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Adrien Dubouloz, Pierre-Marie Poloni

ON A CLASS OF DANIELEWSKI SURFACES IN AFFINE 3-SPACE

ADRIEN DUBOULOZ AND PIERRE-MARIE POLONI

Abstract. L. Makar-Limanov [15] and [16] computed the automorphisms groups of surfaces in $\mathbb{C}^3$ defined by the equations $x^n z - P(y) = 0$, where $n \geq 1$ and $P(y)$ is a nonzero polynomial. Similar results have been obtained by A. Crachiola [3] for surfaces defined by the equations $x^n z - y^2 - h(x)y = 0$, where $n \geq 2$ and $h(0) \neq 0$, defined over an arbitrary base field. Here we consider the more general surfaces defined by the equations $x^n z - Q(x,y) = 0$, where $n \geq 2$ and $Q(x,y)$ is a polynomial with coefficients in an arbitrary base field k. Among these surfaces, we characterize the ones which are Danielewski surfaces in the sense of [8], and we compute their automorphism groups. We study closed embeddings of these surfaces in affine 3-space. We show that in general their automorphisms do not extend to the ambient space. Finally, we give explicit examples of $\mathbb{C}^*$-actions on a surface in $\mathbb{A}^3_{\mathbb{C}}$ which can be extended holomorphically but not algebraically to a $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$.

Introduction

Since they appeared in a celebrated counter-example to the Zariski Cancellation Problem due to W. Danielewski [5], the surfaces defined by the equations $xz - y(y-1) = 0$ and $x^2 z - y(y-1) = 0$ in $\mathbb{C}^3$ and their natural generalizations, such as surfaces defined by the equations $x^n z - P(y) = 0$, where $P(y)$ is a non-constant polynomial, have been studied in many different contexts. During the last decade, it has been progressively realized that they play a central role in certain problems of affine algebraic geometry. One of the reasons of this ubiquity comes from the fact that they can be equipped with nontrivial actions of the additive group $\mathbb{C}_+$. As a consequence, these surfaces S admit a so-called *quotient* $\mathbb{A}^1$*-fibration* $\mathrm{pr}_x : S \to \mathbb{A}^1$, that is, a surjective morphism with generic fiber isomorphic to an affine line. The general fibers of these fibrations coincide with the general orbits of a $\mathbb{C}_+$-action on S. Normal affine surfaces S equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ can be roughly classified into two classes according the following alternative : either $\pi : S \to \mathbb{A}^1$ is a unique $\mathbb{A}^1$-fibration on S up to automorphisms of the base, or there exists a second $\mathbb{A}^1$-fibration $\pi' : S \to \mathbb{A}^1$ with general fibers distinct from the ones of π.

Due to the symmetry between the variables x and z, a surface defined by the equation $xz - P(y) = 0$ admits two distinct fibrations as above. In contrast, L. Makar-Limanov [16] established that on a surface $S_{P,n}$ defined by the equation $x^n z - P(y) = 0$ in $\mathbb{C}^3$, where $n \geq 2$ and where $P(y)$ is a polynomial of degree $r \geq 2$, the projection $\mathrm{pr}_x : S_{P,n} \to \mathbb{C}$ is a unique $\mathbb{A}^1$-fibration up to automorphisms of the base. More precisely, using the correspondence between algebraic $\mathbb{C}_+$-actions on S and locally nilpotent derivations of the algebra $\mathcal{O}(S)$ of regular functions on S, L. Makar-Limanov established that general orbits of an arbitrary nontrivial $\mathbb{C}_+$-action on S coincide with the general fibers of the $\mathbb{A}^1$-fibration $\mathrm{pr}_x : S_{P,n} \to \mathbb{C}$. Over a field k of positive characteristic, additive group actions no longer correspond to locally nilpotent derivations but rather to suitable systems of Hasse-Schmidt derivations. Using this correspondence, A. Crachiola [3] obtained similar results for surfaces with equations $x^n z - y^2 - r(x)y = 0$, where $n \geq 2$ and where $r(x) \in k[x]$ is a polynomial such that $r(0) \neq 0$.

When an affine surface S admits a unique $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$, its study become simpler. For instance, every automorphism of S must preserve this fibration. In this context, a result due to J. Bertin [2] asserts that the identity component of the automorphism group of such a surface is an algebraic pro-group obtained as an increasing union of solvable algebraic subgroups

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of rank ≤ 1. For surfaces defined by the equations $x^n z - P(y) = 0$ in $\mathbb{C}^3$, the picture has been completed by L. Makar-Limanov [16] who gave explicit generators of their automorphisms groups. Again, similar results have been obtained by A. Crachiola [3] for surfaces defined by the equations $x^n z - y^2 - r(x)y = 0$ defined over an arbitrary base field.

The latter surfaces are particular examples of a general class of $\mathbb{A}^1$-fibered surfaces, called *Danielewski surfaces*, introduced and studied by the first author in [8]. We recall *loc. cit.* that a Danielewski surface over a field k is a normal integral affine surface S equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ over an affine line with origin o, such that every fiber $\pi^{-1}(x)$, where $x \in \mathbb{A}^1_k \setminus \{o\}$, is geometrically integral, and such that every irreducible component of $\pi^{-1}(o)$ is geometrically integral. For instance, Crachiola surfaces S defined by the equations $x^n z - y^2 - r(x)y = 0$, where $r(0) \neq 0$ are Danielewski surfaces $\pi = \mathrm{pr}_x : S \to \mathbb{A}^1_k$ whereas a surface $S_{P,n} \subset \mathbb{A}^3_k$ defined by the equation $x^n z - P(y) = 0$ is a Danielewski surface $\pi = \mathrm{pr}_x : S_{P,n} \to \mathbb{A}^1_k$ if and only if the polynomial P splits with $r \geq 1$ simple roots in k.

In this article, we obtain similar descriptions for a class of Danielewski surface which contains the previously mentioned ones, namely the surfaces $S_{Q,n} \subset \mathbb{A}^3_k$ defined by an equation of the form $x^n z - Q(x,y) = 0$, where $n \geq 2$ and where $Q(x,y) \in k[x,y]$ is a polynomial such that $Q(0,y)$ splits with $r \geq 2$ simple roots in k.

The paper is organized as follows. In section 1, we recall the main facts on Danielewski surfaces and their classification by means of weighted rooted trees in a form appropriate to our needs. We also generalize to an arbitrary base field k some results which are only stated for fields of characteristic zero in [7] and [8]. In particular, we establish the following result (Theorem 1.15) which generalizes Theorem 4.2 in [9].

Theorem. *For a Danielewski surface $\pi : S \to \mathbb{A}^1_k$, the following are equivalent :*

1) S admits two $\mathbb{A}^1$-fibrations with distinct general fibers.

2) There exists an integer $h \geq 1$ and a collection of monic polynomials $P_0, \ldots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \ldots, h-1$, $j = 1, \ldots, \deg_t(P_i)$, such that S is isomorphic to the surface $S_{P_0,\ldots,P_{h-1}} \subset \mathrm{Spec}(k[x][y_{-1}, \ldots, y_{h-2}][z])$ defined by the equations

$$\begin{cases} xz - y_{h-2}\prod_{l=0}^{h-1} P_l(y_{l-1}) = 0 & \\ zy_{i-1} - y_i y_{h-2}\prod_{l=i+1}^{h-1} P_l(y_{l-1}) = 0 \quad xy_i - y_{i-1}\prod_{l=0}^{i} P_l(y_{l-1}) = 0 & 0 \leq i \leq h-2 \\ y_{i-1}y_j - y_i y_{j-1}\prod_{l=i+1}^{j} P_l(y_{l-1}) = 0 & 0 \leq i < j \leq h-2. \end{cases}$$

In section 2, we characterize Danielewski surfaces among the surfaces defined by the equations of the form $x^n z - Q(x,y) = 0$ in $\mathbb{A}^3_k$ in terms of their associated weighted trees. This leads to introduce a special class of trees Γ, with the property that they contain a unique element $\bar{e}_0 \in \Gamma$ such that $\Gamma \setminus \{\bar{e}_0\}$ is a nonempty disjoint union of chains. We call them *rakes*. We obtain the following characterization (Theorem 2.1).

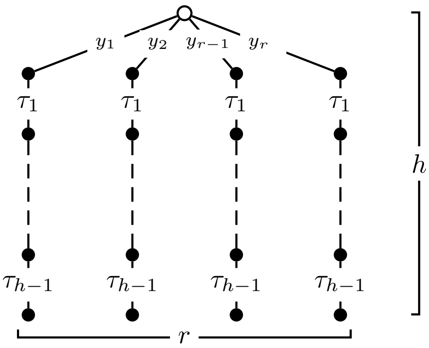
Theorem. *For a normal affine surface S, the following are equivalent :*

1) S is isomorphic to a surface $S_{Q,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$ for a suitable polynomial $Q(x,y)$ such that $Q(0,y)$ splits with $r \geq 2$ simple roots in k.

2) S is isomorphic to a surface $S_{\sigma,h} \subset \mathbb{A}^3_k$ defined by the equation $x^h z - \prod_{i=1}^{r}(y - \sigma_i(x)) = 0$ for a suitable collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ such that $\sigma_i(0) \neq \sigma_j(0)$ for every $i \neq j$.

3) S is isomorphic to a Danielewski surface defined by a fine k-weighted rake of height h with $r \geq 2$ leaves.

In particular, we combine this characterization with a result of M.H. Gizatullin [12] and J. Bertin [2] (see also [4], [14] or [7])) to deduce that if $h \geq 2$ then the natural $\mathbb{A}^1$-fibration $\mathrm{pr}_x : S_{Q,h} \to \mathbb{A}^1_k$ on a Danielewski surface defined by the equation $x^h z - Q(x,y) = 0$ is a unique $\mathbb{A}^1$-fibration on $S_{Q,h}$ up to automorphisms of the base. In Lemma 2.3, we give an algorithm to determine explicitly the rake corresponding to a given Danielewski surface $S_{Q,h}$. For instance, the surfaces $S_{P,h}$ defined by the equations $x^h z - P(y) = 0$ considered by L. Makar-Limanov correspond to fine k-weighted rakes of the following type



where $y_1, \ldots, y_r, \tau_1, \ldots, \tau_{h-1} \in k$.

Then we compute the automorphism groups of Danielewski surfaces $S_{Q,h}$. In particular, we arrive at the following description (Theorem 2.5) which contains the ones previously obtained by L. Makar-Limanov and A. Crachiola in their particular cases.

Theorem. *The automorphism group of a Danielewski surface $S_{\sigma,h}$ with equation*

$$x^h z - \prod_{i=1}^{r} (y - \sigma_i(x)) = 0$$

is generated by the restrictions of the following automorphisms of $\mathbb{A}^3_k$:

(a) $\Delta_b(x,y,z) = \left(x, y + x^h b(x), z + x^{-h}\left(Q\left(x, y + x^h b(x)\right) - Q(x,y)\right)\right)$, *where $b(x) \in k[x]$.*

(b) *If there exists a polynomial $\tau(x)$ such that $Q(x, y + \tau(x)) = \tilde{Q}(y)$ then the automorphisms $H_a(x,y,z) = \left(ax, y + \tau(ax) - \tau(x), a^{-h} z\right)$, where $a \in k^*$, should be added.*

(c) *If there exists a polynomial $\tau(x)$ such that $Q(x, y + \tau(x)) = \tilde{Q}(x^{q_0}, y)$, then the automorphisms $\tilde{H}_a(x,y,z) = \left(ax, y + \tau(ax) - \tau(x), a^{-h} z\right)$, where $a \in k^*$ and $a^{q_0} = 1$, should be added.*

(d) *If there exists a polynomial $\tau(x)$ such that $Q(x, y + \tau(x)) = y^i \tilde{Q}(x, y^s)$, where $i = 0, 1$ and $s \geq 2$, then the automorphisms $S_\mu(x,y,z) = \left(x, \mu y + (1-\mu)\tau(x), \mu^i z\right)$, where $\mu \in k^*$ and $\mu^s = 1$, should be added.*

(e) *If $char(k) = s > 0$ and $Q(x,y) = \tilde{Q}\left(y^s - c(x)^{s-1} y\right)$ for a certain polynomial $c(x) \in k[x]$ such that $c(0) \neq 0$, then the automorphism $T(x,y,z) = (x, y + c(x), z)$ should be added.*

(f) If $h = 1$ then the involution $I(x,y,z) = (z,y,x)$ should be added.

In section 3, we study embeddings of Danielewski surfaces in $\mathbb{A}^3_k$ as surfaces $S_{Q,h}$ up to algebraic and holomorphic equivalence. This question is motivated by an example due to G. Freudenburg and L. Moser-Jauslin [11] of two closed embeddings i and i' in $\mathbb{A}^3_{\mathbb{C}}$ of the Danielewski surface defined by the equation $x^2 z - y^2 - 1 = 0$ for which there does not exist an algebraic automorphism Φ of $\mathbb{A}^3_{\mathbb{C}}$ such that $i' = \Phi \circ i$. We combine our classification results with general methods introduced by the second author and L. Moser-Jauslin in [17] to construct new examples of this phenomenon. For instance, we establish that for every $h \geq 2$ and every polynomial $P(y)$ with $r = \deg(P) \geq 2$ simple roots in k, the surfaces defined by the equations $x^h z - P(y) = 0$ and $x^h z - (1-x) P(y) = 0$ are algebraically inequivalent embeddings in $\mathbb{A}^3_k$ of a same Danielewski surface S. However, for complex surfaces, we prove the following result (Theorem 3.9), which generalizes Theorem 2.2 in [17].

Theorem. *The closed embeddings $i : S \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$ of a Danielewski surface S as a surface $S_{Q,h}$ defined by the equation $x^h z - Q(x,y) = 0$ are all* holomorphically *equivalent.*

Our description of the automorphism groups of Danielewski surfaces defined by the equations

$x^h z - \prod_{i=1}^{r}(y-\sigma_i(x)) = 0$ implies that every automorphism of such a surface arises as the restriction of an automorphism of the ambient space. More generally, we establish in Proposition 2.11, that *every* additive group action on a Danielewski surface $S_{Q,h}$ arise as the restriction of such an action on the ambient space $\mathbb{A}_k^3$. In contrast, for multiplicative group actions, we have the following result. (Theorem 3.12 and Corollary 3.16).

Theorem. *Every Danielewski surface $S \subset \mathbb{A}_{\mathbb{C}}^3$ with equation $x^h z - (1-x)P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{C}^*$-action which is algebraically non-extendable but holomorphically extendable to $\mathbb{A}_{\mathbb{C}}^3$.*

In particular, the endomorphism $J(x,y,z) = (-x, y, (1+x)((1+x)z + P(y)))$ of $\mathbb{A}_{\mathbb{C}}^3$ defines an involution of the surface S defined by the equation $x^2 z - (1-x)P(y) = 0$ which does not extend to an algebraic automorphism of $\mathbb{A}_k^3$.

1. Danielewski surfaces

For certain authors (see e.g. [11]), a Danielewski surface is an affine surface S which is algebraically isomorphic to a surface in $\mathbb{C}^3$ defined by an equation of the form $x^n z - P(y) = 0$, where $n \geq 1$ and $P(y) \in \mathbb{C}[y]$. These surfaces have been studied in many different contexts over the past 15 years. Of particular interest is the fact that they come equipped with a surjective morphism $\pi = pr_x|_S : S \to \mathbb{A}^1$ restricting to a trivial $\mathbb{A}^1$-bundle over the complement of the origin. Moreover, if the roots $y_1, \ldots, y_r \in \mathbb{C}$ of $P(y)$ are simple, then $\pi = pr_x|_S : S \to \mathbb{A}^1$ factors through a locally trivial fiber bundle over the affine line with an r -fold origin (see *e.g.* [5] and [10]). In [8], the first author used the term Danielewski surface to refer to an affine surface S equipped with a morphism $\pi : S \to \mathbb{A}^1$ which factors through a locally trivial fiber bundle in a similar way as above. In what follows, we keep this point of view, which leads to a natural geometric generalization of the surfaces constructed by W. Danielewski [5]. We recall that an $\mathbb{A}^1$*-fibration* over an integral scheme is a faithfully flat (*i.e* flat and surjective) affine morphism $\pi : X \to Y$ with generic fiber isomorphic to the affine line $\mathbb{A}_{K(Y)}^1$ over the function field $K(Y)$ of Y. The following definition is a generalization to arbitrary base fields k of the one introduced in [8].

Definition 1.1. A *Danielewski surface* is an integral affine surface S defined over a field k, equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}_k^1$ restricting to a trivial $\mathbb{A}^1$-bundle over the complement of the origin o of $\mathbb{A}_k^1$ and such that the fiber $\pi^{-1}(o)$ is reduced, consisting of a disjoint union of affine lines $\mathbb{A}_k^1$ over k.

1.2. For instance, a surface $S \subset \mathbb{A}_k^3$ defined by the equation $x^n z - Q(x,y) = 0$ is a Danielewski surface $\pi = pr_x : S \to \mathbb{A}_k^1$ if and only if the polynomial $P(y) = Q(0,y)$ splits with simples roots in k. In the following subsections, we recall the main facts about the correspondence between Danielewski surfaces and weighted rooted trees established by the first author in [8] in a form appropriate to our needs. Although the results given in *loc. cit.* are formulated for surfaces defined over a field of characteristic zero, most of them remain valid without any changes over a field of arbitrary characteristic. We provide full proofs only when additional arguments are needed. Then we consider the class of Danielewski surfaces S with trivial canonical sheaf $\omega_{S/k} = \Lambda^2 \Omega_{S/k}^1$, which we call *special Danielewski surfaces.* We give a complete classification of these surfaces in terms of their associated weighted trees.

1.1. Danielewski surfaces, weighted trees and $\mathbb{A}^1$-fibrations.

The first author constructed in [8] a correspondence between Danielewski surfaces and certain weighted trees. Hereafter we review the main steps of this construction.

Basic facts on weighted rooted trees.

Definition 1.3. A *tree* is a nonempty finite partially ordered set $\Gamma = (\Gamma, \leq)$ with a unique minimal element e_0 called the *root*, and such that for every $e \in \Gamma$ the subset $(\downarrow e)_\Gamma = \{e' \in \Gamma,\ e' \leq e\}$ is a chain for the induced ordering.

1.4. A minimal sub-chain $\overleftarrow{e'e} = \{e' < e\}$ with two elements of a tree Γ is called *an edge* of Γ. We denote the set of all edges in Γ by $E(\Gamma)$. An element $e \in \Gamma$ such that $\mathrm{Card}(\downarrow e)_\Gamma = m$ is said to be *at level* m. The maximal elements $e_i = e_{i,m_i}$, where $m_i = \mathrm{Card}(\downarrow e_i)_\Gamma$ of Γ are called the *leaves* of Γ. We denote the set of those elements by $L(\Gamma)$. The maximal chains of Γ are the chains

$$\Gamma_{e_{i,m_i}} = (\downarrow e_{i,m_i})_\Gamma = \{e_{i,0} = e_0 < e_{i,1} < \cdots < e_{i,m_i}\}, \quad e_{i,m_i} \in L(\Gamma). \tag{1.1}$$

We say that Γ has *height* $h = \max(m_i)$. The *children* of an element $e \in \Gamma$ are the elements of Γ at relative level 1 with respect to e, *i.e.* the maximal elements of the subset $\{e' \in \Gamma,\ e' > e\}$ of Γ.

Definition 1.5. A *fine k-weighted tree* $\gamma = (\Gamma, w)$ is a tree Γ equipped with a weight function $w : E(\Gamma) \to k$ with values in a field k, which assigns an element $w\left(\overleftarrow{e'e}\right)$ of k to every edge $\overleftarrow{e'e}$ of Γ, in such a way that $w\left(\overleftarrow{e'e_1}\right) \neq w\left(\overleftarrow{e'e_2}\right)$ whenever e_1 and e_2 are distinct children of a same element e'.

In what follows, we frequently consider the following classes of trees.

Definition 1.6. A tree Γ rooted in e_0 is called a *rake* if there exists a unique element $\bar{e}_0 \in \Gamma$ such that $\Gamma \setminus \{\bar{e}_0\}$ is a nonempty disjoint union of chains.

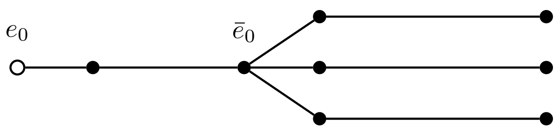


FIGURE 1.1. A rake Γ rooted in e_0.

Definition 1.7. A *comb* is a rooted tree Γ such that $\Gamma \setminus L(\Gamma)$ is a chain. Equivalently, Γ is a comb if every $e \in \Gamma \setminus L(\Gamma)$ has at most one child which is not a leaf of Γ.

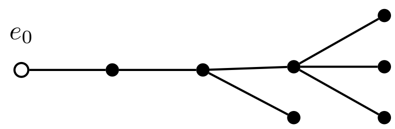


FIGURE 1.2. A comb Γ rooted in e_0.

Danielewski surfaces and weighted trees.
Here we recall the correspondence between Danielewski surfaces and fine k-weighted trees established by the first author in [8].

1.8. To every tree $\gamma = (\Gamma, w)$, one can associate a Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ which arises as the total space of an $\mathbb{A}^1$-bundle over the scheme $\delta : X(r) \to \mathbb{A}^1_k$ obtained from $\mathbb{A}^1_k$ by replacing its origin o by $r \geq 1$ origins $o_1, \ldots, o_r$. In what follows we denote by $\mathcal{U}_r = (X_i(r))_{i=1,\ldots,r}$ the canonical open covering of $X(r)$ by means of the subsets $X_i(r) = \delta^{-1}\left(\mathbb{A}^1_k \setminus \{o\}\right) \cup \{o_i\} \simeq \mathbb{A}^1_k$. Given a fine k-weighted tree $\gamma = (\Gamma, w)$ of height h, with leaves e_i at levels $n_i \leq h$, $i = 1, \ldots, r$, we associate to every maximal sub-chain $\gamma_i = (\downarrow e_i)$ of γ (see 1.4 for the notation) a polynomial

$$\sigma_i(x) = \sum_{j=0}^{n_i - 1} w\left(\overleftarrow{e_{i,j}e_{i,j+1}}\right) x^j \in k[x], \quad i = 1, \ldots, r.$$

We let $\rho : S(\gamma) \to X(r)$ be the unique $\mathbb{A}^1$-bundle over $X(r)$ which is trivial on the canonical open covering $\mathcal{U}_r$, and is defined by pairs of transition functions

$$(f_{ij}, g_{ij}) = \left(x^{n_j - n_i}, x^{-n_i}\left(\sigma_j(x) - \sigma_i(x)\right)\right) \in k\left[x, x^{-1}\right]^2, \quad i, j = 1, \ldots, r.$$

This means that $S(\gamma)$ is obtained by gluing n copies $S_i = \mathrm{Spec}(k[x][u_i])$ of the affine plane $\mathbb{A}^2_k$ over $\mathbb{A}^1_k \setminus \{0\} \simeq \mathrm{Spec}\left(k\left[x, x^{-1}\right]\right)$ by means of the transition isomorphisms induced by the $k\left[x, x^{-1}\right]$-algebras isomorphisms

$$k\left[x, x^{-1}\right][u_i] \xrightarrow{\sim} k\left[x, x^{-1}\right][u_j], \quad u_i \mapsto x^{n_j - n_i} u_j + x^{-n_i}(\sigma_j(x) - \sigma_i(x)) \qquad i \neq, i, j = 1, \ldots, r.$$

Note that this definition makes sense as the transition functions g_{ij} satisfy the twisted cocycle $g_{ik} = g_{ij} + x^{n_j - n_i} g_{jk}$ in $k\left[x, x^{-1}\right]$ for every triple $i \neq j \neq k$. The fact that $g_{ij}(x) \in k\left[x, x^{-1}\right] \setminus k[x]$ for every $i \neq j$, $i, j = 1, \ldots, r$, guarantees that the scheme $S(\gamma)$ is separated, whence an affine surface by virtue of Fieseler's criterion (see proposition 1.4 in [10]). Therefore, $\pi_\gamma = \delta \circ \rho : S(\gamma) \to \mathbb{A}^1_k = \mathrm{Spec}(k[x])$ is a Danielewski surface, the fiber $\pi^{-1}(o)$ being the disjoint union of affine lines

$$C_i = \pi_\gamma^{-1}(o) \cap S_i \simeq \mathrm{Spec}(k[u_i]), \quad i = 1, \ldots, r.$$

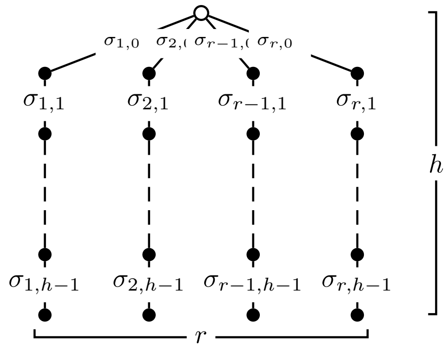
1.9. The Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ above comes canonically equipped with a birational morphism $(\pi, \psi_\gamma) : S \to \mathbb{A}^1_k \times \mathbb{A}^1_k = \mathrm{Spec}(k[x][t])$ restricting to an isomorphism over $\mathbb{A}^1_k \setminus \{o\}$, corresponding to the unique regular function ψ_γ on $S(\gamma)$ whose restrictions to the open subsets $S\mid_{X_i(r)} \simeq \mathrm{Spec}(k[x][u_i])$, are given by the polynomials $\psi_{\gamma,i} = x^{n_i} u_i + \sigma_i(x)$, $i = 1, \ldots, r$. This function is referred to as the *canonical function* on $S(\gamma)$. The morphism $(\pi_\gamma, \psi_\gamma) : S(\gamma) \to \mathbb{A}^2_k$ is called the *canonical birational morphism* from $S(\gamma)$ to $\mathbb{A}^2_k$.

1.10. It was established by the first author in [8] that there exists a one-to-one correspondence between pairs $(S, (\pi, \psi))$ consisting of a Danielewski surface $\pi : S \to \mathbb{A}^1_k$ and a birational morphisms $(\pi, \psi) : S \to \mathbb{A}^2_k$ restricting to an isomorphism outside the fiber $\pi^{-1}(o)$ and fine k-weighted trees γ. More precisely, Proposition 3.4 in [8], which remains valid over arbitrary base fields k, implies the following result.

Theorem 1.11. *For every pair consisting of a Danielewski surface $\pi : S \to \mathbb{A}^1_k$ and a birational morphism $(\pi, \psi) : S \to \mathbb{A}^1_k \times \mathbb{A}^1_k$ restricting to an isomorphism over $\mathbb{A}^1_k \setminus \{o\}$, there exists a unique fine k-weighted tree γ and an isomorphism $\phi : S \xrightarrow{\sim} S(\gamma)$ such that $\psi = \psi_\gamma \circ \phi$.*

Remark 1.12. If $\gamma = (\Gamma, w)$ is not the trivial tree with one element then the canonical function $\psi_\gamma : S(\gamma) \to \mathbb{A}^1_k$ on the corresponding Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ is locally constant on the fiber $\pi^{-1}(o)$. It takes the same value on two distinct irreducible components of $\pi^{-1}(o)$ if and only if the corresponding leaves of γ belong to a same subtree of γ rooted in an element at level 1 of γ. Different birational morphisms $(\pi, \psi) ; S \to \mathbb{A}^1_k \times \mathbb{A}^1_k$ on a given Danielewski surfaces $\pi : S \to \mathbb{A}^1_k$ may lead to different weighted trees. For instance, given a fine k-weighted tree $\gamma = (\Gamma, w)$, an integer $n \geq 1$ and an arbitrary polynomial $b(x) = b_0 + b_1 x + \ldots b_{n-1} x^{n-1}$, we can equip the Danielewski surface $S(\gamma)$ with the birational morphism $(\pi_\gamma, \psi_\gamma(n, b) = x^n \psi_\gamma + b(x)) : S(\gamma) \to \mathbb{A}^2_k$. The weighted tree $\gamma(n, b)$ corresponding to this datum is obtained from γ by replacing its root e_0 by a chain $\{\tilde{e}_0 < \cdots < \tilde{e}_{n-1} < \tilde{e}_n = e_0\}$ weighted by $\tilde{w}\left(\overleftarrow{\tilde{e}_i \tilde{e}_{i+1}}\right) = b_i$ for every $i = 0, \ldots, n-1$. However, every Danielewski surface nonisomorphic to $\mathbb{A}^2_k$ admits a birational morphism (π, ψ) for which ψ is locally constant but not constant on the fiber $\pi^{-1}(o)$. By construction, the tree γ corresponding to such a morphism ψ will have at least two elements at level 1.

Example 1.13. Let $h \geq 1$ an be integer and let $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ be a collections of $r \geq 2$ polynomials $\sigma_i(x) \in k[x]$ of degree $\deg(\sigma_i(x)) < h$ such that $\sigma_i(0) \neq \sigma_j(0)$ for every $i \neq j$. Then the surface $S = S_{\sigma,h} \subset \mathbb{A}^3_k$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$ is a Danielewski surface for the projection $\pi = \mathrm{pr}_x \mid_S : S \to \mathbb{A}^1_k$. The degenerate fiber $\pi^{-1}(0)$ consists of r copies C_i of the affine line defined by the equations $\{x = 0, y = \sigma_i(0)\}_{i=1,\ldots,r}$. For every index $i = 1, \ldots, r$, the open subset $S_i = S \setminus \bigcup_{j \neq i} C_i$ of S is isomorphic to $\mathbb{A}^2_k = \mathrm{Spec}(k[x, u_i])$, where u_i denotes the regular function on S_i induced by the rational function $u_i = x^{-h}(y - \sigma_i(x)) = z \prod_{j \neq i} (y - \sigma_j(x))^{-1}$ on S. So we conclude that $\pi : S \to \mathbb{A}^1_k$ factors through an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ isomorphic to the one with transition pairs $(f_{ij}, g_{ij}) = \left(1, x^{-h}(\sigma_j(x) - \sigma_i(x))\right)$, $i, j = 1, \ldots, r$. Letting $\sigma_i = \sum_{i=0}^{h-1} \sigma_{i,n} x^n$, these pairs are exactly the ones associated with the following fine k-weighted rake γ:



The associated canonical function ψ_γ is the unique regular function on $S(\gamma)$ with restrictions $\psi_\gamma|_{S_i} = x^h u_i + \sigma_i(x)$ for every $i = 1, \ldots, r$. So it coincides with the regular function y on S under the isomorphism above. In the setting of Theorem 1.11, this means that the pair $(S(\gamma), \psi_\gamma)$ is isomorphic to the pair $\left(S, \mathrm{pr}_{x,y}|_S\right)$ where $\mathrm{pr}_{x,y} : S \to \mathbb{A}^1_k \times \mathbb{A}^1_k$ denotes the birational morphism obtained as the restriction to S of the projection $\mathrm{pr}_{x,y} : \mathbb{A}^3_k \to \mathbb{A}^2_k$, $(x, y, z) \mapsto (x, y)$.

$\mathbb{A}^1$-fibrations on Danielewski surfaces.
To determine isomorphism classes of Danielewski surfaces, the first step is to decide if the structural $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ on a given surface is unique up to automorphisms of the base. Indeed, in the case that $\pi : S \to \mathbb{A}^1_k$ is unique, then a second Danielewski surface $\pi' : S' \to \mathbb{A}^1_k$ will be isomorphic to S as an abstract surface if and only if it is isomorphic to S as a fibered surface.

1.14. Using a geometric characterization of quasi-homogeneous affine surfaces due to M.H. Gizatullin [12], T. Bandman and L. Makar-Limanov [1] established that a complex Danielewski surface S with a trivial canonical sheaf ω_S admits two $\mathbb{A}^1$-fibrations with distinct general fibers if and only if it is isomorphic to a surface $S_{P,1} \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $xz - P(y) = 0$ for a certain nonconstant polynomial P with simple roots. This result was further generalized by the first author in [8] and [9] to a characterization of Danielewski surfaces admitting two $\mathbb{A}^1$-fibrations with distinct general fibers in terms of their associated trees. In turns out that the main result of [9], which is stated for surfaces defined over a field of characteristic zero, remains valid over arbitrary base fields.

Theorem 1.15. *For a Danielewski surface $\pi : S \to \mathbb{A}^1_k$, the following are equivalent :*
1) S admits two $\mathbb{A}^1$-fibrations with distinct general fibers.
2) S isomorphic to a Danielewski surface $S(\gamma)$ defined by a fine k-weighted comb $\gamma = (\Gamma, w)$.
3) There exists an integer $h \geq 1$ and a collection of monic polynomials $P_0, \ldots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \ldots, h-1$, $j = 1, \ldots, \deg_t(P_i)$, such that S is isomorphic to the surface $S_{P_0,\ldots,P_{h-1}} \subset$ Spec $(k[x][y_{-1}, \ldots, y_{h-2}][z])$ defined by the equations

$$\left\{\begin{array}{lll} xz - y_{h-2}\prod_{l=0}^{h-1} P_l(y_{l-1}) = 0 & & \\ zy_{i-1} - y_i y_{h-2}\prod_{l=i+1}^{h-1} P_l(y_{l-1}) = 0 & xy_i - y_{i-1}\prod_{l=0}^{i} P_l(y_{l-1}) = 0 & 0 \leq i \leq h-2 \\ y_{i-1}y_j - y_i y_{j-1}\prod_{l=i+1}^{j} P_l(y_{l-1}) = 0 & & 0 \leq i < j \leq h-2 \end{array}\right.$$

Proof. One checks in a similar way as in the proof Theorem 2.9 in [9] that every surface $S = S_{P_0,\ldots,P_{h-1}}$ as above is irreducible and becomes a Danielewski surface for the projection $\pi = \mathrm{pr}_x|_S : S \to \mathbb{A}^1_k$. Furthermore, the projection $\pi' = \mathrm{pr}_z|_S : S \to \mathbb{A}^1_k$ induces a second $\mathbb{A}^1$-fibration on S restricting to a trivial $\mathbb{A}^1$-bundle $(\pi')^{-1}\left(\mathbb{A}^1_k \setminus \{0\}\right) \simeq \mathrm{Spec}\left(k\left[z, z^{-1}\right][y_{h-2}]\right)$ over $\mathbb{A}^1_k \setminus \{0\}$. So 3) implies 1). To show that 1) implies 2), we use the characterization of quasi-homogeneous surfaces obtained by M.H. Gizatullin [12]. We recall that an integral affine surface S is said to be quasi-homogeneous if the automorphism group $\mathrm{Aut}(S)$ acts on S with a dense orbit with finite complement. It was established by M.H. Gizatullin that a nonsingular integral affine surface S defined over an algebraically closed field k is quasi-homogeneous if and only if it can be completed into a nonsingular projective surface $\bar{S}$ in such a way that the boundary divisor $\bar{S} \setminus S$ is a *zigzag*, that is, a chain of nonsingular complete rational curves. In [7], the first author reworked Gizatullin's arguments to prove that a normal complex affine surface S admits a completion by a zigzag if and only if it admits two $\mathbb{A}^1$-fibrations over the affine line with distinct general fibers. The proof given in *loc. cit.* only depends of certain properties of $\mathbb{A}^1$-fibrations which remain valid over an algebraically closed field of arbitrary characteristic. In [8], the first author constructed canonical completions $\bar{S}$ of a Danielewski surface $S(\gamma)$ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$ for which the dual graph Γ' of the boundary divisor $\bar{S} \setminus S(\gamma)$ is isomorphic to the tree obtained from Γ be deleting its leaves and replacing its root by a chain with two elements. Clearly, Γ' is a chain if and only if Γ is a comb. This construction, which only depends on the existence of an $\mathbb{A}^1$-bundle structure $\rho : S(\gamma) \to X(r)$ on a Danielewski surface $S(\gamma)$, remains valid over an arbitrary base field k. Now let S be a Danielewski surface admitting two distinct $\mathbb{A}^1$-fibrations. We may suppose that S isomorphic to a one $S(\gamma)$ associated with a certain fine k-weighted tree γ. Letting $\bar{k}$ be an algebraic closure of k, the surface $S_{\bar{k}} = S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ is again a Danielewski surface, isomorphic to the one defined by the tree γ consider as a fine $\bar{k}$-weighted tree via the inclusion $k \subset \bar{k}$. Since every $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ lifts to an $\mathbb{A}^1$-fibration $\pi_{\bar{k}} : S_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ it follows that $S_{\bar{k}}$ admits two $\mathbb{A}^1$-fibrations with distinct general fibers. We deduce from the above discussion that γ is a comb, and so that 1) implies 2). Note that if k is already algebraically closed, then above argument shows that 1) is actually equivalent to 2). However, if k is arbitrary, then, in general, there is no no guarantee that a given $\mathbb{A}^1$-fibration on $S_{\bar{k}}$ comes as the lifting of such a fibration on S. Therefore, we can not conclude directly from the above criterion. So we are led to prove by hand that if $S_{\bar{k}}$ admits two distinct $\mathbb{A}^1$-fibrations, then this also holds for S. One way to proceed is by constructing a closed embedding of S in an affine space as a surface $S_{P_0,\ldots,P_{h-1}}$ for which two such fibrations appear explicitly. This is done in Lemma 1.17 below. □

1.16. Over a base field of characteristic zero, the first author gave in [9] a canonical procedure to construct a closed embeddings of a Danielewski surface S defined by a fine k-weighted comb γ of height h as a surface $S_{P_0,\ldots,P_{h-1}} \subset \mathbb{A}^{h+2}_k$. However, the proof given in *loc. cit.* seems to depend on Taylor Formula for polynomials, which may lead one to suspect that the construction fails in positive characteristic. Rather than letting the reader convince himself that the use of the Taylor Formula in the construction of 4.6 in [9] is just a trick to avoid the introduction of cumbersome notations, we shall provide a complete proof of the following result.

Lemma 1.17. *For every Danielewski surface $S(\gamma)$ defined by a fine k-weighted comb $\gamma = (\Gamma, w)$ of height $h \geq 1$, there exists a collection of monic polynomials $P_0, \ldots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \ldots, h-1$, $j = 1, \ldots, \deg_t(P_i)$, and a closed embedding $S \hookrightarrow \mathbb{A}^{h+2}_k$ of S in $\mathbb{A}^{h+2}_k = \mathrm{Spec}(k[x][y_{-1}, \ldots, y_{h-2}][z])$ as the surface $S_{P_0,\ldots,P_{h-1}}$ defined in Theorem 1.15.*

Proof. In view of Remark 1.12 above, we may suppose that the comb $\gamma = (\Gamma, w)$ has at least two elements at level 1. We denote by $e_{0,0} < e_{1,0} < \cdots < e_{h-1,0}$ the elements of the sub-chain $C = \Gamma \setminus L(\Gamma)$ of Γ consisting of elements of Γ which are not leaves of Γ. For every $l = 1, \ldots, h$, the elements of Γ at level l distinct from $e_{l,0}$ will be denoted by $e_{l,1}, \ldots, e_{l,r_l}$ provided that they exist. Replacing the weight function w by the one w_0 defined by

$$w_0(\overleftarrow{e_i e_{i+1}}) = \begin{cases} w(\overleftarrow{e_i e_{i+1}}) - w(\overleftarrow{e_{i,0} e_{i+1,0}}) & \text{if } i = 0, \ldots, h-2 \\ w(\overleftarrow{e_i e_{i+1}}) & \text{if } i = h-1 \end{cases}$$

yields the same pairs of transition functions $(f_{ij}, g_{ij}) \in k[x, x^*]^2$ (see 1.8 above). So may assume from the very beginning that $w = w_0$ and that $w_0(\overleftarrow{e_{h-1,0} e_{h-1,1}}) = 0$. We consider the associated Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ as an $\mathbb{A}^1$-bundle $\rho : S(\gamma) \to X(r)$ and we denote by $S_i = \mathrm{Spec}(k[x][u_i])$ the trivializing open subsets of $S(\gamma)$. For every $l = 0, \ldots, h-1$ and every $i = 1, \ldots, s_l$, we let $\tau_{l,i}(x, u_i) = x u_i + w(\overleftarrow{e_{l-1,0} e_{l,i}}) \in k[x][u_i]$. With this notation, the canonical function ψ_γ on $S(\gamma)$ restricts on an open subset S_i corresponding to a leaf $e_{l,i}$ of γ at level l to the polynomial $x^{l-1}\tau_{l,i}(x,u_i) \in k[x][u_i]$. Starting with $y_{-1} = \psi_\gamma$, we will construct regular functions $y_{-1},\ldots,y_{h-2}$ and $y_{h-1} = z$ on the associated Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ and polynomials $P_0,\ldots,P_{h-1}$ satisfying the following properties:

a) For every $l = 0,\ldots,h-1$, and every $l \leq m \leq h$, y_{l-1} restricts on an open subset S_i corresponding to a leaf $e_{m,i}$ of γ at level m to a polynomial $y_{l-1,i} \in k[x][u_i]$ such that

$$y_{l-1,i} = \begin{cases} L_{l,i}(u_i) & \mod x & \text{if } m = l \\ a_{l,i} + xL_{l+1,i}(u_i) & \mod x^2 & \text{if } m = l+1 \\ \xi_m x^{m-l-1}\tau_{m,i}(x,u_i) + \nu_{m,i}x^{m-l} & \mod x^{m-l+1} & \text{if } m > l+1, \end{cases}$$

where $\xi_m \in k^*$ for every $m \geq l$, $\lambda_i, a_{l,i}, \in k^*$ and $\nu_i \in k$ for every i, and where $L_{l,i}(u_i), L_{l+1,i}(u_i) \in k[u_i]$ are a polynomial of degree 1 for every i. Furthermore $a_{l,i} \neq a_{l,j}$ for every $i \neq j$.

b) For every $l = 0,\ldots,h-1$, P_l is the unique monic polynomial with simple roots $a_{l,1},\ldots,a_{l,r_l}$ such that $x^{-1}y_{l-1}\prod_{i=0}^{l-1} P_i(y_{i-1}) P_l(y_{l-1})$ is a regular function on $S(\gamma)$.

To construct these functions, we proceed by induction as follows:

Step 0. By construction, $y_{-1} = \psi_\gamma$ is constant with the value $a_{0,i} = w(\overleftarrow{e_{0,0}e_{1,i}}) \in k^*$ on the irreducible component $\pi^{-1}(o)$ corresponding to a leaf $e_{1,i}$, $i = 1,\ldots,r_1$, at level 1 and vanishes identically on every irreducible component of $\pi_\gamma^{-1}(o)$ corresponding to a leaf of γ at level $l \geq 2$. Letting $P_0(t) = \prod_{i=1}^{r_1}(t - a_{0,i})$ it follows that $y_0 = x^{-1}y_{-1}P_0(y_{-1}) \in \mathcal{O}(S(\gamma))_x$ is a regular function on $S(\gamma)$ whose restriction to an open subset S_i corresponding to a leaf $e_{l,i}$ of γ at level l is given by a polynomial of the following form :

$$y_{0,i} = \begin{cases} a_{0,i}P_{0,i}(a_{0,i})u_i & \mod x & \text{if } l = 1 \\ \tau_{2,i}(0,u_i)P_0(0) + x\left(P_0(0)u_i + P'(0)(\tau_{2,i}(0,u_i))^2\right) & \mod x^2 & \text{if } l = 2 \\ P_0(0)x^{l-2}\tau_{l,i}(x,u_i) & \mod x^{m-l+1} & \text{if } l > 2, \end{cases}$$

where $P_{0,i}(t) = (t - a_{0,i})^{-1}P_0(t)$ for every $i = 1,\ldots,r_1$. So $P_0(t)$ and y_0 satisfy conditions a) and b) above. In particular, y_0 is constant with the value $a_{1,i} = \tau_{2,i}(0,u_i)P_0(0) \in k^*$ on the irreducible component $\pi^{-1}(o)$ corresponding to a leaf $e_{2,i}$, $i = 1,\ldots,r_2$, of γ at level 2. Furthermore $a_{1,i} \neq a_{1,j}$ for every $i \neq j$ as γ is a fine k-weighted comb.

Step l, $1 \leq l \leq h-1$. We assume that the regular functions $y_{-1}, y_0,\ldots,y_{l-1}$ and the polynomials $P_0,\ldots,P_{l-1}$ satisfying conditions a) and b) above have been constructed. In particular, if $l < h-1$ then y_{l-1} is constant with the value $a_{l,i} = k^*$ on the irreducible component $\pi^{-1}(o)$ corresponding to a leaf $e_{l+1,i}$, $i = 1,\ldots,r_l$, of γ at level $l+1$. Otherwise, if $l = h-1$ then it follows from our hypothesis that y_{h-2} is constant with the value $a_{h-2,i} = k^*$ on the irreducible component $\pi^{-1}(o)$ corresponding to a leaf $e_{h,i}$, $i = 2,\ldots,r_h$ and vanishes identically on the component corresponding to the leaf $e_{h,1}$. Letting

$$P_l(t) = \begin{cases} \prod_{i=1}^{r_l}(t - a_{l,i}) & \text{if } l < h-1 \\ \prod_{i=2}^{r_h}(t - a_{l,i}) & \text{if } l = h-1, \end{cases}$$

it follows that $\tilde{y}_l = x^{-1}y_{l-1}P_l(y_{l-1}) \in \mathcal{O}(S(\gamma))_x$ extend to a regular function on the open complement in $S(\gamma)$ of the irreducible components of $\pi^{-1}(o)$ corresponding to leaves of γ at levels lower or equal to l. Furthermore, the restriction of $\tilde{y}_l$ to an open subset S_i corresponding to a leaf $e_{m,i}$ of γ at level $m \geq l+1$ is given by a polynomial $\tilde{y}_{l,i}$ such that

$$\tilde{y}_{l,i} = \begin{cases} a_{l,i}P_{l,i}(a_{l,i})L_{l+1,i}(u_i) & \mod x & \text{if } m = l+1 \\ \xi_m\tau_{m,i}(0,u_i)P_l(0) + x(P_l(0)\xi_m u_i + \tilde{\nu}_{l+2,i}) & \mod x^2 & \text{if } m = l+2 \\ P_l(0)\left(\xi_m x^{m-l-2}\tau_{m,i}(x,u_i) + \nu_i x^{m-l-1}\right) & \mod x^{m-l+1} & \text{if } m > l+2, \end{cases}$$

where $P_{l,i}(t) = (t - a_{l,i})^{-1}P_l(t)$. One checks easily that $y_l = P_0(y_{-1})P_1(y_0)\cdots P_{l-1}(y_{l-2})\tilde{y}_l$ extends to a regular function on $S(\gamma)$ whose restrictions to the open subset S_i corresponding to the leaves $e_{m,i}$ of γ at level $m \geq l+1$ satisfy condition a) above. By construction, condition b) is also satisfied.

After exactly $h-1$ steps, we obtain a collection of regular functions $y_{-1},\ldots,y_{h-2},y_{h-1}=z$ which distinguish the irreducible components of the fiber $\pi^{-1}(o)$ and induce coordinate functions on them. Furthermore y_{-1} induces an isomorphism $\pi^{-1}\left(\mathbb{A}^1\setminus\{o\}\right)\xrightarrow{\sim}\mathrm{Spec}\left(k\left[x,x^{-1}\right][y_{-1}]\right)$. So we conclude that the morphism $i=(\pi,y_{-1},\ldots,y_{h-1},z):S(\gamma)\hookrightarrow\mathbb{A}_k^{h+2}$ is an embedding. The same argument as in the proof of Lemma 3.6 in [9] shows that i is actually a closed embedding whose image is contained in the surface $S_{P_0,\ldots,P_{h-1}}\subset\mathbb{A}_k^{h+2}$ defined in Theorem 1.15 above. By construction, the induced morphism $\phi:S(\gamma)\rightarrow S_{P_0,\ldots,P_{h-1}}$ defines a bijection between the sets of closed points of $S(\gamma)$ and $S_{P_0,\ldots,P_{h-1}}$, whence is quasi-finite. Furthermore, ϕ is also birational, restricting to an isomorphism between the complements of the fibers $\pi^{-1}(o)$ and $\mathrm{pr}_x^{-1}(0)$ of $\pi:S(\gamma)\rightarrow\mathbb{A}_k^1$ and $\mathrm{pr}_x:S_{P_0,\ldots,P_{h-1}}\rightarrow\mathbb{A}_k^1$ respectively. Since $S_{P_0,\ldots,P_{h-1}}$ is nonsingular, we finally conclude that ϕ an isomorphism by virtue of Zariski Main Theorem (see *e.g.* 4.4.9 in [13]). □

1.2. **Special Danielewski surfaces.**
A Danielewski surface $\pi:S\rightarrow\mathbb{A}_k^1$ with a trivial canonical sheaf $\omega_{S/k}$, or equivalently with a trivial sheaf of relative differential forms $\Omega^1_{S/\mathbb{A}_k^1}$, will be called *special*. For instance, it follows from the Adjunction Formula that every Danielewski surface $S\subset\mathbb{A}_k^3$ is special. It turns out that special Danielewski surfaces correspond to a distinguished class of weighted trees, as shown by the following result.

Proposition 1.18. *A Danielewski surface $\pi:S\rightarrow\mathbb{A}_k^1$ is special if and only if it is isomorphic to a Danielewski surface $\pi_\gamma:S(\gamma)\rightarrow\mathbb{A}_k^1$ defined by a fine k-weighted tree γ with all its leaves at the same level $h=h(\gamma)$.*

Proof. It follows from 1.8 above that every Danielewski surface is isomorphic to the total space of an $\mathbb{A}^1$-bundle $\rho:S\rightarrow X(r)$ obtained by gluing r copies of $\mathbb{A}_k^1\times\mathbb{A}_k^1$ over $\mathbb{A}_k^1\setminus\{o\}\times\mathbb{A}_k^1$ by means of transition isomorphisms

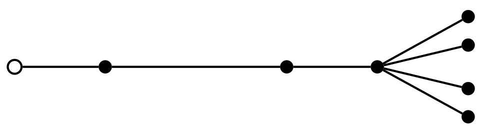
$$\tau_{ij}:k\left[x,x^{-1}\right][u_i]\rightarrow k\left[x,x^{-1}\right][u_j],\quad u_i\mapsto x^{n_j-n_i}u_j+x^{-n_i}\left(\sigma_j(x)-\sigma_i(x)\right),\quad i,j=1,\ldots,r.$$

Every nowhere vanishing 2-form $q\in\omega_{S/k}$ must restrict on a open subset S_i of S to a multiple $\lambda_i\omega_i$, where $\lambda_i\in k^*$, of the 2-form $\omega_i=dx\wedge du_i$. In view of the transition isomorphisms above, this possible if and only if $\lambda_i=\lambda_j=\lambda$ and $n_j=n_i=h(\gamma)$ for every $i,j=1,\ldots,r$. □

As a consequence of Theorem 1.15, we obtain the following result which is due to T. Bandman and L. Makar-Limanov [1] for complex surfaces.

Corollary 1.19. *A special Danielewski surface admits two distinct $\mathbb{A}^1$-fibrations over the affine line if and only if it is isomorphic to a surface $S_{P,1}\subset\mathbb{A}_k^3$ defined by the equation $xz-P(y)=0$ for a certain nonconstant polynomial P, with simple roots in k.*

Proof. Indeed, a tree with all its leaves at the same level h is a comb if and only if it is isomorphic to a chain or a tree of the following form :



So the result follows from 2.12 below. □

Isomorphism classes of special Danielewski surfaces.
To determine isomorphism classes of special Danielewski surfaces, we can exploit the fact established above that every such surface is isomorphic to the total space of an $\mathbb{A}^1$-bundle $\rho:S\rightarrow X(r)$ over $X(r)$ defined by means of transition isomorphisms

$$\tau_{ij}:k\left[x,x^{-1}\right][u_i]\rightarrow k\left[x,x^{-1}\right][u_j],\quad u_i\mapsto u_j+g_{ij}\left(x,x^{-1}\right),\quad i,j=1,\ldots,r,$$

where $g=\{g_{ij}\}_{i,j}$ is a Čech cocycle with values in $\mathcal{O}_{X(r)}$ for the canonical open covering $\mathcal{U}_r$. The group $\mathrm{Aut}(X(r))\simeq\mathrm{Aut}\left(\mathbb{A}_k^1\setminus\{0\}\right)\times\mathfrak{S}_r$ acts on the set $\mathbb{P}H^1\left(X(r),\mathcal{O}_{X(r)}\right)$ of isomorphism classes of $\mathbb{A}^1$-bundles as above in such a way that for every $\phi\in\mathrm{Aut}(X(r))$, the image $\phi\cdot[g]$ of a class $[g] \in \mathbb{P}H^1\left(X(r), \mathcal{O}_{X(r)}\right)$ represented by a bundle $\rho : S \to X(r)$ is the isomorphism class of the fiber product bundle $pr_2 : \phi^* S = S \times_{X(r)} X(r) \to X(r)$. The following criterion generalizes a result of J. Wilkens [18].

Theorem 1.20. *Two special Danielewski surfaces $\pi_1 : S_1 \to \mathbb{A}^1_k$ and $\pi_2 : S_2 \to \mathbb{A}^1_k$ with underlying $\mathbb{A}^1$-bundle structures $\rho_1 : S_1 \to X(r_1)$ and $\rho_2 : S_2 \to X(r_2)$ are isomorphic as abstract surfaces if and only if $r_1 = r_2 = r$ and their isomorphism classes in $\mathbb{P}H^1\left(X(r), \mathcal{O}_{X(r)}\right)$ belongs to the same orbit under the action of $Aut(X(r))$.*

Proof. Clearly, the condition guarantees that S_1 and S_2 are isomorphic. Suppose conversely that there exists an isomorphism $\Phi : S_1 \xrightarrow{\sim} S_2$. The divisor class group of a special Danielewski surface $\pi : S \to \mathbb{A}^1_k$ is generated by the classes of the connected components $C_1, \ldots, C_r$ of $\pi^{-1}(o)$ modulo the relation $C_1 + \cdots + C_r = \pi^{-1}(o) \sim 0$, whence is isomorphic to $\mathbb{Z}^{r-1}$. Therefore, $r_1 = r_2 = r$ for a certain $r \geq 1$. If one of the S_i's, say S_1 is isomorphic to a surface $S_{P,1} \subset \mathbb{A}^3_k$ defined by the equation $xz - P(y) = 0$, then the result follows from [16]. Otherwise, we deduce from Corollary 1.19 that the $\mathbb{A}^1$-fibrations $\pi_1 : S_1 \to \mathbb{A}^1_k$ and $\pi_2 : S_2 \to \mathbb{A}^1_k$ are unique up to automorphisms of the base. In turn, this implies that Φ induces an isomorphism $\phi : X(r) \xrightarrow{\sim} X(r)$ such that $\phi \circ \rho_1 = \rho_2 \circ \Phi$. Therefore, $\Phi : S_1 \xrightarrow{\sim} S_2$ factors through an isomorphism of $\mathbb{A}^1$-bundles $\tilde{\phi} : S_1 \xrightarrow{\sim} \phi^* S_2$, where $\phi^* S_2$ denotes the the fiber product $\mathbb{A}^1$-bundle $pr_2 : \phi^* S_2 = S_2 \times_{X(r)} X(r) \to X(r)$. This completes the proof as $\phi^* S_2 \simeq S_2$. □

As an application of Theorem 1.20, we have the following characterization which generalizes a result obtained by Makar-Limanov [16] for complex surfaces $S_{P,h}$ defined by the equations $x^h z - P(y) = 0$.

Example 1.21. *Two Danielewski surfaces in $\mathbb{A}^3_k$ defined by the equations*

$$x^{h_1} z = Q_1(x, y) = \prod_{i=1}^{r_1} (y - \sigma_{1,i}(x)) \quad \text{and} \quad x^{h_2} z = Q_2(x, y) = \prod_{i=1}^{r_2} (y - \sigma_{2,i}(x))$$

are isomorphic if and only if $h_1 = h_2 = h$, $r_1 = r_2 = r$ and there exists a triple $(a, \mu, \tau(x)) \in k^* \times k^* \times k[x]$ such that $Q_2(ax, y) = \mu^r Q_1\left(x, \mu^{-1} y + \tau(x)\right)$.

Indeed, the condition is clearly sufficient. Conversely, Theorem 1.20 above implies that $r_1 = r_2 = r$. If $r = 1$ then S_1 and S_2 are isomorphic to $\mathbb{A}^2_k$ and we are done. If one of the h_i's is equal to 1 then $h_1 = h_2 = 1$ by virtue of Corollary 1.19 and so, the assertion follows from [15]. So we may suppose from now on that $r, h_1, h_2 \geq 2$. According to example 1.13 above the $\mathbb{A}^1$-bundle structures $\rho_1 : S_1 \to X(r)$ and $\rho_2 : S_2 \to X(r)$ are determined by the data $\left(r, h_1, \sigma_1 = \{\sigma_{1,i}(x)\}_{i=1,\ldots,r}\right)$ and $\left(r, h_2, \sigma_2 = \{\sigma_{2,i}(x)\}_{i=1,\ldots,r}\right)$, which correspond to fine k-weighted rakes which are not combs as $h_1, h_2 \geq 2$. So it follows from Corollary 1.19 that every isomorphism $\Phi : S_1 \xrightarrow{\sim} S_2$ induces an automorphism ϕ of $X(r)$, determined by a k-algebra automorphism $x \mapsto ax$ and a permutation $\alpha \in \mathfrak{S}_r$ of the origins of $X(r)$, such that $\phi \circ \rho_1 = \rho_2 \circ \Phi$. In turn, this implies that the $\mathbb{A}^1$-bundle $\rho_1 : S_1 \to X(r)$ is isomorphic to the one $\mathrm{pr}_2 : \phi^* S_2 = S_2 \times_{X(r)} X(r) \to X(r)$ determined by the datum $\left(r, h_2, \tilde{\sigma}_2 = \left\{a^{-h_2} \sigma_{2,\alpha(i)}(ax)\right\}_{i=1,\ldots,r_2}\right)$. One checks in a similar way as in the proof of Theorem 3.11 in [8] that the corresponding transition functions have the same class in $\mathbb{P}\left(H^1\left(X(r), \mathcal{O}_{X(r)}\right)\right)$ if and only if $h_1 = h_2 = h$ and there exists a pair $(\lambda, b(x)) \in k^* \times k[x]$ such that $\sigma_{2,i}(ax) = \lambda a^h \sigma_{1,\alpha(i)}(x) + b(x)$ for every $i = 1, \ldots, r$. So the assertion follows with $\mu = \lambda a^h$ and $\tau(x) = \mu^{-1} b(x)$.

2. Danielewski surfaces in $\mathbb{A}^3_k$ defined by an equation of the form $x^h z - Q(x, y) = 0$ and their automorphisms

In this section, we apply the above results to study Danielewski surfaces $\pi : S \to \mathbb{A}^1_k$ which admit a closed embedding $i : S \hookrightarrow \mathbb{A}^3_k$ in the affine 3-space as a surface $S_{Q,h}$ defined by the equation $x^h z - Q(x, y) = 0$ for a certain polynomial $Q(x, y) \in k[x, y]$ in such a way that $\pi = \mathrm{pr}_x |_{S_{Q,n}} \circ i$. We compute their automorphism groups. We also characterize among them the ones which are isomorphic to surfaces $S_{P,h} \subset \mathbb{A}^3_k$ defined by the equations $x^h z - P(y) = 0$.

2.1. **Characterization of Danielewski surfaces $S_{Q,h}$.**
A surface $S = S_{Q,h} \subset \mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$ is a Danielewski surface with respect to the fibration $\pi = \mathrm{pr}_x \mid_S : S \to \mathbb{A}^1_k$ if and only if the polynomial $Q(0,y)$ splits with simple roots $y_1, \ldots, y_r \in k$, where $r = \deg_y(Q(0,y))$. If $r = 1$, then $\pi^{-1}(0) \simeq \mathbb{A}^1_k$ and $\pi : S \to \mathbb{A}^1_k$ is isomorphic to a trivial $\mathbb{A}^1$-bundle. Thus S is isomorphic to the affine plane. Otherwise, if $r \geq 2$, then S is not isomorphic to $\mathbb{A}^2_k$, as follows for instance from the fact that the divisor class group $\mathrm{Div}(S)$ of S is isomorphic to $\mathbb{Z}^{r-1}$, generated by the classes of the connected components $C_1, \ldots, C_r$ of $\pi^{-1}(0)$, with a unique relation $C_1 + \ldots + C_r = \mathrm{div}(\pi^* x) \sim 0$. The following result shows that these surfaces $S_{Q,h}$ can be thought as the simplest generalization of surfaces $S_{P,h}$ defined by the equations $x^h z - \prod_{i=1}^r (y - y_i) = 0$, obtained by replacing the constants $y_i \in k$ by polynomials $\sigma_i(x) \in k[x]$.

Theorem 2.1. *For a normal affine surface S, the following are equivalent:*
1) S is isomorphic to a surface $S_{Q,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$ for a suitable polynomial $Q(x,y)$ such that $Q(0,y)$ splits defined by the $r \geq 2$ simple roots $y_1, \ldots, y_r$ in k.
2) S is isomorphic to a surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$ for a suitable collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ such that $\sigma_i(0) \neq \sigma_j(0)$ for every $i \neq j$.

Proof. The projection $\mathrm{pr}_{x,y} : \mathbb{A}^3_k \to \mathbb{A}^2_k$ restricts on a Danielewski surface $S = S_{Q,h}$ to a birational morphism $\psi : S \to \mathbb{A}^2_k$ inducing an isomorphism outside the fiber $\pi^{-1}(0)$. So it follows from Theorem 1.11 and Proposition 1.18 above that there exists a fine k-weighted tree $\gamma = (\Gamma, w)$ with all its leaves at a same level h' and an isomorphism $\phi : S \xrightarrow{\sim} S(\gamma)$ such that $\psi = \psi_\gamma \circ \phi$. Since $\psi \mid_{\pi^{-1}(0)}$ is locally constant on $\pi^{-1}(0)$ and takes distinct values y_i on the $r = \deg(Q(0,y))$ irreducible components of $\pi^{-1}(0)$, it follows from Remark 1.12 above that Γ has r leaves and that every element e of Γ but the root e_0 has at most one child, whereas e_0 has exactly r children. Thus γ is a rake of a certain height h'. To determine h', we observe that by construction, $h' = \mathrm{ord}_{C_i}(dx \wedge dy \mid_S) = \mathrm{ord}_{C_j}(dx \wedge dy \mid_S)$ for every $i, j = 1, \ldots, r$. On the other hand, the 2-form $dx \wedge dy \mid_S \in \Lambda^2 \Omega^1_{S/k}$ is supported on the fiber $\pi^{-1}(0)$ as $\pi^{-1}(\mathbb{A}^1_k \setminus \{0\}) \simeq \mathrm{Spec}(k[x, x^{-1}][y])$. Moreover, since the roots of $Q(0,y)$ are simple, q vanishes on $\pi^{-1}(0)$ with the same orders as the 2-form $p = \dfrac{\partial}{\partial y} Q(x,y)\, dx \wedge dy \mid_S = x^h dx \wedge dz \mid_S$. Since z induces a coordinate function on every irreducible components of $\pi^{-1}(0)$, we eventually conclude that $\mathrm{div}(dx \wedge dy \mid_S) = h \cdot \pi^{-1}(0)$ whence that $h' = h$. So the assertion follows from example 1.13 above which guarantees that every rake γ of height h with r leaves corresponds to a certain Danielewski surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$. □

2.2. The above result implies that every Danielewski surface S isomorphic to a surface $S_{Q,h}$ admits a closed embedding $i_{\mathrm{norm}} : S \hookrightarrow \mathbb{A}^3_k$ in $\mathbb{A}^3_k$ as a surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$. We say that i_{norm} is a *normalized embedding* of S in $\mathbb{A}^3_k$ and that $S_{\sigma,h}$ is a *normal form of S in $\mathbb{A}^3_k$*. Starting from a Danielewski surface $S_{Q,h}$ defined by the equation $x^h z - Q(x,y) = 0$, where $Q(0,y)$ splits with $r \geq 2$ simple roots $y_1, \ldots, y_r$ in k, the collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ corresponding to a normal form can be determined as follows.

Proposition 2.3. *In the above setting, there exists a unique triple $(R_1, R_2, \sigma) \in k[x,y]^2 \times k[x]^r$ satisfying the following properties:*
a) $R_1(x,y) \in k[x,y] \setminus (x^h k[x,y])$ is a polynomial such that $R_1(0,y)$ is a nonzero constant,
b) $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ is a collection of polynomials of degrees $\deg(\sigma_i(x)) < h$ such that $\sigma_i(0) = y_i$ for every $i = 1, \ldots, r$
c) $Q(x,y) = R_1(x,y) \prod_{i=1}^r (y - \sigma_i(x)) + x^h R_2(x,y)$.

Furthermore, the endomorphism ϕ^{norm} *of* $\mathbb{A}^3_k$ *defined by* $(x,y,z)\mapsto(x,y,R_1(x,y)z+R_2(x,y))$ *restricts to an isomorphism* $\phi:S_{Q,h}\xrightarrow{\sim}S_{\sigma,h}$ *whose inverse is induced by the endomorphism*

$$\phi_{\mathrm{norm}}:\mathbb{A}^3_k\to\mathbb{A}^3_k,\quad (x,y,z)\mapsto\left(x,y,f(x,y)z+g(x,y)\prod_{i=1}^{r}(y-\sigma_i(x))-f(x,y)R_2(x,y)\right)$$

where (f,g) *denotes a pair of polynomial such that* $R_1(x,y)f(x,y)+x^hg(x,y)=1$.

Proof. We let $Q(x,y)=c\prod_{i=1}^{r}(y-y_i)+x\tilde{Q}(x,y)$ where $c\in k^*$ and $\tilde{Q}\in k[x,y]$. Provided that they exist, the polynomials $\sigma_i(x)$ must satisfy the condition $Q(x,\sigma_i(x))\equiv 0\mod x^s$ for every $i=1,\ldots,r$ and every $s=1,\ldots,h$. This condition is satisfied for $s=1$ by letting $\sigma_i(0)=y_i$ for every $i=1,\ldots,r$. We proceed by induction, assuming that the σ_i's have been constructed up to the degree s. For every $\lambda\in k$ we have

$$\begin{aligned}Q\left(x,\sigma_i(x)+\lambda x^{s+1}\right) &= c\prod_{j=1}^{r}\left(\sigma_i(x)+\lambda x^{s+1}-y_j\right)+x\tilde{Q}\left(x,\sigma_i(x)+\lambda x^{s+1}\right)\\ &\equiv c\lambda x^{s+1}\prod_{j\neq i}(y_i-y_j)+Q(x,\sigma_i(x))\mod x^{s+2}\\ &\equiv c\lambda b_ix^{s+1}+a_ix^{s+1}\mod x^{s+2}\end{aligned}$$

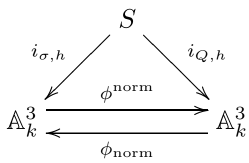
for a certain element $a_i\in k$. Since $b_i\neq 0$ as $y_j\neq y_i$ for every $j\neq i$, we conclude that $\tilde{\sigma}_i(x)=\sigma_i(x)-a_i(cb_i)^{-1}x^{s+1}$ is a unique polynomial of degree at most $s+1$ which is congruent to $\sigma_i(x)$ modulo x^{s+1} and such that $Q(x,\tilde{\sigma}_i(x))\equiv 0\mod x^{s+2}$. So existence and uniqueness of the σ_i's follows. In turn, this implies that there exist polynomials $S_0(x,y)=Q(x,y)$ and $S_i(x,y)$, $i=1,\ldots,r$ such that $S_i(x,y)=S_i(x,\sigma_{i+1}(x))+(y-\sigma_{i+1}(x))S_{i+1}(x,y)$ for every $i=0,\ldots,r-1$. By construction, $S_i(x,\sigma_{i+1}(x))\equiv 0\mod x^h$ for every $i=0,\ldots,r-1$. Therefore, letting $S_r(x,y)=R_1(x,y)+x^h\tilde{R}(x,y)$, where $\deg_x(R_1(x,y))<h$, it follows that $Q(x,y)-R_1(x,y)\prod_{i=1}^{r}(y-\sigma_i(x))\equiv 0\mod x^h$. This completes the proof of the first assertion as $R_1(0,y)=c$ by construction.

For the second one, one checks that $(\phi^{\mathrm{norm}})^*\left(x^hz-Q(x,y)\right)=R_1(x,y)\left(x^hz-\prod_{i=1}^{r}(y-\sigma_i(x))\right)$ and that $(\phi_{\mathrm{norm}})^*\left(x^hz-\prod_{i=1}^{r}(y-\sigma_i(x))\right)=f(x,y)\left(x^hz-Q(x,y)\right)$. This means that the images of the restrictions of ϕ^{norm} and ϕ_{norm} to $S_{\sigma,h}$ and $S_{Q,h}$ respectively are contained in $S_{Q,h}$ and $S_{\sigma,h}$. Finally, the identities $(\phi^{\mathrm{norm}}\circ\phi_{\mathrm{norm}})^*(z)=z-g(x,y)\left(x^hz-Q(x,y)\right)$ and $(\phi_{\mathrm{norm}}\circ\phi^{\mathrm{norm}})^*(z)=z-g(x,y)\left(x^hz-\prod_{i=1}^{r}(y-\sigma_i(x))\right)$ guarantee that ϕ is an isomorphism. □

Remark 2.4. It follows from example 1.13 above that the restriction of y to a Danielewski surface $S=S_{\sigma,h}$ defined by the equation $x^hz-\prod_{i=1}^{r}(y-\sigma_i(x))=0$ coincides with the canonical function ψ_γ on S considered as an $\mathbb{A}^1$-bundle $\rho:S\to X(r)$ with transition functions $g_{ij}=x^{-h}(\sigma_j(x)-\sigma_i(x))$ defined by means of the fine k-weighted rake γ of example 1.13. Similarly, the restriction of z to S coincides with the regular function $\xi_\gamma=x^{-h}\prod_{i=1}^{r}(\psi_\gamma-\sigma_i(x))$ on S. On the other hand, for every pair $(R_1,R_2)\in k[x,y]^2$ of polynomials such that $R_1(0,y)$ is a nonzero constant, the rational function

$$x^{-h}\left(R_1(x,\psi_\gamma)\prod_{i=1}^{r}(y-\sigma_i(x))\right)+R_2(x,\psi_\gamma)\in\mathcal{O}(S)\otimes_{k[x]}k\left[x,x^{-1}\right]$$

extends to a section $\xi_{(R_1,R_2),\gamma}\in\mathcal{O}(S)$ which still induces a coordinate function on every irreducible component of the fiber $\pi^{-1}(0)$. By construction, the regular functions π, ψ_γ and $\xi_{(R_1,R_2),\gamma}$ on S define a new closed embedding $i_{(R_1,R_2)}=\left(\pi,\psi_\gamma,\xi_{(R_1,R_2),\gamma}\right):S\hookrightarrow\mathbb{A}^3_k$ of S in $\mathbb{A}^3_k$ inducing an isomorphism between S and the surface defined by the equation $x^hz-R_1(x,y)\prod_{i=1}^{r}(y-\sigma_i(x))+x^hR_2(x,y)=0$. This means that a closed embedding $i_{Q,h}:S\hookrightarrow\mathbb{A}^3_k$ of a Danielewski surface S in $\mathbb{A}^3_k$ as a surface defined by the equation $x^hz-Q(x,y)=0$ can be interpreted as a twisted form of one of its normalized embeddings $i_{\sigma,h}:S\hookrightarrow\mathbb{A}^3_k$ as a surface defined by the equation $x^hz-\prod_{i=1}^{r}(y-\sigma_i(x))$ obtained by modification of the function inducing a coordinate on every irreducible component of the fiber $\pi^{-1}(0)$. Proposition 2.3 above says that a twisted embedding $i_{Q,h}$ is related to its associated normalized embedding defined in 2.2 by a commutative diagram



where ϕ^{norm} and ϕ_{norm} are the $\mathbb{A}^2_k$-endomorphisms of $\mathbb{A}^3_k$ introduced above.

2.2. **Automorphisms of Danielewski surfaces $S_{Q,h}$ in $\mathbb{A}^3_k$.**
In [15] and [16], Makar-Limanov computed the automorphism groups of surfaces in $\mathbb{A}^3$ defined by the equation $x^h z - P(y) = 0$, where $h \geq 1$ and where $P(y)$ is an arbitrary polynomial. In particular, he established that every automorphism of such a surface is induced by the restriction of an automorphism of the ambient space. Recently, Crachiola established that this also holds for surfaces defined by the equations $x^h z - y^2 - r(x)y = 0$, where $h \geq 1$ and where $r(x)$ is an arbitrary polynomial such that $r(0) \neq 0$. This subsection is devoted to the proof of the following more general result.

Theorem 2.5. *Let $S_{Q,h} \subset \mathbb{A}^3_k$ be a Danielewski surface defined by the equation $x^h z - Q(x,y) = 0$ and let $(\phi^{\mathrm{norm}}, \phi_{\mathrm{norm}})$ be a pair of endomorphisms of $\mathbb{A}^3_k$ as in Proposition 2.3 above inducing isomorphisms between $S_{Q,h}$ and the Danielewski surface $S_{\sigma,h}$ defined by the equation $x^h z - P(x,y) = 0$, where $P(x,y) = \prod_{i=1}^{r}(y - \sigma_i(x))$. Then every automorphism of $S_{Q,h}$ is induced by the restriction of an endomorphism of $\mathbb{A}^3_k$ of the form $\phi_{\mathrm{norm}}\psi \circ \phi^{\mathrm{norm}}$, where ψ belongs to the subgroup $G_{\sigma,h}$ of $\mathrm{Aut}(\mathbb{A}^3_k)$ generated by the following automorphisms:*

(a) $\Delta_b(x,y,z) = \left(x, y + x^h b(x), z + x^{-h}\left(P\left(x, y + x^h b(x)\right) - P(x,y)\right)\right)$, *where $b(x) \in k[x]$.*

(b) *If there exists a polynomial $\tau(x)$ such that $P(x, y + \tau(x)) = \tilde{P}(y)$ then the automorphisms $H_a(x,y,z) = \left(ax, y + \tau(ax) - \tau(x), a^{-h}z\right)$, where $a \in k^*$ should be added.*

(c) *If there exists a polynomial $\tau(x)$ such that $P(x, y + \tau(x)) = \tilde{P}(x^{q_0}, y)$, then the automorphisms $\tilde{H}_a(x,y,z) = \left(ax, y + \tau(ax) - \tau(x), a^{-h}z\right)$, where $a \in k^*$ and $a^{q_0} = 1$ should be added.*

(d) *If there exists a polynomial $\tau(x)$ such that $P(x, y + \tau(x)) = y^i \tilde{P}(x, y^s)$, where $i = 0, 1$ and $s \geq 2$, then the automorphisms $S_\mu(x,y,z) = \left(x, \mu y + (1-\mu)\tau(x), \mu^i z\right)$, where $\mu \in k^*$ and $\mu^s = 1$ should be added.*

(e) *If $char(k) = s > 0$ and $P(x,y) = \tilde{P}\left(y^s - c(x)^{s-1} y\right)$ for a certain polynomial $c(x) \in k[x]$ such that $c(0) \neq 0$, then the automorphism $T_c(x,y,z) = (x, y + c(x), z)$ should be added.*

(f) If $h = 1$ then the involution $I(x,y,z) = (z,y,x)$ should be added.

2.6. In order to prove the Theorem, it suffices, in view of Proposition 2.3 above, to show that every automorphism of the Danielewski surface $S_{\sigma,h}$ is induced by an automorphism $\psi \in G_{\sigma,h}$. It is clear that every automorphism of $\mathbb{A}^3_k$ of types (a)-(f) above leaves $S_{\sigma,h}$ invariant, whence induces an automorphism of $S_{\sigma,h}$. If $h = 1$, then the converse follows from [15]. Otherwise, if $h \geq 2$, then the fact that every automorphism of $S_{\sigma,h}$ is obtained as the restriction of a composition of automorphisms of types (a)-(e) above is established in 2.7 and 2.9 below. In what follows, we let S be a special Danielewski surface considered as an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ with transition functions defined by a collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ corresponding to a fine k-weighted rake γ of height h with $r \geq 2$ elements at level one.

Proposition 2.7. *If $h \geq 2$ then every automorphism Φ of S is uniquely determined by a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x)) \in \mathfrak{S}_r \times k^* \times k^* \times k[x]$ such that the following hold:*

1) The polynomial $c_\Phi(x) = \sigma_{\alpha(i)}(ax) - \mu\sigma_i(x) + (ax)^h b(x) \in k[x]$ does not depend on the index $i = 1, \ldots, r$.

2a) The permutation α is either trivial or has at most a unique fixed point. If it is nontrivial then all nontrivial cycles with disjoint support occurring in a decomposition of α have the same length $s \geq 2$.

2b) If α is trivial then $\mu = 1$ and the converse also holds provided that char $(k) \neq s$. Otherwise, if α is nontrivial and char $(k) \neq s$ then $(\mu)^s = 1$ but $(\mu)^{s'} \neq 1$ for every $1 \leq s' < s$.

Furthermore, letting $i_{\sigma,h} : S \hookrightarrow \mathbb{A}^3_k$ be the closed embedding of S in $\mathbb{A}^3_k$ as the surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$, every automorphism of S with datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x))$ is induced by the restriction to $S_{\sigma,h}$ of the following automorphism Ψ of $\mathbb{A}^3_k$:

$$\Psi(x,y,z) = \left(ax, \mu y + c_\Phi(x), a^{-h}\mu^r z + (ax)^{-h}\left(\prod_{i=1}^r (\mu y + c_\Phi(x) - \sigma_i(ax)) - \mu^r \prod_{i=1}^r (y - \sigma_i(x))\right)\right)$$

Proof. With the notation of 1.8, the fact that $h \geq 2$ and $r \geq 2$ guarantees that every automorphism Φ of $S = S(\gamma)$ induces an automorphism ϕ of $X(r)$ such that $\phi \circ \rho = \rho \circ \Phi$ (see 1.20 above). So Φ is determined by a collection of local isomorphisms $\Phi_i : S_i \xrightarrow{\sim} S_{\alpha(i)}$ where $\alpha \in \mathfrak{S}_r$, defined by k-algebra isomorphisms

$$\Phi_i^* : k[x]\left[u_{\alpha(i)}\right] \longrightarrow k[x][u_i], \quad x \mapsto a_i x, \quad u_{\alpha(i)} \mapsto \lambda_i u_i + b_i(x), \quad i = 1, \ldots, r$$

where $a_i, \lambda_i \in k^*$ and where $b_i \in k[x]$. These local isomorphisms are compatible with the transition isomorphisms $\tau_{ij} : S_j \ni (x, u_j) \mapsto (x, u_j + g_{ij}(x)) \in S_i$, $i \neq j$, $i, j = 1, \ldots, r$ defining S as an $\mathbb{A}^1$-bundle over $X(r)$ if and only if $a_i = a$ and $\lambda_i = \lambda$ for every $i = 1, \ldots, r$, and the relation $\lambda g_{ij}(x) + b_i(x) = g_{\alpha(i)\alpha(j)}(ax) + b_j(x)$ hold in $k\left[x, x^{-1}\right]$ for every $i, j = 1, \ldots, r$. Since $g_{ij}(x) = x^{-h}(\sigma_j(x) - \sigma_i(x))$, where the σ_i's have degrees strictly lower than h, we conclude that the latter condition is equivalent to the fact that $b_i(x) = b(x)$ for every $i = 1, \ldots, r$ and the polynomial $\tilde{c}(x) = \sigma_{\alpha(i)}(ax) - \lambda a^h \sigma_i(x)$ does not depend on the index $i = 1, \ldots, r$. So the first assertion follows with $\mu = \lambda a^h$. To simplify the notation, we let $y_i = \sigma_i(0)$ for every $i = 1, \ldots, r$. Note that by hypothesis, $y_i \neq y_j$ for every $i \neq j$.

If $\alpha \in \mathfrak{S}_r$ has a least two fixed points, say i_0 and i_1, then $y_{i_0}(1-\mu) = y_{i_1}(1-\mu) = \tilde{c}(0)$, and so, $\mu = 1$ and $\tilde{c}(0) = 0$ as $y_{i_0} \neq y_{i_1}$. In turn, this implies that α is trivial. Indeed, otherwise there would exist an index i such that $\alpha(i) \neq i$ but $y_{\alpha(i)} = y_i$, in contradiction with our hypothesis. If α is nontrivial, then we let $s \geq 2$ be the infimum of the length's of the nontrivial cycles occurring in decomposition of α into a product of cycles with disjoint supports. We deduce that $y_i(1-\mu^s) = y_j(1-\mu^s)$ for every pair of distinct indices i and j in the support of a same cycle of length s. Thus $\mu^s = 1$ as $y_i \neq y_j$ for every $i \neq j$.

If $\mu = 1$ then $s' \cdot \tilde{c}(0) \neq 0$ for every $s' = 1, \ldots, s-1$. Indeed, otherwise we would have $y_{\alpha^{s'}(i)} = y_i + s' \cdot \tilde{c}(0) = y_i$ for every index $i = 1, \ldots, r$ which is impossible since α is nontrivial. In particular, α is fixed-point free. On the other hand $s \cdot \tilde{c}(0) = 0$ as $y_i = y_{\alpha^s(i)} = y_i + s \cdot \tilde{c}(0)$ for every index i in the support of a cycle of length s in α. This is possible if and only if the characteristic of the base field k is exactly s. We also conclude that every cycle in α have length s for otherwise there would exist an index i such that $\alpha^s(i) \neq i$ but $y_{\alpha^s(i)} = y_i + s \cdot \tilde{c}(0) = y_i$ in contradiction with our hypothesis. Otherwise, if $\mu \neq 1$ then $\mu^{s'} \neq 1$ for every $s' < s$. Indeed, otherwise there would exist an index i such that $\alpha^{s'}(i) \neq i$ but $y_{\alpha^{s'}(i)} = \mu^{s'} y_i + \tilde{c}(0) \sum_{p=0}^{s'-1} \mu^p = y_i$, which is impossible. The same argument also implies that all the nontrivial cycles in α have length s.

For the second assertion, we recall that $i_{\sigma,h} : S \hookrightarrow \mathbb{A}^3_k$ is defined by means of the function x, the canonical function ψ_γ on S and the function $\xi_\gamma = x^{-h} \prod_{i=1}^r (\psi_\gamma - \sigma_i(x))$ on S. By definition, ψ_γ restricts on a open subset $S_{\alpha(i)} = \operatorname{Spec}\left(k[x]\left[u_{\alpha(i)}\right]\right)$ of S to the polynomial $x^h u_i + \sigma_i(x) \in k[x][u_i]$, whereas $\Phi^*\left(u_{\alpha(i)}\right) = \lambda u_i + b(x) \in k[x][u_i]$ for every $i = 1, \ldots, r$. So we conclude that $\Phi^*(\psi_\gamma) = \mu \psi_\gamma + c_\Phi(x) \in \mathcal{O}(S)$. Letting $P(x,y) = \prod_{i=1}^r (y - \sigma_i(x))$, it follows that

$$\Phi^*(\xi_\gamma) = (ax)^{-h} P(ax, \mu\psi_\gamma + c_\Phi(x)) = a^{-h}\mu^r \xi_\gamma + (ax)^{-h}\left(P(ax, \mu\psi_\gamma + c_\Phi(x)) - \mu^r P(x, \psi_\gamma)\right).$$

This completes the proof. □

Remark 2.8. Given two automorphisms Φ_1 and Φ_2 of S defined by data $\mathcal{A}_{\Phi_1} = (\alpha_1, \mu_1, a_1, b_1)$ and $\mathcal{A}_{\Phi_2} = (\alpha_2, \mu_2, a_2, b_2)$ respectively, one checks that $\Phi = \Phi_2 \circ \Phi_1$ is the automorphism of S with corresponding datum $\mathcal{A}_\Phi = \left(\alpha_2 \circ \alpha_1, \mu_2\mu_1, a_2 a_1, a_2^{-h}\mu_2 b_1(x) + b_2(a_1 x)\right)$.

The following result completes the proof of Theorem 2.5.

Lemma 2.9. *Every automorphism of Danielewski surface $S_{\sigma,h} \subset \mathbb{A}^3_k$ defined by the equation*

$$x^h z - \prod_{i=1}^{r} (y - \sigma_i(x)) = 0, \quad \textit{where } h \geq 2,$$

is the restriction of a composition of automorphisms of types (a)-(e) in Theorem 2.5.

Furthermore the restriction map $G_{\sigma,h} \to \mathrm{Aut}(S)$ is an isomorphism of group.

Proof. The second assertion follows immediately form the first one and the composition rule given in remark 2.8 above. We let $P(x,y) = \prod_{i=1}^{r} (y - \sigma_i(x))$. By 2.7 above, every automorphism of $S_{\sigma,h}$ is determined by a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x))$ such that that the polynomial $c(x) = \sigma_{\alpha(i)}(ax) - \mu\sigma_i(x) + x^h b(x)$ does not depend on the index $i = 1, \ldots, r$. Clearly, automorphisms of type (a) in Theorem 2.5 coincide with the ones determined by a datum $\mathcal{A} = (\mathrm{Id}, 1, 1, b(x))$. In view of the composition rule given in 2.8, it suffices to consider from now the automorphisms corresponding to data $\mathcal{A} = (\alpha, \mu, a, 0)$.

1°) If α is trivial, then $\mu = 1$ by virtue of Proposition 2.7, and so $\mathcal{A} = (\mathrm{Id}, 1, a, 0)$. Then, the relation $c(x) = \sigma_i(ax) - \sigma_i(x)$ holds for every $i = 1, \ldots, r$.

1°a) If $a^q \neq 1$ for every $q = 1, \ldots, h-1$, then there exists a polynomial $\tau(x) \in k[x]$ such that $\sigma_i(x) = \sigma_i(0) + \tau(x)$ for every $i = 1, \ldots, r$. Thus $c(x) = \tau(ax) - \tau(x)$ and $P(x, y + \tau(x)) = \tilde{P}(y) = \prod_{i=1}^{r} (y - \sigma_i(0))$. The corresponding automorphism is of type (b) in Theorem 2.5, given by $(x,y,z) \mapsto \left(ax, y + \tau(ax) - \tau(x), a^{-h}z\right)$.

1°b) If $a \neq 1$ but $a^{q_0} = 1$ for a minimal $q_0 = 2, \ldots, h-1$, then there exists polynomials $\tau(x)$ and $\tilde{\sigma}_i(x)$, $i = 1, \ldots, r$, such that $\sigma_i(x) = \tilde{\sigma}_i(x^{q_0}) + \tau(x)$ for every $i = 1, \ldots, r$. So there exists a polynomial $\tilde{Q}$ such that $P(x, y + \tau(x)) = \tilde{P}(x^{q_0}, y)$. Moreover, $c(x) = \tau(ax) - \tau(x)$ and the corresponding automorphism is of type (c) in Theorem 2.5, given by $(x,y,z) \mapsto \left(ax, y + \tau(ax) - \tau(x), a^{-h}z\right)$.

2°) If α is not trivial then $\mu^s = 1$. Since $\Phi = \Phi_2 \circ \Phi_1$ where $\Phi_1$and Φ_2 denote the automorphisms with data $\mathcal{A}_{\Phi_1} = (\mathrm{Id}, 1, a, 0)$ and $\mathcal{A}_{\Phi_2} = (\alpha, \mu, 1, 0)$ respectively, it suffices to consider the situation that Φ is determined by a datum $\mathcal{A}_\Phi = (\alpha, \mu, 1, 0)$, where $\mu \in k^*$ and $\mu^s = 1$. So the relation $\sigma_{\alpha(i)}(x) = \mu\sigma_i(x) + c(x)$ holds for every $i = 1, \ldots, r$.

2°a) $\mu^s = 1$ but $\mu^{s'} \neq 1$ for every $s' = 1, \ldots, s-1$. Letting $\tau(x) = (1-\mu)^{-1} c(x)$ and $\tilde{\sigma}_i(x) = \sigma_i(x) - \tau(x)$ for every $i = 1, \ldots, r$, we arrive at the relation $\tilde{\sigma}_{\alpha(i)}(x) = \mu\tilde{\sigma}_i(x)$ for every $i = 1, \ldots, r$. Furthermore, if i_0 is a unique fixed point of α then $\tilde{\sigma}_{i_0}(x) = 0$ as $\sigma_{i_0}(x) = \tau(x)$. So we conclude that $P(x, y + \tau(x)) = y^i \tilde{P}(x, y^s)$ where $i = 0, 1$ and where s denotes the length of the nontrivial cycles in α. The corresponding automorphism is of type (d) in Theorem 2.5, given by $(x,y,z) \mapsto \left(x, \mu y + (1-\mu)\tau(x), \mu^i z\right)$.

2°b) If $\mu = 1$ then α is fixed point free by virtue of Proposition 2.7 and $\mathrm{char}(k) = s$, where s denotes the common length's of the cycle occurring in α. Moreover, $s' \cdot c(0) \neq 0$ for every $s' = 1, \ldots, s-1$ and $\sigma_{i_m}(x) = \sigma_{i_1}(x) + m \cdot c(x)$ for every index i_k occurring in a cycle $(i_1, \ldots, i_s)$ of length s in α. Letting $r = ds$, we may suppose up to a reordering that α decomposes as the product of the standard cycles $(is+1, is+2 \ldots, (i+1)s)$, where $i = 0, \ldots, d-1$. Letting $R(x,y) = \prod_{m=1}^{s} (y - m \cdot c(x)) = y^s - c(x)^{s-1} y$, we conclude that

$$P(x,y) = \prod_{i=0}^{d-1} R(x, y - \sigma_{is}(x)) = \tilde{P}\left(x, y^s - c(x)^{s-1} y\right)$$

for a suitable polynomial $\tilde{P}(x,y) \in k[x,y]$. The corresponding automorphism is of type (e) in Theorem 2.5, given by $(x,y,z) \mapsto (x, y + c(x), z)$. This completes the proof. □

2.10. A special Danielewski surface $S \simeq S_{Q,h}$ always admits nontrivial actions of the multiplicative group $\mathbb{G}_{a,k}$, corresponding to the fact that the associated $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ is actually the structural morphism of a principal $\mathbb{G}_{a,k}$-bundle over $X(r)$ (see 1.18). More precisely, it follows from the gluing construction described in 1.8 that for every polynomial $b(x) \in k[x]$, the local $\mathbb{G}_{a,k}$-actions on the canonical open subset $S_i \simeq \mathrm{Spec}(k[x][u_i])$ of S defined by the comorphisms

$$\mathrm{Id} \otimes 1 + b(x) \otimes \mathrm{Id} : k[x][u_i] \to k[x][u_i] \otimes_{k[x]} k[x][t] \simeq k[x][u_i, t]$$

glue to a global $\mathbb{G}_{a,k}$-action $\mathbf{t}_b : \mathbb{G}_{a,k} \times S \to S$ on S. If $h \geq 2$, then the associated algebraic quotient morphism $\pi : S \to S//\mathbb{G}_{a,k} \simeq \mathbb{A}^1$ is a unique $\mathbb{A}^1$-fibration on S up to automorphisms of the base (see Theorem 1.15 above). Furthermore, Proposition 2.12 in [8] implies that every $\mathbb{G}_{a,k}$-action on S is obtained by the above procedure. In the setting of in Theorem 2.5 above, the $\mathbb{G}_{a,k}$-action $\mathbf{t}_b$ on S coincides in a normalized embedding $i_{\sigma,h} : S \hookrightarrow \mathbb{A}^3_k$ of S as a surface defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$ with the restriction to $S_{\sigma,h}$ of the $\mathbb{G}_{a,k}$-action on $\mathbb{A}^3_k$ defined by the comorphism $\Delta^*_{tb}(x, y, z) : k[x, y, z] \to k[x, y, z, t]$

$$(x, y, z) \mapsto \left(x, y + x^h tb(x), z + x^{-h}\left(\prod_{i=1}^r (y + x^h tb(x) - \sigma_i(x)) - \prod_{i=1}^r (y - \sigma_i(x))\right)\right).$$

More generally, we have the following result.

Proposition 2.11. *Let $S \simeq S_{Q,h}$ be a Danielewski surface and let $\mathbf{t}_b : \mathbb{G}_{a,k} \times S \to S$, where $b \in k[x]$ be a nontrivial $\mathbb{G}_{a,k}$-action on S. Then for every closed embedding $i : S \hookrightarrow \mathbb{A}^3_k$ of S as a surface defined by the equation*

$$x^h z - R_1(x, y) \prod_{i=1}^r (y - \sigma_i(x)) + x^h R_2(x, y) = 0$$

there exists an algebraic $\mathbb{G}_{a,k}$-action on $\mathbb{A}^3_k$ for which i becomes an equivariant embedding.

Proof. Recall that the embedding $i_{(R_1,R_2)}$ of S is defined by the regular functions x, ψ_γ and $\zeta = x^{-h} Q(x, \psi_\gamma)$, where $Q(x, \psi_\gamma) = R_1(x, \psi_\gamma) \prod_{i=1}^r (\psi_\gamma - \sigma_i(x)) + x^h R_2(x, \psi_\gamma)$. Since $\psi_\gamma \mid_{S_i} = x^h u_i + \sigma_i(x) \in k[x][u_i]$,we conclude that $\mathbf{t}^*_b(\psi_\gamma) = \psi_\gamma + x^h f(x) t \in \mathcal{O}(S(\gamma))[t]$. Similarly, one checks that $\mathbf{t}^*_b(\zeta) = \zeta + x^{-h}\left(Q\left(x, \psi_\gamma + x^h f(x) t\right) - Q(x, \psi_\gamma)\right) \in \mathcal{O}(S(\gamma))[t]$. So it follows that i is equivariant when we equip $\mathbb{A}^3_k = \mathrm{Spec}(k[x, y, z])$ with the $\mathbb{G}_{a,k}$-action defined by the comorphism

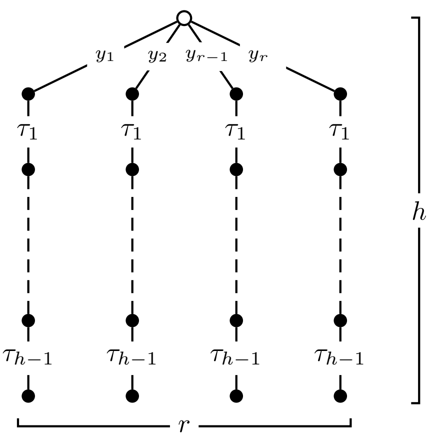
$$\mu^* : k[x, y, z] \to k[x, y, z], (x, y, z) \mapsto \left(x, y + x^h b(x) t, z + x^{-h}\left(Q\left(x, y + x^h b(x) t\right) - Q(x, y)\right)\right).$$

□

2.3. Danielewski surfaces $S_{Q,h}$ with quasi-homogeneous normal forms.

As an application of the above results, we study special Danielewski surfaces isomorphic to a surface with equation $x^h z - P(y) = 0$ in $\mathbb{A}^3_k$. When $k = \mathbb{C}$, we show that up to isomorphisms, there are more surfaces of this type in the holomorphic category than in the algebraic one.

2.12. Lemma 2.3 above implies that the collection of polynomials $\sigma_i(x)$, $i = 1, \ldots, r$, corresponding to a surface $S_{P,h} \subset \mathbb{A}^3_k$ is given by $\sigma_i(x) = y_i$ for every $i = 1, \ldots, r$. In turn, we deduce from Theorem 1.20 and Example 1.21 above that a Danielewski surface $S_{Q,h}$ with a normal form $S_{\sigma,h}$ defined by a datum $\left(r, h, \sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}\right)$ is isomorphic to a surface $S_{P,h}$ as above if and only if there exists a polynomial $\tau(x) \in k[x]$ such that $\sigma_i(x) = \sigma_i(0) + \tau(x)$ for every $i = 1, \ldots, r$. So we conclude that every such surface correspond to a fine k-weighted rake γ of the following type.



2.13. A Danielewski surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(0) - \tau(x)) = 0$ admits a nontrivial action of the multiplicative group $\mathbb{G}_{m,k}$ which arises as the restriction of the $\mathbb{G}_{m,k}$-action Ψ on $\mathbb{A}^3_k$ defined by $\Psi(a;x,y,z) = H_a(x,y,z) = \left(ax, y+\tau(ax) - \tau(x), a^{-h}z\right)$. In the setting of Proposition 2.7 above, the automorphisms H_a correspond to data $\mathcal{A}_{\phi_a} = (1,1,a,0)$, where $a \in k^*$. Over an infinite base field k, it turns out that Danielewski surfaces isomorphic to a surface $S_{P,h}$ are exactly the ones which admit a nontrivial $\mathbb{G}_{m,k}$-action. More precisely, we have the following result.

Proposition 2.14. *If a special Danielewski surface $S_{Q,h}$ defined by the equation $x^h z - Q(x,y) = 0$ admits an automorphism Φ determined by a datum $\mathcal{A}_\Phi = (\alpha,\mu,a,0)$ as in Proposition 2.7 such that $a^s \neq 1$ for every $s = 1,\ldots,h-1$ then $S_{Q,h}$ is isomorphic to a surface $S_{P,h} \subset \mathbb{A}^3_k$ defined by the equation $x^h z - P(y) = 0$.*

Proof. We let $\left(r, h, \sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}\right)$ be the datum associated to the Danielewski surface $S_{Q,h}$ by means of the procedure described in Lemma 2.3. Under our hypothesis, it follows from 1°a) in the proof of Lemma 2.9 that there exists a polynomial $\tau(x) \in k[x]$ such that $\sigma_i(x) = \sigma_i(0) + \tau(x)$ as desired. □

Definition 2.15. A Danielewski surface $S \simeq S_{Q,h}$ is said to be of *quasi-homogeneous normal type* if it is isomorphic to a surface defined by the equation $x^h z - P(y) = 0$ for a suitable polynomial $P(y)$.

Remark 2.16. It follows from the above discussion that a Danielewski surface S of quasi-homogeneous normal type admits a closed embedding in $\mathbb{A}^3_k$ as a surface $S_{P,h}$ for which the $\mathbb{G}_{m,k}$-action on S arises as the restriction of a $\mathbb{G}_{m,k}$-action on the ambient space. However, we will see in 3.12 below that there exists other closed embeddings of S in $\mathbb{A}^3_k$ for which this action on S do not extend to the whole of $\mathbb{A}^3_k$.

2.17. If $k = \mathbb{C}$ then it turns out that there exists Danielewski surfaces which are not algebraically isomorphic but holomorphically isomorphic to a surface $S_{P,h}$. For instance, G. Freudenburg and L. Moser-Jauslin [11] established that the surfaces S_0 and S_1 in $\mathbb{A}^3_\mathbb{C}$ defined by the equations $x^2z - y^2 - 1 = 0$ and $x^2 z - y^2 - 1 + x = 0$ respectively are algebraically non-isomorphic but holomorphically isomorphic. One checks using Lemma 2.3 that the second one is isomorphic to the surface defined by the equation $x^2 z - (y - i + ix/2)(y + i - ix/2) = 0$, which is not of quasi-homogeneous normal type by virtue of the above criterion. In [11], the above example is obtained by giving an explicit holomorphic automorphism Φ of $\mathbb{A}^3_\mathbb{C}$ such that $\Phi(S_0) = S_1$. In what follows, we give an alternative interpretation using the fact that certain $\mathbb{A}^1$-bundles over a fixed base scheme are holomorphically isomorphic but not algebraically isomorphic.

2.18. We let $y_1,\ldots,y_r$ be a collection of $r \geq 2$ pairwise distinct complex numbers. For every integer $h \geq 1$, we let $\eta_h(u) = \sum_{n=0}^{h-1} \frac{u^n}{n!} \in \mathbb{C}[u]$. For every $t \in \mathbb{C}$, we consider the Danielewski surface $S_t \subset \mathbb{A}^3_\mathbb{C}$ defined by the equation $x^h z - \prod_{i=1}^r (y - y_i \eta_h(tx)) = 0$. One checks easily using the criterion given in 2.12 above that for every $t \neq 0$, S_t is not of quasi-homogeneous normal type, whence is not isomorphic to S_0. However, we have the following result.

Proposition 2.19. *For every $t \neq 0$, the surface S_t is biholomorphic to S_0.*

Proof. Recall that by construction S_t is isomorphic to the total space of the $\mathbb{A}^1$-bundle $\rho : S_t \to X(r)$ obtained by gluing r copies $S_{t,i} = \operatorname{Spec}(\mathbb{C}[x,u_i])$ of $\mathbb{A}^2_\mathbb{C}$ outside the line $L = \{x = 0\}$ by means of transition isomorphisms $S_{t,j} \ni (x,u_j) \mapsto \left(x, u_j + x^{-h}\eta_h(tx)(y_j - y_i)\right) \in S_{t,i}$ for every $i,j = 1,\ldots,r$. As an holomorphic $\mathbb{A}^1$-bundle, $\rho : S_t \to X(r)$ is biholomorphic to the one $\tilde{\rho} : \tilde{S}_t \to X(r)$ defined by the transition functions $\left\{\tilde{g}_{ij} = x^{-h} e^{tx}(y_j - y_i)\right\}_{i,j=1,\ldots,r}$. Indeed, since by construction, $x \mapsto x^{-h}(e^{tx} - \eta_h(tx))$ is an holomorphic function on $\mathbb{A}^1_\mathbb{C}$ for every $t \in \mathbb{C}$, the corresponding biholomorphism $\tilde{\phi}_t : \tilde{S}_t \xrightarrow{\sim} S_t$ is simply defined at the level of the open subsets $S_{t,i}$ and $\tilde{S}_{t,i}$ by means of the local biholomorphisms $\tilde{\phi}_{t,i} : \tilde{S}_{t,i} \xrightarrow{\sim} S_{t,i}$, $(x,u) \mapsto \left(x, u + x^{-h}(e^{tx} - \eta_h(tx))\right)$.

On the other hand, the local biholomorphisms $\phi_{t,i} : S_{0,i} \xrightarrow{\sim} \tilde{S}_{t,i}$, $(x, u_i) \mapsto (x, e^{tx}u_i)$, $i = 1, \ldots, r$, glue to a global one $\phi_t : S_0 \xrightarrow{\sim} \tilde{S}_t$, and we conclude that $\tilde{\phi}_t \circ \phi_t : S_0 \xrightarrow{\sim} S_t$ is a biholomorphism. □

Remark 2.20. Letting $\mathcal{X}_h \subset \mathbb{A}^3_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[x, y, z\right]\left[t\right]\right)$ be the algebraic variety defined by the equation $x^h z - \prod_{i=1}^r (y - y_i \eta_h(tx)) = 0$, one can consider the Danielewski surfaces S_t as the members of a nontrivial flat family $\Xi = \mathrm{pr}_t : \mathcal{X}_h \to \mathbb{A}^1_{\mathbb{C}}$ of surfaces in $\mathbb{A}^3_{\mathbb{C}}$. It follows from the proof of Proposition 2.19 above that this family $\Xi : \mathcal{X}_h \to \mathbb{A}^1_{\mathbb{C}}$ is holomorphically trivial, a trivialization being given by the map $\Phi : S_0 \times \mathbb{A}^1_{\mathbb{C}} \to \mathcal{X}_h$, $(s, t) \mapsto \Phi(s, t) = \tilde{\phi}_t \circ \phi_t(s)$.

2.21. The above result implies in particular that for every $t \neq 0$ the Danielewski surface $S_t \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^h z - \prod_{i=1}^r (y - y_i \eta_h(tx)) = 0$ admits a holomorphic $\mathbb{C}^*$-action but no algebraic ones.

3. Embeddings of Danielewski surfaces $S_{Q,h}$ in affine 3-space

It follows from the above discussions that a Danielewski surface S isomorphic to a surface $S_{Q,h}$ in $\mathbb{A}^3_k$ admits for every polynomial $R(x, y)$ such that $R(0, y)$ is a nonzero constant a closed embedding $i_R : S \hookrightarrow \mathbb{A}^3_k$ in $\mathbb{A}^3_k$ as a surface defined by the equation $x^h z - R(x, y) \prod_{i=1}^r (y - \sigma_i(x)) = 0$ for a suitable collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$. In this section, we compare these embeddings up to the action of certain automorphisms of $\mathbb{A}^3_k$. We show that if R is not constant then the automorphisms of these surfaces do not always extend to algebraic automorphisms of the ambient space. In contrast, we establish that if $k = \mathbb{C}$ all the above embeddings are equivalent under the action of the group of holomorphic automorphisms of $\mathbb{A}^3_{\mathbb{C}}$ and that every automorphism of such a surface extend to an holomorphic automorphism of $\mathbb{A}^3_{\mathbb{C}}$.

3.1. **Algebraic and analytic equivalence.**

Here we briefly discuss the notions of algebraic and analytic equivalences of closed embeddings of a given affine algebraic surface in the affine 3-space.

3.1. Given two irreducible affine algebraic surfaces S_1 and S_2 defined over a field k, we consider closed embeddings $i_{P_1} : S_1 \hookrightarrow \mathbb{A}^3_k$ and $i_{P_2} : S_2 \hookrightarrow \mathbb{A}^3_k$ of S_1 and S_2 into a fixed affine 3-space $\mathbb{A}^3_k = \mathrm{Spec}(k[x, y, z])$ as surfaces $V(P_1) = \mathrm{Spec}(k[x, y, z] / (P_1))$ and $V(P_2) = \mathrm{Spec}(k[x, y, z,] / (P_2))$ defined by the equations $P_1 = 0$ and $P_2 = 0$ respectively. In this setting, a morphism $\phi : S_2 \to S_1$ is said to be *algebraically extendable* to $\mathbb{A}^3_k$ with respect to the pair (i_{P_1}, i_{P_2}) if there exists an endomorphism $\Phi : \mathbb{A}^3_k \to \mathbb{A}^3_k$ such that $i_{P_1} \circ \phi = \Phi \circ i_{P_2}$.

Definition 3.2. Two closed embeddings $i_{P_1} : S \hookrightarrow \mathbb{A}^3_k$ and $i_{P_2} : S \hookrightarrow \mathbb{A}^3_k$ of a same irreducible affine algebraic surface S are called *algebraically equivalent* if one of the following equivalent conditions is satisfied:

1) $\mathrm{id}_S : S \to S$ is algebraically extendable to an automorphism Φ of $\mathbb{A}^3_k$.

2) There exists an algebraic automorphism Φ of $\mathbb{A}^3_k$ such that $\Phi^* P_1 = \lambda P_2$ for a certain nonzero constant $\lambda \in k^*$.

3) The algebraic families $P_1 : \mathbb{A}^3_k \to \mathbb{A}^1_k$ and $P_2 : \mathbb{A}^3_k \to \mathbb{A}^1_k$ of level surfaces of the polynomials P_1 and P_2 respectively are isomorphic in the sense that they fit in a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^3_k & \xrightarrow{\Phi} & \mathbb{A}^3_k \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_2} \\ \mathbb{A}^1 & \xrightarrow{\phi} & \mathbb{A}^1 \end{array}$$

where Φ and ϕ are automorphisms of $\mathbb{A}^3_k$ and $\mathbb{A}^1_k$ respectively.

Remark 3.3. Over a base field k of characteristic zero, the above characterisations of algebraic equivalence are also equivalent to the following one:

4) There exists an automorphism ϕ of $\mathbb{A}^1_k$ and an algebraic family of endomorphisms of $\mathbb{A}^3_k$

$$\tilde{\Phi} : \mathbb{A}^1_k \times \mathbb{A}^3_k \to \mathbb{A}^3_k, \quad (t, x, y, z) \mapsto \Phi_t(x, y, z) = \tilde{\Phi}(t; x, y, z),$$

parametrized by $\mathbb{A}^1_k$ and inducing isomorphisms $\Phi_t : P_1^{-1}(t) \xrightarrow{\sim} P_2^{-1}(\phi(t))$ for every $t \in \mathbb{A}^1_k$.

Indeed, since the level surfaces of P_1 and P_2 cover $\mathbb{A}^3_k$, the condition guarantees that $\Phi(x,y,z) = \tilde{\Phi}(P_1(x,y,z), x, y, z)$ is a bijective endomorphism of $\mathbb{A}^3_k$, whence an isomorphism by virtue of Zariski Main Theorem, such that $P_2 \circ \Phi = \psi \circ P_1$.

3.4. I what follows, we will also discuss extensions of an algebraic morphism $\phi : S_2 \to S_1$ to an holomorphic map $\Phi : \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$. For closed embeddings of a given complex affine variety, this leads to the notion of *analytic equivalence* that we use in 3.9 below. Note that analytic equivalence is a weaker requirement than algebraic equivalence. Indeed, two hypersurfaces $V(P_1)$ and $V(P_2)$ of $\mathbb{A}^3_{\mathbb{C}}$ as in Definition 3.2 above are analytically equivalent if and only if there exists an automorphism $\Phi^* : \mathcal{H}(\mathbb{A}^3_{\mathbb{C}}) \overset{\sim}{\to} \mathcal{H}(\mathbb{A}^3_{\mathbb{C}})$, where $\mathcal{H}(\mathbb{A}^3_{\mathbb{C}})$ denotes the algebra of holomorphic functions on $\mathbb{A}^3_{\mathbb{C}}$, such that $\Phi^*(P) = \lambda Q$ for a certain nowhere vanishing holomorphic function $\lambda \in \mathcal{H}(\mathbb{A}^3_{\mathbb{C}})$. Since there are many nonconstant holomorphic functions with this property on $\mathbb{A}^3_{\mathbb{C}}$, the corresponding holomorphic map $\Phi : \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$ need not preserve the algebraic families of level surfaces $P_1 : \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ and $P_2 : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$.

3.2. **Algebraic and analytic equivalence of embeddings of Danielewski surfaces $S_{Q,h}$.** Here we consider equivalence of embeddings in $\mathbb{A}^3_k$ of a Danielewski surface isomorphic to a surface $S_{Q,h}$. The study of certain of these embeddings has been initiated by G. Freudenburg and L. Moser-Jauslin [11] and developed further by Moser-Jauslin and the second author [17]. Our purpose here is to illustrate by explicit examples of algebraically inequivalent embeddings what kind of phenomena can arise in the algebraic context. Indeed, we establish in contrast that if $k = \mathbb{C}$ then the embeddings of a given Danielewski surface S as a surface $S_{Q,h}$ in $\mathbb{A}^3_{\mathbb{C}}$ are all holomorphically equivalent.

3.5. It follows from Lemma 2.3 above that every Danielewski surface S isomorphic to a surface $S_{Q,h}$ in $\mathbb{A}^3_k$ admits a closed embedding in $\mathbb{A}^3_k$ as a surface defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$ for a suitable collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$. It is natural to ask if every other closed embedding of S as a surface $S_{Q,h}$ is algebraically equivalent to the previous one. In 2002, G. Freudenburg and L. Moser-Jauslin [11] discovered that the answer is negative in general.

Example 3.6. Given an integer $h \geq 2$ and polynomial $P(y) \in \mathbb{C}[y]$ with $r \geq 2$ simples roots $y_1, \ldots, y_r$, we consider the Danielewski surfaces S_0 and S_1 in $\mathbb{A}^3_{\mathbb{C}}$ defined respectively by the following equations :

$$f_0 = x^h z - P(y) = 0 \qquad f_1 = x^h z - (1-x) P(y) = 0.$$

Note that S_0 is simply a normalized embedding of S_1 in $\mathbb{A}^3_{\mathbb{C}}$. We claim that these embeddings are not algebraically equivalent. Indeed, for every $t \in \mathbb{C}^*$ such that the polynomial $P(y) - t$ has a multiple root, the level surface $f_0^{-1}(t)$ of f_0 is a singular surface. On the other hand, all the level surfaces of f_1 are nonsingular as follows for instance from the Jacobian Criterion. So condition 3) in Definition 3.2 cannot be satisfied.

3.7. In view on the above example, it is very tempting to believe that every closed embedding of S_0 as a surface with equation $x^h z - R(x,y) P(y) = 0$, where $R(x,y)$ is nonconstant polynomial, is algebraically inequivalent to the original one. It turns out that it is not the case in general and that the classification of these embeddings up to algebraic equivalence is a difficult problem. For instance, a slight modification of the above example leads to the following surprising fact which generalizes a result of L. Moser-Jauslin and the second author [17].

Proposition 3.8. *For every polynomial $P(y) \in \mathbb{C}[y]$ of degree $r \geq 2$ and every polynomial $R \in \mathbb{C}[y]$, the surfaces in $\mathbb{A}^3_{\mathbb{C}}$ defined by the equations $f_R(x,y,z) = x^2 z - (1 - xP'(y) R(P(y))) P(y) = 0$ are algebraically equivalent embeddings of a same surface S.*

Proof. It is enough to show that for every R, there exists an automorphism $\Psi^* = \Psi^*_R$ of $\mathbb{C}[x,y,z]$ such that $\Psi^* f_R = f_0$. To construct Ψ^* we first define a one-parameter family $(\psi^*_t)_{t \in \mathbb{C}}$ of endomorphisms of $\mathbb{C}[x,y,z]$ such that for every $t \in \mathbb{C}$, $\psi^*_t(f_R + t)$ belongs to the ideal generated by $f_0 + t$. For every $t \in \mathbb{C}$, we let $S_t(y) = (P(y) - t)^{-1}(tR(t) - P(y) R(P(y))) \in \mathbb{C}[y,t]$, and we let ψ^*_t be the $\mathbb{C}[x]$-endomorphism of $\mathbb{C}[x,y,z]$ $y\mapsto y+xtR(t)$, $z\mapsto (1+xP'(y)S_t(y))z+s_t(x,y)$, where $s_t(x,y)\in\mathbb{C}[x,y,t]$ is the unique polynomial such that

$$P(\psi_t^*(y))-xP'(\psi_t^*(y))R(P(\psi_t^*(y)))P(\psi_t^*(y))-t \quad = \quad (P(y)-t)(1+xP'(y)S_t(y))+x^2s_t(x,y).$$

It follows that $\psi_t^*(f_R+t)=(1+xP'(y)S_t(y))(f_0+t)$ for every $t\in\mathbb{C}$. One checks that the $\mathbb{C}[x]$-endomorphism $\Psi^*=\psi_{-f_0}^*$ of $\mathbb{C}[x,y,z]$ satisfies $\Psi^*f_R=f_0$. To show that Ψ^* is an automorphism, it is enough to check that $\mathbb{C}[\Psi^*x,\Psi^*y,\Psi^*z]=\mathbb{C}[x,y,z]$. Since $\Psi^*x=x$ and $\Psi^*f_R=f_0$, it follows that $y=\Psi^*(y+xf_RR(-f_R))$. In turn, this implies that the polynomial $(1+xP'(y)S_{-f_0}(y))z=\Psi^*z-s_{-f_0}(x,y)$ and $(1-xP'(y)S_{-f_0}(y))$, whence the polynomial $(1-xP'(y)S_{-f_0}(y))(1+xP'(y)S_{-f_0}(y))z$ belong to $\mathbb{C}[\Psi^*x,\Psi^*y,\Psi^*z]$. We eventually conclude that $z-x^2z(P'(y)S_{-f_0})^2=z-(f_0+P(y))(P'(y)S_{-f_0})^2$ is in the image of Ψ^* and so is z too. This completes the proof. □

It turns out that things become simpler if one works in the holomorphic category. Indeed, we have the following result.

Theorem 3.9. *The embeddings $i_{Q,h}:S\hookrightarrow\mathbb{A}^3_{\mathbb{C}}$ of a Danielewski surface S as a surface defined by the equation $x^hz-Q(x,y)=0$ are all* analytically *equivalent.*

Proof. It suffices to show that every embedding $i_{Q,h}$ is analytically equivalent to a normalized one $i_{\sigma,h}$. In view of Propositon 2.3, we can let $Q(x,y)=R_1(x,y)\prod_{i=1}^r(y-\sigma_i(x))+x^hR_2(x,y)$. It is enough to construct an holomorphic automorphism Ψ of $\mathbb{A}^3_{\mathbb{C}}$ such that $\Psi^*\left(x^hz-\prod_{i=1}^r(y-\sigma_i(x))\right)=\alpha\left(x^hz-Q(x,y)\right)$ for a suitable invertible holomorphic function α on $\mathbb{A}^3_{\mathbb{C}}$. We let $R_1(0,y)=\lambda\in\mathbb{C}^*$ and we let $f(x,y)\in\mathbb{C}[x,y]$ be a polynomial such that $\lambda\exp(xf(x,y))\equiv R_1(x,y) \bmod x^h$. Now the result follows from the fact that the holomorphic automorphism Ψ of $\mathbb{A}^3_{\mathbb{C}}$ defined by

$$\Psi(x,y,z) \quad = \quad \left(x,y,\lambda\exp(xf(x,y))z-x^{-h}(\lambda\exp(xf(x,y))-R_1(x,y))\prod_{i=1}^{r}(y-\sigma_i(x))+R_2(x,y)\right)$$

satisfies $\Psi^*\left(x^hz-Q(x,y)\right)=\lambda\exp(xf(x,y))\left(x^hz-\prod_{i=1}^r(y-\sigma_i(x))\right)$. □

Example 3.10. In Example 3.6 above, we observed that the surfaces S_0 and S_1 defined by the equations $f_0=x^2z-P(y)=0$ and $f_1=x^2z-(1-x)P(y)=0$ are algebraically inequivalent embeddings of a same surface S. However, they are analytically equivalent via the automorphism $(x,y,z)\mapsto\left(x,y,e^{-x}z-x^{-2}(e^{-x}-1+x)P(y)\right)$ of $\mathbb{A}^3_{\mathbb{C}}$.

3.3. **Algebraic and analytic extension of automorphisms.**

We have seen in Theorem 2.5 above that every automorphism of a Danielewski surface $S_{\sigma,h}\subset\mathbb{A}^3_k$ defined by the equation $x^hz-\prod_{i=1}^r(y-\sigma_i(x))=0$ can be extended to an automorphism of $\mathbb{A}^3_k$. In this subsection, we investigate the case of surfaces $S_{Q,h}$ in general and we show that some of them admit automorphisms which are not algebraically extendable to the whole of $\mathbb{A}^3_k$.

3.11. In what follows, we work over an algebraically closed field k of characteristic zero. We consider a surface $S\subset\mathbb{A}^3_k$ defined by the equation $x^hz-(1-x)P(y)=0$, where $h\geq 2$ and where $P(y)$ is a nonconstant polynomial of degree $r\geq 2$. Letting, ψ^* and ϕ^* be the $k[x,y]$-algebra endomorphisms of $k[x,y,z]$ defined by $\psi^*(z)=(1-x)z$ and $\phi^*(z)=\left(\sum_{i=0}^{h-1}x^i\right)z+P(y)$, one checks that the corresponding endomorphisms ϕ and ψ of $\mathbb{A}^3_k$ induce isomorphisms between S and the surface $S_{P,h}$ defined by the equation $x^hz-P(y)=0$. The latter admits an action $\theta:\mathbb{G}_{m,k}\times S_{P,h}\to S_{P,h}$ of the multiplicative group $\mathbb{G}_{m,k}$ defined by $\theta(a,x,y,z)=H_a(x,y,z)=\left(ax,y,a^{-h}z\right)$ for every $a\in k^*$. It turns out that corresponding action $\tilde{\theta}$ on S defined by $\tilde{\theta}(a,x,y,z)=\tilde{\theta}_a(x,y,z)=\phi\circ H_a(x,y,z)\mid_{S_{P,h}}\circ\psi$ does not extend to a $\mathbb{G}_{m,k}$-action on $\mathbb{A}^3_k$. Indeed, we have the following result.

Theorem 3.12. *Every Danielewski surface $S\subset\mathbb{A}^3_k$ defined by the equation $x^hz-(1-x)P(y)=0$, where $h\geq 2$ and where $P(y)$ has $r\geq 2$ simple roots, admits a nontrivial $\mathbb{G}_{m,k}$-action $\tilde{\theta}:\mathbb{G}_{m,k}\times$*

$S \to S$ which is not algebraically extendable to $\mathbb{A}^3_k$. More precisely, for every $a \in k \setminus \{0,1\}$ the automorphism $\tilde{\theta}_a = \tilde{\theta}(a, \cdot)$ of S do not extend to an algebraic automorphism of $\mathbb{A}^3_k$.

Since by construction, $\tilde{\theta}^*_a(x) = ax$ for every $a \in k^*$, the assertion is a consequence of the following Lemma which guarantees that the automorphism $\tilde{\theta}_a$ of S is not algebraically extendable to an automorphism of $\mathbb{A}^3_k$.

Lemma 3.13. *If Φ is an algebraic automorphism of $\mathbb{A}^3_k$ extending an automorphism of S, then $\Phi^*(x) = x$.*

Proof. Our proof is similar as the one of Theorem 2.1 in [17]. We let Φ be an automorphism of $\mathbb{A}^3_k$ extending an arbitrary automorphism of S. Since $f_1 = x^h z - (1-x) P(y)$ is an irreducible polynomial, there exists $\mu \in k^*$ such that $\Phi^*(f_1) = \mu f_1$. Therefore, for every $t \in k$, the automorphism Φ induces an isomorphism between the level surfaces $f_1^{-1}(t)$ and $f_1^{-1}(\mu^{-1}t)$ of f_1. There exists an open subset $U \subset \mathbb{A}^1_k$ such that for every $t \in U$, $f_1^{-1}(t)$ is a special Danielewski surfaces isomorphic to a one defined by a fine k-weighted rake γ whose underlying tree Γ is isomorphic to the one associated with S. Since Γ is not a comb, it follows from Theorem 1.15 that for every $t \in U$, the projection $\mathrm{pr}_x : f_1^{-1}(t) \to \mathbb{A}^1_{\mathbb{C}}$ is a unique $\mathbb{A}^1$-fibration on $f_1^{-1}(t)$ up to automorphisms of the base. Furthermore, $\mathrm{pr}_x : f_1^{-1}(t) \to \mathbb{A}^1_k$ has a unique degenerate fiber, namely $\mathrm{pr}_x^{-1}(0)$. Therefore, for every $t \in U$, the image of the ideal $(x, f_1 - t)$ of $k[x,y,z]$ by Φ^* is contained in the ideal $(x, \mu f_1 - t) = (x, P(y) + \mu^{-1}t)$, and so $\Phi^*(x) \in \bigcap_{t \in U} (x, P(y) + \mu^{-1}t) = (x)$. Since Φ is an automorphism of $\mathbb{A}^3_k$, we conclude that there exists $c \in k^*$ such that $\Phi^*(x) = cx$. In turn, this implies that for every $t, u \in k$, Φ induces an isomorphism between the surfaces $S_{t,u}$ and $\tilde{S}_{t,u}$ defined by the equations $f_1 + tx + u = x^h z - (1-x) P(y) + tx + u = 0$ and $f_1 + \mu^{-1}ctx + \mu^{-1}u = x^h z - (1-x) P(y) + \mu^{-1}ctx + \mu^{-1}u = 0$ respectively. Since $\deg(P) \geq 2$ there exists $y_0 \in k$ such that $P'(y_0) = 0$. Note that y_0 is not a root of P as these ones are simple . We let $t = -u = -P(y_0)$. Since $h \geq 2$, it follows from the Jacobian Criterion that $S_{t,u}$ is singular, and even non normal along the nonreduced component of the fiber $\mathrm{pr}_x^{-1}(0)$ defined by the equation $\{x = 0;\, y = y_0\}$. Therefore $\tilde{S}_{t,u}$ must be singular along a multiple component of the fiber $\mathrm{pr}_x^{-1}(0)$. This the case if and only if the polynomial $P(y) - \mu^{-1}cP(y_0)$ has a multiple root, say y_1, such that $P(y_1) - \mu^{-1}P(y_0) = 0$. Since $P(y_0) \neq 0$ this condition is satisfied if and only if $c = 1$. This completes the proof. □

Example. In particular, even the involution $J(x,y,z) = (-x, y, (1+x)((1+x)z + P(y)))$ of the surface S defined by the equation $x^2 z - (1-x) P(y) = 0$ does not extend to an algebraic automorphism of $\mathbb{A}^3_k$.

3.14. By virtue of Proposition 2.14 above a Danielewski surface $S \simeq S_{Q,h}$ admits a nontrivial $\mathbb{C}^*$-action if and only if it is isomorphic to one defined by the equation $x^h z - P(y) = 0$. If this holds then S admits a normalized closed embedding in $\mathbb{A}^3_{\mathbb{C}}$ for which this $\mathbb{C}^*$-action comes as the restriction of a linear $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$. On the other hand, Theorem 3.12 above shows that S may admit other embeddings, which are not equivalent to their associated normalized one, for which this action does not extend to the ambient space. This leads to the following natural question:

Conjecture : For a closed embedding $i_{Q,h} : S \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$ of a Danielewski surface S of quasi-homogeneous normal type (see Definition 2.15 above) as a surface $S_{Q,h}$ the following are equivalent:

1) $i_{Q,h}$ is algebraically equivalent to a normalized embedding $i_{\sigma,h}$ of S.
2) There exists a $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$ extending the one on $S_{Q,h}$.

3.15. It follows from 3.9 above that every closed embedding in $\mathbb{A}^3_{\mathbb{C}}$ of a Danielewski surface as a surface $S_{Q,h}$ is analytically equivalent to a normalized one $i_{\sigma,h}$. In turn, Theorem 2.5 implies that every automorphism of $S_{\sigma,h}$ can be extended to an algebraic automorphism of $\mathbb{A}^3_{\mathbb{C}}$. This leads to the following result.

Corollary 3.16. *Every algebraic automorphism of a Danielewski surface $S_{Q,h}$ in $\mathbb{A}^3_{\mathbb{C}}$ is extendable to a* holomorphic *automorphism of $\mathbb{A}^3_{\mathbb{C}}$.*

This implies in particular that the $\mathbb{C}^*$-action on the surface S defined by the equation $x^h z - (1-x)\,P(y) = 0$ defined in 3.11 extends to an holomorphic $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$. This contrasts with an example of a non-extendable $\mathbb{C}_+$-action on an hypersurface in $\mathbb{A}^5_{\mathbb{C}}$ which is even holomorphically inextendable given by H. Derksen, F. Kutzschebauch and J. Winkelmann [6].

Corollary 3.17. *Every surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^h z - (1-x)\,P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{C}^*$-action which is algebraically inextendable but holomorphically extendable to $\mathbb{A}^3_{\mathbb{C}}$.*

Example 3.18. We let $\tilde{\theta} : \mathbb{C}^* \times S \to S$ be the $\mathbb{C}^*$-action on the surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^2 z - (1-x)\,P(y) = 0$ constructed in 3.11. For every $a \in \mathbb{C}^*$, the automorphism $\tilde{\theta}(a,\cdot)$ of S maps a closed point $(x,y,z) \in S$ to the point $\tilde{\theta}(a,x,y,z) = \left(ax, y, a^{-2}(1-ax)\left((1+x)\,z + P(y)\right)\right)$. One checks that the holomorphic automorphism Φ_a of $\mathbb{A}^3_{\mathbb{C}}$ such that $\Phi_a\mid_S = \tilde{\theta}(a,\cdot)$ is the following one:

$$\Phi_a(x,y,z) = \left(ax, y, a^{-2}e^{(1-a)x}z + (ax)^{-2}\,P(y)\left(e^{(1-a)x}(x-1) - ax + 1\right)\right).$$

Clearly, the holomorphic map $\Phi : \mathbb{C}^* \times \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$, $(a,(x,y,z)) \mapsto \Phi_a(x,y,z)$ defines a $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$ extending the one $\tilde{\theta}$ on S.

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Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 Saint-Martin d'Hères Cedex, France
E-mail address: adrien.dubouloz@ujf-grenoble.fr

Institut de Mathématiques de Bourgogne, CNRS-UMR 5584, 9 avenue Alain Savary, BP 47870, 21078 Dijon, France
E-mail address: pierre-marie.poloni@u-bourgogne.fr