop mathematical models for the flow during this conforming process and to develop specific numerical strategies to solve the resulting equations.

The numerical simulation of the flow has been carried out in several different ways, including a series of simplifying hypotheses. This modeling is usually achieved within the framework of dilute or semi-dilute suspensions of non-spherical particules in a Newtonian fluid. The resulting system of equations involves the coupling of an elliptic boundary value problem with a convection-type equation. The elliptic problem is associated with the equations of motion, whereas the advection equation describes the time evolution of the anisotropic viscosity tensor. The second problem, associated with the orientation equation, presents two difficulties: it is non-linear and hyperbolic.

The method of characteristics, applicable as long as every trajectory crosses the inflow boundary, has been applied many times (see for example Rosenberg et al., 1990). When there are recirculations, specific methods especially well adapted to hyperbolic equations in eulerian variables are needed (e.g., SUPG or discontinuous finite volume methods).

Many of the experimental and industrial flows show recirculating areas or recirculate themselves. For example, in a lid-driven cavity problem, the flow recirculates under the influence of a moving plate. Contraction or expansion flows (as encountered in extrusion processes), standard rotative rheometers, etc., proceed with a steady recirculating flow. In this flow type, the advection equation is supposed to have a steady solution but neither boundary conditions nor initial conditions are known.

In Chinesta and Chaidron (2001), a numeric procedure to determine a steady solution of the fibers orientation in steady recirculating flows was proposed, which took into account that neither initial condition nor boundary conditions are given. This procedure may be used in the numerical simulation of SFRT flows involving recirculating parts, as

Numerical modeling of short fiber suspensions flows involves the coupling between motion equations, which define an elliptic problem, and the fluid constitutive equation, which introduces a non-linear advection problem related to the fiber orientation (induced anisotropy). In a previous work these authors have proposed a numerical procedure to determine a steady solution of the fibers orientation in steady recirculating flows, taking into account that neither initial nor boundary conditions are given. This procedure may be used in the numerical simulation of SFRT flows involving recirculating parts as encountered in the simulation of industrial processes, as well as in inverse rheological identification using, for example, rotative rheometric devices.

La modélisation numérique des suspensions de fibres courtes implique le couplage entre les équations de mouvement (qui définissent un problème elliptique) et l'équation constitutive qui introduit un problème de transport non linéaire associé à l'orientation des fibres. Les auteurs ont proposé, dans des travaux précédents, une technique numérique pour le calcul de l'orientation des fibres dans un écoulement stationnaire recirculant pour lequel les conditions aux limites et les conditions initiales ne sont pas connues. Cette technique peut être utilisée dans la simulation d'écoulements de fibres courtes présentant des recirculations, comme c'est le cas dans les écoulements industriels en contraction ainsi que dans les instruments rhéométriques rotatifs.

Keywords: non-linear advection problems, steady recirculating flows, short fibers suspensions.

encountered when simulating industrial processes as well as in inverse rheological identification, for 2D or 3D steady kinematics and orientation fields.

An example of a 2D kinematics flow and 3D orientation field will be presented below in a section devoted to applications for inverse rheological identification. In this paper, the procedure will be applied in different steady recirculating flows of short fibers suspensions.

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Mechanical and Numerical Modeling

The flow model associated with a short fibers reinforced thermoplastic model is defined by the following equations (Batchelor, 1970; Hand, 1962; Hinch, and Leal, 1975, 1976; Meslin, 1997):

The balance of momentum equations, without inertia and mass terms, is:

$$\text{Div } \underline{\underline{\sigma}} = 0 \quad (1)$$

where $\underline{\underline{\sigma}}$ is the stress tensor.

The incompressibility condition is:

$$\text{Div } \underline{\underline{v}} = 0 \quad (2)$$

where $\underline{\underline{v}}$ represents the velocity field.

The constitutive equation, with a quadratic closure relation for the fourth-order orientation tensor and other simplifying assumptions (see Tucker, 1991), is:

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\{\underline{\underline{D}} + N_p \text{Tr}(\underline{\underline{aD}})\underline{\underline{a}}\} \quad (3)$$

where P denotes the pressure, $\underline{\underline{I}}$ the unit tensor, μ the equivalent suspension viscosity, $\underline{\underline{D}}$ the strain rate tensor, N_p a scalar parameter depending on both the fiber concentration and the fiber aspect ratio, and $\underline{\underline{a}}$ the second order orientation tensor defined by:

$$\underline{\underline{a}} = \oint \underline{\underline{\rho}} \otimes \underline{\underline{\rho}} \Psi(\underline{\underline{\rho}}) d\underline{\underline{\rho}} \quad (4)$$

where $\underline{\underline{\rho}}$ is the unit vector associated with the direction of the fiber axis, and $\Psi(\underline{\underline{\rho}})$ is the orientation distribution function, verifying:

$$\oint \Psi(\underline{\underline{\rho}}) d\underline{\underline{\rho}} = 1 \quad (5)$$

If $\Psi(\underline{\underline{\rho}}) = \delta(\underline{\underline{\rho}} - \hat{\underline{\underline{\rho}}})$ with $\delta()$ the Dirac's function, all the orientation probability is concentrated in the direction defined by $\hat{\underline{\underline{\rho}}}$, and the corresponding orientation tensor is $\underline{\underline{a}} = \hat{\underline{\underline{\rho}}} \otimes \hat{\underline{\underline{\rho}}}$. In a planar case, the isotropic orientation state is defined by the uniform distribution function:

$$\Psi(\underline{\underline{\rho}}) = \frac{1}{2\pi} \quad (6)$$

The orientation tensor related to a planar isotropic orientation state is then:

$$\underline{\underline{a}} = \frac{\underline{\underline{I}}}{2} \quad (7)$$

From a physical point of view, we can consider that the eigenvalues of the second order orientation tensor ($\underline{\underline{a}}$) represent the probability of finding the fibers in the direction of the corresponding eigenvectors.

With a quadratic closure relation for the fourth order orientation tensor, the orientation equation is expressed as:

$$\frac{d\underline{\underline{a}}}{dt} = \underline{\underline{\Omega}}\underline{\underline{a}} - \underline{\underline{a}}\underline{\underline{\Omega}} + k(\underline{\underline{D}}\underline{\underline{a}} + \underline{\underline{a}}\underline{\underline{D}} - 2\text{Tr}(\underline{\underline{aD}})\underline{\underline{a}}) + D_r\left(\underline{\underline{a}} - \frac{\underline{\underline{I}}}{2}\right) \quad (8)$$

where $\underline{\underline{a}}$ satisfies

$$\begin{cases} \underline{\underline{a}} = \underline{\underline{a}}^t \\ \text{Tr}(\underline{\underline{a}}) = 1 \end{cases} \quad (9)$$

$\underline{\underline{D}}$ and $\underline{\underline{\Omega}}$ are the symmetric and skew-symmetric components of $\text{Grad } \underline{\underline{v}}$, k is a constant that depends on the fiber aspect ratio: $k=(r^2-1)(r^2+1)$, with the fiber aspect ratio defined by $r=L/\phi$ where L is the fibers length and ϕ its diameter, and D_r a diffusion coefficient.

The quadratic closure relation is exact as long as the fibers are perfectly aligned locally. If we consider other closure relations, the orientation equation is changed but its hyperbolic character is maintained and the iteration procedure proposed in this paper can be used. The comparison of different closure relations is an important feature of this equation and will be the main subject of a future work.

The flow model is defined in $\Omega \in \mathbb{R}^2$. On the boundary, $\Gamma = \partial\Omega$ either the velocity or the traction are imposed:

$$\underline{\underline{v}}(\underline{\underline{x}} \in \Gamma_1) \underline{\underline{v}}_g \quad (10)$$

and

$$\underline{\underline{\sigma}} \underline{\underline{n}}(\underline{\underline{x}} \in \Gamma_2) = \underline{\underline{F}}_d \quad (11)$$

with $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $\underline{\underline{n}}(\underline{\underline{x}})$ is the unit outwards vector, normal to the boundary at the point $\underline{\underline{x}}$. The inflow boundary will be denoted by Γ_- :

$$\Gamma_- = \{\underline{\underline{x}} \in \Gamma_1, +\underline{\underline{v}}'(\underline{\underline{x}}) \underline{\underline{n}}(\underline{\underline{x}}) < 0\} \quad (12)$$

As the orientation equation has a hyperbolic character, the integration of the orientation equation only requires an orientation boundary condition on the inflow boundary in a steady state flow simulation:

$$\underline{\underline{a}}(\underline{\underline{x}} \in \Gamma_-) = \underline{\underline{a}}_0 \quad (13)$$

Kinematics and Fiber Orientation Coupling

When one is looking for a steady solution of the flow problem defined by Equations (1), (2), (3) and (8), a fixed-point iterative strategy is usually considered (Lipscomb et al., 1988; Chiba and Nakamura, 1998). In this form the flow kinematics given by the

solution of Equations (1), (2) and (3) are obtained at time $t_m = m\Delta t$ from the orientation field at the previous iteration $\underline{a}(t_{m-1}) = \underline{a}_{m-1}$.

$$\begin{cases} \text{Div} \underline{\sigma}_m = 0 \\ \text{Div} \underline{v}_m = 0 \\ \underline{\sigma}_m = -p_m \underline{I} + 2\mu \left\{ \underline{D}_m + N_p \text{Tr}(\underline{a}_{m-1} \underline{D}_m) \underline{a}_{m-1} \right\} \end{cases} \quad (14)$$

Now, from the actualized flow kinematics, $(\underline{v}(t_m) = \underline{v}_m)$, the fiber orientation is recalculated solving Equation (8):

$$\begin{aligned} \frac{d\underline{a}_m}{dt} &= \underline{\Omega}_m \underline{a}_m - \underline{a}_m \underline{\Omega}_m + \\ &+ k \left(\underline{D}_m \underline{a}_m + \underline{a}_m \underline{D}_m - 2\text{Tr}(\underline{a}_m \underline{D}_m) \underline{a}_m \right) + D_r \left(\underline{a}_m - \frac{\underline{I}}{2} \right) \end{aligned} \quad (15)$$

This iterative process must be continued until the convergence is reached. The flow kinematics define an elliptic problem, whereas Equation (8) is non-linear and hyperbolic. When the solution of Equations (1), (2) and (3) defines a recirculating flow, the resolution of Equation (8) is more difficult to obtain, because neither boundary nor initial conditions are known. In the next section we will describe the numerical procedure proposed in Chinesta and Chaidron (2001) to solve linear advection problems in steady recirculating flows, which must be coupled with a scheme to treat the non-linearity of Equation (8).

Remarks

The flow kinematics are easily solved using a velocity-pressure mixed formulation with a functional approximation verifying the inf-sup condition. In this way, part of the problem is solved without difficulty.

As we will only solve Equation (8) at time t_m in a steady recirculating flow, the index m will be excluded from the notation.

In forming processes such as extrusion or in rheological identification using any rotating viscosimetric flow, the fibers' orientation has an important impact on the fluid kinematics, and vice-versa. However, to give an accurate description of the coupling between fluid kinematics and fibers' orientation, this orientation has to be precisely computed. This paper advances an original strategy for obtaining an accurate solution for the orientation in any steady recirculating flow kinematics, which poses the major difficulty in this kind of problem.

As shown in Chaidron and Chinesta (2001), the finite element solution of advection equations in steady recirculating flows are not necessarily periodic. To avoid this problem, we use the Finite Element Method (FEM) to resolve the flow kinematics and a specific and original Lagrangian method to solve the orientation problem.

Numerical Treatment for Non-Linearity

Equation (8) can be linearized by using a fixed point strategy (Chaidron and Chinesta, 2002), which looks a periodic solution of Equation (16) at each iteration n .

$$\begin{aligned} \frac{d\underline{a}^n}{dt} &= \underline{\Omega} \underline{a}^n - \underline{a}^n \underline{\Omega} + k \left(\underline{D} \underline{a}^n + \underline{a}^n \underline{D} - 2\text{Tr}(\underline{a}^{n-1} \underline{D}) \underline{a}^n \right) + \\ &+ D_r \left(\underline{a}^n - \frac{\underline{I}}{2} \right) \end{aligned} \quad (16)$$

As $\underline{a}^{n-1}$ is known in present iteration n , the transport equation becomes linear, and the numerical procedure proposed in Chinesta and Chaidron (2001) allows us to compute a steady solution of Equation (16).

Steady Solution of the Linear Orientation Equation

Equation (16) can now be written taking into account the linearity and symmetry of the orientation tensor in the Lagrangian vectorial form:

$$\frac{d\underline{a}}{dt} = \underline{A}(\underline{x}(t)) \underline{a} + \underline{B}(\underline{x}(t)) \quad (17)$$

where

$$\underline{a} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{22} \end{pmatrix} \quad (18)$$

The matrix $\underline{A}$ and the vector $\underline{B}$ depend only on the flow kinematics or on the material parameters k and D_r , and consequently on the particle position given by the integration of:

$$\frac{d\underline{x}}{dt} = \underline{v}(\underline{x}(t)) \quad (19)$$

The differential system in Equation (17) is linear and consequently its general solution may be written as an addition to the general solution of the homogeneous differential system:

$$\frac{d\underline{a}}{dt} = \underline{A}(\underline{x}(t)) \underline{a} \quad (20)$$

and as a particular solution of the non-homogeneous differential system (Equation 17).

If we denote by $\underline{a}_h^{(11)}(t)$, $\underline{a}_h^{(12)}(t)$ and $\underline{a}_h^{(22)}(t)$, the solutions of the differential system (20) associated with the initial conditions given by $\underline{e}_1 \otimes \underline{e}_1$, $\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1$ and $\underline{e}_2 \otimes \underline{e}_2$, and by $\underline{a}_c(t)$, a particular solution of (17), then the general solution of (17) is given by:

$$\underline{a}(t) = \alpha_{11} \underline{a}_h^{(11)}(t) + \alpha_{12} \underline{a}_h^{(12)}(t) + \alpha_{22} \underline{a}_h^{(22)}(t) + \underline{a}_c(t) \quad (21)$$

with $(\alpha_{11}, \alpha_{12}, \alpha_{22}) \in \mathbb{R}^3$.

For steady recirculating flows, the only natural condition which can be imposed on the solution is the periodicity of that solution along the closed trajectories, as proved in Poitou et al. (2000). In order to obtain this periodic solution numerically, Chinesta and Chaidron (2001) propose to impose the periodicity in

Equation (21). Thus, the solution after a spatial period T (a complete rotation) must be the initial value:

$$\underline{a}(t=0) = \alpha_{11}\underline{a}_h^{(11)}(T) + \alpha_{12}\underline{a}_h^{(12)}(T) + \alpha_{22}\underline{a}_h^{(22)}(T) + \underline{a}_c(T) \quad (22)$$

This defines the linear system:

$$\underline{\underline{C}}\underline{\alpha} = \underline{\underline{D}} \quad (23)$$

From that system, we can obtain the values of the three components of the vector $\underline{\alpha}$: α_{11} , α_{12} and α_{22} . In Chinesta and Chaidron (2001), the authors proved that the periodicity condition implies the solution stationarity.

Numerical Results

In order to verify the numerical procedure we are going to analyze different simple recirculating flows. In the integration of differential equations, a fourth-order Runge-Kutta scheme has been considered.

SFRT Flow without Diffusion Effects

We consider the model given in Equation (8) with a diffusion coefficient close to zero. In this case, Poitou et al. (2000) have proved that for fibers with a quasi-infinite aspect ratio in a general flow, the only solution is given by the local alignment of the fibers with the streamlines.

Rotation of a Rigid Body

We now consider the rigid motion given by the velocity field:

$$\underline{v}^T = (-y, x) \quad (24)$$

This equation describes the rigid body rotation around the center point (0,0). The velocity is constant on a circle trajectory centered on this point. The velocity is linearly dependent on the radius. We solve the problem for the point $\underline{x}^t = (0,1)$. The matrix of coefficients given in (23) is singular. This means that any solution may be considered at the chosen point as a periodic initial solution.

The expression of this solution in a cylindrical coordinates system remains the same along a closed trajectory. To prove this, we can verify that for this flow the evolution of the orientation tensor in cylindrical coordinates becomes:

$$\frac{d\underline{a}}{dt} = \underline{0} \quad (25)$$

Planar and Circular Flow

We will consider here the planar flow given by the following velocity field:

$$\underline{v}^T = \left(-y\sqrt{x^2 + y^2}, x\sqrt{x^2 + y^2} \right) \quad (26)$$

We are looking for a periodic orientation solution with a unit trace at the point $\underline{x}^t = (0,1)$ using the strategy described above.

The solution found corresponds to the local alignment of the fibers with the flow, i.e.:

$$\underline{a}_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (27)$$

When inextensible fibers with a finite aspect ratio are considered, the local alignment of the fibers with the flow is not a solution. Moreover, in Chinesta (2000), it is proved that for general flows the periodic solution cannot have a unit eigenvalue. We can verify that result using our strategy.

If we are looking for a periodic fiber orientation in the flow defined by Equation (26) for fibers with $k = 0.6$, and with the diffusion coefficient $D_r = 0$, then the solution found at the point $\underline{x}^t = (0,1)$ is given by:

$$\underline{a}_{(0,1)} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} \quad (28)$$

Increasing the isotropic behavior with the k parameter, until reaching the isotropic orientation for circular particles $k = 0$.

The solution along the streamline is depicted in Figure 1. We represent the orientation tensor by means of an ellipse. The ellipse axis directions correspond to the eigenvectors of the orientation tensor, and the semi-axis sizes correspond to the eigen values. In this figure we can see that ellipses have a certain thickness corresponding to a non-null eigenvalue in the radial direction. That means that the most probable direction of a fiber is the direction of the streamline but there is a light misalignment. In this flow, each fiber rotates along its trajectory with a period that depends on both the fiber aspect ratio and the shear rate, and the orientation average is given by the orientation tensor shown in Figure 1.

Figure 1. Steady fiber orientation for fibers with finite aspect ratio $k = 0.6$, $D_r = 0$, in a circular and planar flow.

Figure 2. Steady fibers orientation in lid-driven cavity flow for fibers with quasi-infinite aspect ratio $k = 1$ and $D_r = 0$.

Figure 4. Steady fibers orientation with a diffusion coefficient $D_r \approx 0$, $k = 1$, in a planar and circular flow.

Figure 3. Steady fibers orientation in lid-driven cavity flow for fibers with infinite aspect ratio $k = 0.9$ and $D_r = 0$.

Figure 5. Steady fibers orientation with high diffusion coefficient $D_r = 10$, $k = 1$, in a planar and circular flow.

Lid-Driven Cavity Flow

Finally, we consider the lid-driven cavity flow problem, with the flow kinematics given by the solution of the Stokes problem, as defined by:

$$\begin{aligned} \nabla p &= \mu \Delta \underline{u} & \text{in }]0, 1[\times]0, 1[\\ \underline{u} &= \underline{0} & \text{on } x = 0; \ x = 1; \ y = 0 \\ \underline{u}^T &= \left(\frac{2}{5} x(x-1), \ 0 \right) & \text{on } y = 1 \end{aligned} \quad (29)$$

In order to get accurate incompressible solutions, we use a stream function formulation and solve the resulting biharmonic equation with a pseudospectral Chebyshev technique, neglecting the fiber orientation effects on the kinematics of the flow. Figures 2 and 3 show the steady orientation solution along

some closed trajectories for fibers with quasi-infinite and finite ($k = 0.9$) aspect ratios, respectively.

In the first case (Figure 2), the local alignment of the fibers with the flow is identified, except in the vicinity of the center of rotation, where the kinematics are very close to a rigid motion. However, even in this area, the local alignment may be considered as a steady orientation solution.

In the second case (Figure 3), as indicated in Poitou et al. (2000), both high gradients near the corners and elongational flow induce high deviations from the local alignment in some areas of the fluid domain. In order to emphasize this deviation, we have superimposed the orientation field with the unit vectors in the streamline direction.

SFRT Flow with Diffusion Effects

Planar and Circular Flow

In this case we consider the more general model given by Equation (8) with $D_r \neq 0$ for fibers with a quasi-infinite aspect ratio immersed in a steady recirculating flow defined by Equation (26).

Figure 6. Steady fibers orientation with a small diffusion coefficient $D_r = 0$, $k = 1$, in a lid-driven cavity flow.

Figure 7. Steady fibers orientation with high diffusion coefficient $D_r = 10$, $k = 1$, in a lid-driven cavity flow.

Figures 4 and 5 show the steady fiber orientation for two different diffusion coefficients. In the first case the diffusion coefficient is assumed to be small, and consequently the steady solution is very close to the local alignment of the fibers with the flow. In the other case, the higher diffusion coefficient induces more isotropy.

Lid-driven Cavity Flow

For this more complex flow and for fibers with a quasi-infinite aspect ratio, high extensions are induced by the flow. In consequence, the term which preserves the fiber inextensibility ($Tr(\underline{q}\underline{D})\underline{q}$) becomes greater than for previous flows. The high non-linearity makes the fixed point strategy less efficient. Figures 6 and 7 show the steady fiber orientation for two different diffusion coefficients.

Application to Inverse Rheological Identification

In this section we consider the simplest rheological device, the cone and plate rheometer and a dilute short fibers suspension. We will denote ω as the angular velocity of the cone, R as the

external radius, and α as the cone angle. As indicated previously, the constitutive equation is given by:

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + 2\mu\left\{\underline{\underline{D}} + N_p Tr(\underline{\underline{q}}\underline{\underline{D}})\underline{\underline{q}}\right\} \quad (30)$$

where μ and N_p are two rheological parameters to determine.

The angular velocity on the cone surface for a radius r results in $v_\theta = \omega r$, and at a point (r, θ, z) in the fluid domain, $v_\theta = \omega z / \tan(\alpha)$. Thus, the shear strain rate is constant in the fluid domain $\dot{\gamma}_{\theta z} = \omega / \tan(\alpha) \approx \omega / \alpha$.

The expression of the strain rate tensor in cylindrical coordinates results in:

$$\underline{\underline{D}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\dot{\gamma}_{\theta z}}{2} \\ 0 & \frac{\dot{\gamma}_{\theta z}}{2} & 0 \end{pmatrix} \quad (31)$$

For these kinematics, the steady orientation tensor obtained with the proposed strategy results in:

$$\underline{\underline{q}} = \begin{pmatrix} a_{rr} & 0 & 0 \\ 0 & a_{\theta\theta} & a_{\theta z} \\ 0 & a_{\theta z} & a_{zz} \end{pmatrix} \quad (32)$$

which is also constant in the fluid domain. Thus, for example, if we take $\omega = 0.01$ rad/s, $k = 1$ and $D_r = 10$ and $\alpha = 3^\circ$, the steady fiber orientation tensor results in:

$$\underline{\underline{q}} = \begin{pmatrix} 0.184 & 0 & 0 \\ 0 & 0.632 & 0.203 \\ 0 & 0.203 & 0.184 \end{pmatrix} \quad (33)$$

With $\underline{\underline{D}}$ and $\underline{\underline{q}}$ as constants, and if we assume a constant pressure, the stress tensor will also be constant and, consequently, the equilibrium equation $Div \underline{\underline{\sigma}} = 0$, will be automatically verified. On the other hand, the kinematics verify the incompressibility condition $Div \underline{\underline{v}} = 0$, as well as the boundary conditions. To prove that $\underline{\underline{v}}$, $\underline{\underline{\sigma}}$ and $\underline{\underline{q}}$ define the problem solution, we need to prove that the stress boundary condition is verified, i.e.:

$$\underline{\underline{\sigma}} \underline{\underline{e}}_r|_{r=R} = 0 \quad (34)$$

where $\underline{e}_r$ is the unit vector aligned in the radial direction. This condition gives us the pressure value:

$$p = 2\mu N_p a_{rr} \dot{\gamma}_{\theta z} \quad (35)$$

Now, if we know the angular torque C and the normal force F applied on the cone, we can identify both rheological parameters μ and N_p from the following relations:

$$F = (\underline{\underline{\sigma}} \underline{e}_z) \underline{e}_z \pi R^2 \quad (36)$$

$$C = \int_0^R (\underline{\underline{\sigma}} \underline{e}_z) \underline{e}_{\theta} r 2\pi r \, dr \quad (37)$$

More precisely:

$$F = 2\pi R^2 \mu N_p \dot{\gamma}_{\theta z} (a_{\theta z} a_{zz} - a_{rr}) \pi R^2 \quad (38)$$

$$C = \frac{2\pi R^3}{3} \mu \dot{\gamma}_{\theta z} (1 + 2N_p a_{\theta z}^2) \quad (39)$$

In the previous analysis we considered the lateral fluid surface $r=R$ to be perfectly cylindrical. When we consider a convex lateral surface, a 3D computation with the free boundary condition must be carried out. However, the procedure proposed to compute steady orientation solutions can be applied to obtain the steady orientation associated with the kinematics at each iteration. This is the general situation found in the analysis of rotative rheometers, where, in general, homogeneous kinematics are not characteristic.

Remarks

As one of the referees of this paper has pointed out, the diffusion term in Equation (8) was introduced in the context of non-Brownian suspensions, as is the case with short fiber suspensions, to take into account the fibers' interaction in the flow. It has been widely assumed (Folgar and Tucker, 1984; Tucker, 1991), that the diffusion coefficient D_r is proportional to the shear rate, i.e., $D_r = C_f \dot{\gamma}$ with $C_f \dot{\gamma}$ constant. Thus, from that proportionality between the diffusion coefficient and the shear rate, the orientation solution in steady recirculating flows depends also on fiber interaction. In this case, from the values of F and C and Equations (38) and (39) the effect of fiber interaction in rheometry could be analyzed.

Conclusions

We have presented some results concerning the steady solution of the orientation of fibers in steady recirculating flows. We have proposed a solution of the non-linear advection equation governing the fiber orientation evolution, even if neither initial nor boundary conditions are known. We have also given a numerical strategy for finding this solution. We have used the proposed strategy to analyse different simple recirculating flows of short fibers suspensions. This paper has led us to offer a numerical procedure for computing the steady solution of industrial recirculating flows, as are encountered in all rotative rheometric devices, which are generally 3D and couple flow kinematics with fiber orientation.

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Nomenclature

$\underline{\underline{g}}$	second order orientation tensor, with components a_{ij}
$\underline{\underline{g}}_4$	fourth order orientation tensor, with components a_{ijkl}
C	angular torque, (N·m)
$\underline{\underline{D}}$	strain rate tensor, with components D_{ij} , (s^{-1})
D_r	diffusion coefficient
$\underline{e}_m$	unit vector associated with the m^{th} coordinate
F	is the normal force N
$\underline{\underline{I}}$	unit tensor
k	fiber parameter depending on the fiber aspect ratio r
L	fiber length, (m)
$\underline{n}$	outwards unit vector defined on the domain boundary
N_p	particle number
p	pressure, (Pa)
r	fiber aspect ratio
R, r	radius, (m)
t	time, (s)
T	rotation period, (s)
Tr	trace of a tensor
$\underline{v}$	velocity field, with components v_i , ($m \cdot s^{-1}$)

Greek Symbols

$\dot{\gamma}$	shear rate, (s^{-1})
Γ	fluid domain boundary
δ	Dirac's delta distribution
δ_{ij}	Kronecker's delta function
θ	fiber diameter, m
μ	suspension viscosity, (Pa·s)
$\underline{p}$	unit vector which defines the fiber axis direction
$\underline{\underline{\sigma}}$	stress tensor, with components σ_{ij} , (Pa·s)
Ψ	fiber orientation distribution
ω	angular velocity, (s^{-1})
Ω	fluid domain
$\underline{\underline{\Omega}}$	vorticity tensor, with components Ω_{ij} , (s^{-1})

The symbol $\otimes$ denotes the dyadic product. Thus, the components of $\underline{u} \otimes \underline{v}$ are $u_i v_j$.

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