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ADRIEN DUBOULOZ AND PIERRE-MARIE POLONI

Abstract. In [16] and [17], L. Makar-Limanov computed the automorphism groups of surfaces in $\mathbb{C}^3$ defined by the equations $x^n z - P(y) = 0$, where $n \geq 1$ and $P(y)$ is a nonzero polynomial. Similar results have been obtained by A. Crachiola [3] for surfaces defined by the equations $x^n z - y^2 - \sigma(x)\, y = 0$, where $n \geq 2$ and $\sigma(0) \neq 0$, defined over an arbitrary base field. Here we consider the more general surfaces defined by the equations $x^n z - Q(x, y) = 0$, where $n \geq 2$ and $Q(x, y)$ is a polynomial with coefficients in an arbitrary base field k. We characterise among them the ones which are Danielewski surfaces in the sense of [8], and we compute their automorphism groups. We study closed embeddings of these surfaces in affine 3-space. We show that in general their automorphisms do not extend to the ambient space. Finally, we give explicit examples of $\mathbb{C}^*$-actions on a surface in $\mathbb{A}^3_{\mathbb{C}}$ which can be extended holomorphically but not algebraically to $\mathbb{C}^*$-actions on $\mathbb{A}^3_{\mathbb{C}}$.

Introduction

Since they appeared in a celebrated counterexample to the Cancellation Problem due to W. Danielewski [5], the surfaces defined by the equations $xz - y(y-1) = 0$ and $x^2 z - y(y-1) = 0$ in $\mathbb{C}^3$ and their natural generalisations, such as surfaces defined by the equations $x^n z - P(y) = 0$, where $P(y)$ is a nonconstant polynomial, have been studied in many different contexts. Of particular interest is the fact that they can be equipped with nontrivial actions of the additive group $\mathbb{C}_+$. The general orbits of these actions coincide with the general fibers of $\mathbb{A}^1$-fibrations $\pi : S \to \mathbb{A}^1$, that is, surjective morphisms with generic fiber isomorphic to an affine line. Normal affine surfaces S equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ can be roughly classified into two classes according the following alternative : either $\pi : S \to \mathbb{A}^1$ is a unique $\mathbb{A}^1$-fibration on S up to automorphisms of the base, or there exists a second $\mathbb{A}^1$-fibration $\pi' : S \to \mathbb{A}^1$ with general fibers distinct from the ones of π.

Due to the symmetry between the variables x and z, a surface defined by the equation $xz - P(y) = 0$ admits two distinct $\mathbb{A}^1$-fibrations over the affine line. In contrast, it was established by L. Makar-Limanov [17] that on a surface $S_{P,n}$ defined by the equation $x^n z - P(y) = 0$ in $\mathbb{C}^3$, where $n \geq 2$ and where $P(y)$ is a polynomial of degree $r \geq 2$, the projection $\mathrm{pr}_x : S_{P,n} \to \mathbb{C}$ is a unique $\mathbb{A}^1$-fibration up to automorphisms of the base. In his proof, L. Makar-Limanov used the correspondence between algebraic $\mathbb{C}_+$-actions on an affine surface S and locally nilpotent derivations of the algebra of regular functions on S. It turns out that his proof is essentially independent of the base field k provided that we replace locally nilpotent derivations by suitable systems of Hasse-Schmidt derivations when the characteristic of k is positive (see e.g., [3]).

The fact that an affine surface S admits a unique $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1$ makes its study simpler. For instance, every automorphism of S must preserve this fibration. In this context, a result due to J. Bertin [2] asserts that the identity component of the automorphisms group of such a surface is an algebraic pro-group obtained as an increasing union of solvable algebraic subgroups of rank ≤ 1. For surfaces defined by the equations $x^n z - P(y) = 0$ in $\mathbb{C}^3$, the picture has been completed by L. Makar-Limanov [17] who gave explicit generators of their automorphisms groups. Similar results have been obtained over arbitrary base fields by A. Crachiola [3] for surfaces defined by the equations $x^n z - y^2 - \sigma(x)\, y = 0$, where $\sigma(x)$ is a polynomial such that $\sigma(0) \neq 0$.

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The latter surfaces are particular examples of a general class of $\mathbb{A}^1$-fibred surfaces called *Danielewski surfaces* [8], that is, normal integral affine surface S equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ over an affine line with a fixed k-rational point o, such that every fiber $\pi^{-1}(x)$, where $x \in \mathbb{A}^1_k \setminus \{o\}$, is geometrically integral, and such that every irreducible component of $\pi^{-1}(o)$ is geometrically integral. In this article, we consider Danielewski surfaces $S_{Q,n}$ in $\mathbb{A}^3_k$ defined by an equation of the form $x^n z - Q(x,y) = 0$, where $n \geq 2$ and where $Q(x,y) \in k[x,y]$ is a polynomial such that $Q(0,y)$ splits with $r \geq 2$ simple roots in k. This class contains most of the surfaces considered by L. Makar-Limanov and A. Crachiola.

The paper is organised as follows. First, we briefly recall definitions about weighted rooted trees and the notion of equivalence of algebraic surfaces in an affine 3-space. In section 2, we recall from [8] the main facts about Danielewski surfaces and we review the correspondence between these surfaces and certain classes of weighted trees in a form appropriate to our needs. We also generalise to arbitrary base fields k some results which are only stated for fields of characteristic zero in [7] and [8]. In particular, the case of Danielewski surfaces which admit two $\mathbb{A}^1$-fibrations with distinct general fibers is studied in Theorem 2.11. We show that these surfaces correspond to Danielewski surfaces $S(\gamma)$ defined by the fine k-weighted trees γ which are called *combs* and we give explicit embeddings of them. This result generalises Theorem 4.2 in [9].

In section 3, we classify Danielewski surfaces $S_{Q,h}$ in $\mathbb{A}^3_k$ defined by equations of the form $x^h z - Q(x,y) = 0$ and determine their automorphism groups. We remark that such a surface admits many embeddings as a surface $S_{Q,h}$. In particular, we establish in Theorem 3.2 that these surfaces can always be embedded as surface $S_{\sigma,h}$ defined by an equation of the form $x^h z - \prod_{i=1}^{r}(y - \sigma_i(x)) = 0$ for a suitable collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$. We say that these surfaces $S_{\sigma,h}$ are *standard form* of Danielewski surfaces $S_{Q,h}$. Next, we compute (Theorem 3.10) the automorphism groups of Danielewski surfaces in standard form. We show in particular that any of them comes as the restriction of an algebraic automorphism of the ambient space.

Finally we consider the problem of extending automorphisms of a given Danielewski surface $S_{Q,h}$ to automorphisms of the ambient space $\mathbb{A}^3_k$. We show that this is always possible in the holomorphic category but not in the algebraic one. We give explicit examples which come from the study of multiplicative group actions on Danielewski surfaces. For instance, we prove that every surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^h z - (1-x)P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{C}^*$-action which is algebraically inextendable but holomorphically extendable to $\mathbb{A}^3_{\mathbb{C}}$. In particular, even the involution of the surface S defined by the equation $x^2 z - (1-x)P(y) = 0$ induced by the endomorphism $J(x,y,z) = (-x, y, (1+x)((1+x)z + P(y)))$ of $\mathbb{A}^3_k$ does not extend to an algebraic automorphism of $\mathbb{A}^3_k$.

1. Preliminaries

1.1. Basic facts on weighted rooted trees.

Definition 1.1. A *tree* is a nonempty, finite, partially ordered set $\Gamma = (\Gamma, \leq)$ with a unique minimal element e_0 called the *root*, and such that for every $e \in \Gamma$ the subset $(\downarrow e)_\Gamma = \{e' \in \Gamma,\, e' \leq e\}$ is a chain for the induced ordering.

1.2. A minimal sub-chain $\overleftarrow{e'e} = \{e' < e\}$ with two elements of a tree Γ is called *an edge* of Γ. We denote the set of all edges in Γ by $E(\Gamma)$. An element $e \in \Gamma$ such that $\mathrm{Card}(\downarrow e)_\Gamma = m$ is said to be *at level* m. The maximal elements $e_i = e_{i,m_i}$, where $m_i = \mathrm{Card}(\downarrow e_i)_\Gamma$ of Γ are called the *leaves* of Γ. We denote the set of those elements by $L(\Gamma)$. The maximal chains of Γ are the chains

$$\Gamma_{e_{i,m_i}} = (\downarrow e_{i,m_i})_\Gamma = \{e_{i,0} = e_0 < e_{i,1} < \cdots < e_{i,m_i}\}, \quad e_{i,m_i} \in L(\Gamma). \tag{1.1}$$

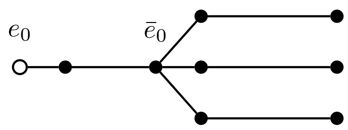
We say that Γ has *height* $h = \max(m_i)$. The *children* of an element $e \in \Gamma$ are the elements of Γ at relative level 1 with respect to e, i.e., the maximal elements of the subset $\{e' \in \Gamma,\, e' > e\}$ of Γ.

Definition 1.3. A *fine k-weighted tree* $\gamma = (\Gamma, w)$ is a tree Γ equipped with a weight function $w : E(\Gamma) \to k$ with values in a field k, which assigns an element $w\left(\overleftarrow{e'e}\right)$ of k to every edge $\overleftarrow{e'e}$

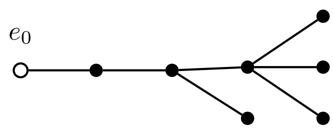
of Γ, in such a way that $w\left(\overleftarrow{e'e_1}\right) \neq w\left(\overleftarrow{e'e_2}\right)$ whenever e_1 and e_2 are distinct children of a same element e'.

In what follows, we frequently consider the following classes of trees.

Definition 1.4. Let Γ be a rooted tree.
a) If all the leaves of Γ are at the same level $h \geq 1$ and if there exists a unique element $\bar{e}_0 \in \Gamma$ for which $\Gamma \setminus \{\bar{e}_0\}$ is a nonempty disjoint union of chains then we say that Γ is a *rake*.
b) If $\Gamma \setminus L(\Gamma)$ is a chain then we say that Γ is a *comb*. Equivalently, Γ is a comb if and only if every $e \in \Gamma \setminus L(\Gamma)$ has at most one child which is not a leaf of Γ.



A rake rooted in e_0.



A comb rooted in e_0.

1.2. **Algebraic and analytic equivalence of closed embeddings.**
Here we briefly discuss the notions of algebraic and analytic equivalences of closed embeddings of a given affine algebraic surface in an affine 3-space.

Let S be an irreducible affine surface and let $i_{P_1} : S \hookrightarrow \mathbb{A}^3_k$ and $i_{P_2} : S \hookrightarrow \mathbb{A}^3_k$ be embeddings of S in a same affine 3-space as closed subschemes defined by polynomial equations $P_1 = 0$ and $P_2 = 0$ respectively.

Definition 1.5. In the above setting, we say that the closed embeddings i_{P_1} and i_{P_2} are *algebraically equivalent* if one of the following equivalent conditions is satisfied:
1) There exists an automorphism Φ of $\mathbb{A}^3_k$ such that $i_{P_2} = i_{P_1} \circ \Phi$.
2) There exists an automorphism Φ of $\mathbb{A}^3_k$ and a nonzero constant $\lambda \in k^*$ such that $\Phi^* P_1 = \lambda P_2$.
3) There exists automorphisms Φ and ϕ of $\mathbb{A}^3_k$ and $\mathbb{A}^1_k$ respectively such that $P_2 \circ \Phi = \phi \circ P_1$.

1.6. Over the field $k = \mathbb{C}$ of complex numbers, one can also consider holomorphic automorphisms. With the notation of definition 1.5, two closed algebraic embeddings i_{P_1} and i_{P_2} of a given affine surface S in $\mathbb{A}^3_{\mathbb{C}}$ are called *holomorphically equivalent* if there exists a biholomorphism $\Phi : \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$ such that $i_{P_2} = i_{P_1} \circ \Phi$. Clearly, the embeddings i_{P_2} and i_{P_1} are holomorphically equivalent if and only if there exists a biholomorphism $\Phi : \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$ such that $\Phi^*(P_1) = \lambda P_2$ for a certain nowhere vanishing holomorphic function λ. Since there are many nonconstant holomorphic functions with this property on $\mathbb{A}^3_{\mathbb{C}}$, Φ need not preserve the algebraic families of level surfaces $P_1 : \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ and $P_2 : \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$. So holomorphic equivalence is a weaker requirement than algebraic equivalence.

2. Danielewski surfaces

For certain authors, a Danielewski surface is an affine surface S which is algebraically isomorphic to a surface in $\mathbb{C}^3$ defined by an equation of the form $x^n z - P(y) = 0$, where $n \geq 1$ and $P(y) \in \mathbb{C}[y]$. These surfaces come equipped with a surjective morphism $\pi = \mathrm{pr}_x |_S : S \to \mathbb{A}^1$ restricting to a trivial $\mathbb{A}^1$-bundle over the complement of the origin. Moreover, if the roots $y_1, \ldots, y_r \in \mathbb{C}$ of $P(y)$ are simple, then the fibration $\pi = \mathrm{pr}_x |_S : S \to \mathbb{A}^1$ factors through a locally trivial fiber bundle over the affine line with an r -fold origin (see e.g., [5] and [11]). In [8], the first author used the term Danielewski surface to refer to an affine surface S equipped with a morphism $\pi : S \to \mathbb{A}^1$ which factors through a locally trivial fiber bundle in a similar way as above. In what follows, we keep this point of view, which leads to a natural geometric generalisation of the surfaces constructed by W. Danielewski [5]. We recall that an $\mathbb{A}^1$-*fibration* over an integral scheme Y is a faithfully flat (i.e., flat and surjective) affine morphism $\pi : X \to Y$ with generic fiber isomorphic to the affine line $\mathbb{A}^1_{K(Y)}$ over the function field $K(Y)$ of Y. The following definition is a generalisation to arbitrary base fields k of the one introduced in [8].

Definition 2.1. A *Danielewski surface* is an integral affine surface S defined over a field k, equipped with an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ restricting to a trivial $\mathbb{A}^1$-bundle over the complement of the a k-rational point o of $\mathbb{A}^1_k$ and such that the fiber $\pi^{-1}(o)$ is reduced, consisting of a disjoint union of affine lines $\mathbb{A}^1_k$ over k.

Notation 2.2. In what follows, we fix an isomorphism $\mathbb{A}^1_k \simeq \mathrm{Spec}(k[x])$ and we assume that the k-rational point o is simply the "origin" of $\mathbb{A}^1_k$, that is, the closed point (x) of $\mathrm{Spec}(k[x])$.

2.3. In the following subsections, we recall the correspondence between Danielewski surfaces and weighted rooted trees established by the first author in [8] in a form appropriate to our needs. Although the results given in *loc. cit.* are formulated for surfaces defined over a field of characteristic zero, most of them remain valid without any changes over a field of arbitrary characteristic. We provide full proofs only when additional arguments are needed. Then we consider Danielewski surfaces S with a trivial canonical sheaf $\omega_{S/k} = \Lambda^2\Omega^1_{S/k}$. We call them *special Danielewski surfaces.* We give a complete classification of these surfaces in terms of their associated weighted trees.

2.1. **Danielewski surfaces and weighted trees.**

Here we review the correspondence which associates to every fine k-weighted tree $\gamma = (\Gamma, w)$ a Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k = \mathrm{Spec}(k[x])$ which is the total space of an $\mathbb{A}^1$-bundle over the scheme $\delta : X(r) \to \mathbb{A}^1_k$ obtained from $\mathbb{A}^1_k$ by replacing its origin o by $r \geq 1$ k-rational points $o_1, \ldots, o_r$.

Notation 2.4. In what follows we denote by $\mathcal{U}_r = (X_i(r))_{i=1,\ldots,r}$ the canonical open covering of $X(r)$ by means of the subsets $X_i(r) = \delta^{-1}\left(\mathbb{A}^1_k \setminus \{o\}\right) \cup \{o_i\} \simeq \mathbb{A}^1_k$.

2.5. Let $\gamma = (\Gamma, w)$ be a fine k-weighted tree $\gamma = (\Gamma, w)$ of height h, with leaves e_i at levels $n_i \leq h$, $i = 1, \ldots, r$. To every maximal sub-chain $\gamma_i = (\downarrow e_i)$ of γ (see 1.2 for the notation) we associate a polynomial

$$\sigma_i(x) = \sum_{j=0}^{n_i-1} w\left(\overleftarrow{e_{i,j}e_{i,j+1}}\right) x^j \in k[x], \quad i = 1, \ldots, r.$$

We let $\rho : S(\gamma) \to X(r)$ be the unique $\mathbb{A}^1$-bundle over $X(r)$ which becomes trivial on the canonical open covering $\mathcal{U}_r$, and is defined by pairs of transition functions

$$(f_{ij}, g_{ij}) = \left(x^{n_j-n_i}, x^{-n_i}(\sigma_j(x) - \sigma_i(x))\right) \in k\left[x, x^{-1}\right]^2, \quad i, j = 1, \ldots, r.$$

This means that $S(\gamma)$ is obtained by gluing n copies $S_i = \mathrm{Spec}(k[x][u_i])$ of the affine plane $\mathbb{A}^2_k$ over $\mathbb{A}^1_k \setminus \{o\} \simeq \mathrm{Spec}\left(k\left[x, x^{-1}\right]\right)$ by means of the transition isomorphisms induced by the $k\left[x, x^{-1}\right]$-algebras isomorphisms

$$k\left[x, x^{-1}\right][u_i] \stackrel{\sim}{\to} k\left[x, x^{-1}\right][u_j], \quad u_i \mapsto x^{n_j-n_i}u_j + x^{-n_i}(\sigma_j(x) - \sigma_i(x)) \qquad i \neq, i, j = 1, \ldots, r.$$

This definition makes sense as the transition functions g_{ij} satisfy the twisted cocycle relation $g_{ik} = g_{ij} + x^{n_j-n_i}g_{jk}$ in $k\left[x, x^{-1}\right]$ for every triple of distinct indices i, j and k. Since γ is a fine weighted tree, it follows that for every pair of distinct indices i and j, the rational function $g_{ij} = x^{-n_i}(\sigma_j(x) - \sigma_i(x)) \in k\left[x, x^{-1}\right]$ does not extend to a regular function on $\mathbb{A}^1_k$. This condition guarantees that $S(\gamma)$ is a separated scheme, whence an affine surface by virtue of Fieseler's criterion (see proposition 1.4 in [11]). Therefore, $\pi_\gamma = \delta \circ \rho : S(\gamma) \to \mathbb{A}^1_k = \mathrm{Spec}(k[x])$ is a Danielewski surface, the fiber $\pi^{-1}(o)$ being the disjoint union of affine lines

$$C_i = \pi_\gamma^{-1}(o) \cap S_i \simeq \mathrm{Spec}(k[u_i]), \quad i = 1, \ldots, r.$$

2.6. A Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ above comes canonically equipped with a birational morphism $(\pi, \psi_\gamma) : S \to \mathbb{A}^1_k \times \mathbb{A}^1_k = \mathrm{Spec}(k[x][t])$ restricting to an isomorphism over $\mathbb{A}^1_k \setminus \{o\}$. Indeed, this morphism corresponds to the unique regular function ψ_γ on $S(\gamma)$ whose restrictions to the open subsets $S_i \simeq \mathrm{Spec}(k[x][u_i])$ of S are given by the polynomials

$$\psi_{\gamma,i} = x^{n_i}u_i + \sigma_i(x) \in k[x][u_i], \quad i = 1, \ldots, r.$$

This function is referred to as the *canonical function* on $S(\gamma)$. The morphism $(\pi_\gamma, \psi_\gamma) : S(\gamma) \to \mathbb{A}^2_k$ is called the *canonical birational morphism* from $S(\gamma)$ to $\mathbb{A}^2_k$.

2.7. It turns out that there exists a one-to-one correspondence between pairs $(S, (\pi, \psi))$ consisting of a Danielewski surface $\pi : S \to \mathbb{A}^1_k$ and a birational morphism $(\pi, \psi) : S \to \mathbb{A}^2_k$ restricting to an isomorphism outside the fiber $\pi^{-1}(o)$ and fine k-weighted trees γ. In particular, Proposition 3.4 in [8], which remains valid over arbitrary base fields k, implies the following result.

Theorem 2.8. *For every pair consisting of a Danielewski surface $\pi : S \to \mathbb{A}^1_k$ and a birational morphism $(\pi, \psi) : S \to \mathbb{A}^1_k \times \mathbb{A}^1_k$ restricting to an isomorphism over $\mathbb{A}^1_k \setminus \{o\}$, there exists a unique fine k-weighted tree γ and an isomorphism $\phi : S \xrightarrow{\sim} S(\gamma)$ such that $\psi = \psi_\gamma \circ \phi$.*

Remark 2.9. If $\gamma = (\Gamma, w)$ is not the trivial tree with one element then the canonical function $\psi_\gamma : S(\gamma) \to \mathbb{A}^1_k$ on the corresponding Danielewski surface $\pi : S(\gamma) \to \mathbb{A}^1_k$ is locally constant on the fiber $\pi^{-1}(o)$. It takes the same value on two distinct irreducible components of $\pi^{-1}(o)$ if and only if the corresponding leaves of γ belong to a same subtree of γ rooted in an element at level 1. Since every Danielewski surface nonisomorphic to $\mathbb{A}^2_k$ admits a birational morphism (π, ψ) for which ψ is locally constant but not constant on the fiber $\pi^{-1}(o)$, it follows that every such surface correspond to a tree γ with at least two elements at level 1.

2.2. $\mathbb{A}^1$-fibrations on Danielewski surfaces.

Suppose that the structural $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ on a Danielewski surface S is unique up to automorphisms of the base. Then a second Danielewski surface $\pi' : S' \to \mathbb{A}^1_k$ will be isomorphic to S as an abstract surface if and only if it is isomorphic to S as a fibered surface, that is, if and only if there exists a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow[\Phi]{\sim} & S' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \mathbb{A}^1_k & \xrightarrow[\phi]{\sim} & \mathbb{A}^1_k \end{array},$$

where $\Phi : S \xrightarrow{\sim} S'$ is an isomorphism and ϕ is an automorphism of $\mathbb{A}^1_k$ preserving the origin o.

2.10. So it is useful to a have characterisation of those Danielewski surfaces admitting two $\mathbb{A}^1$-fibrations with distinct general fibers. The first result toward such a classification has been obtained by T. Bandman and L. Makar-Limanov [1] who established that a complex Danielewski surface S with a trivial canonical sheaf ω_S admits two $\mathbb{A}^1$-fibrations with distinct general fibers if and only if it is isomorphic to a surface $S_{P,1}$ in $\mathbb{A}^3_{\mathbb{C}}$ defined by the equation $xz - P(y) = 0$, where P is a polynomial with simple roots. Over a field of characteristic zero, a complete classification has been given by the first author in [8] and [9]. It turns out that the main result of [9] remains valid over arbitrary base fields. This leads to the following characterisation.

Theorem 2.11. *For a Danielewski surface $\pi : S \to \mathbb{A}^1_k$, the following are equivalent :*

1) S admits two $\mathbb{A}^1$-fibrations with distinct general fibers.

2) S is isomorphic to a Danielewski surface $S(\gamma)$ defined by a fine k-weighted comb $\gamma = (\Gamma, w)$.

3) There exists an integer $h \geq 1$ and a collection of monic polynomials $P_0, \ldots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \ldots, h-1$, $j = 1, \ldots, \deg_t(P_i)$, such that S is isomorphic to the surface $S_{P_0,\ldots,P_{h-1}} \subset Spec(k[x][y_{-1}, \ldots, y_{h-2}][z])$ defined by the equations

$$\left\{\begin{array}{lll} xz - y_{h-2}\prod_{l=0}^{h-1} P_l(y_{l-1}) = 0 & & \\ zy_{i-1} - y_i y_{h-2}\prod_{l=i+1}^{h-1} P_l(y_{l-1}) = 0 & xy_i - y_{i-1}\prod_{l=0}^{i} P_l(y_{l-1}) = 0 & 0 \leq i \leq h-2 \\ y_{i-1}y_j - y_i y_{j-1}\prod_{l=i+1}^{j} P_l(y_{l-1}) = 0 & & 0 \leq i < j \leq h-2 \end{array}\right.$$

Proof. One checks in a similar way as in the proof of Theorem 2.9 in [9] that a surface $S = S_{P_0,\ldots,P_{h-1}}$ is a Danielewski surface $\pi = \mathrm{pr}_x\mid_S : S \to \mathbb{A}^1_k$. Furthermore, the projection $\pi' = \mathrm{pr}_z\mid_S : S \to \mathbb{A}^1_k$ is a second $\mathbb{A}^1$-fibration on S restricting to a trivial $\mathbb{A}^1$-bundle $(\pi')^{-1}\left(\mathbb{A}^1_k \setminus \{0\}\right) \simeq \mathrm{Spec}\left(k\left[z,z^{-1}\right][y_{h-2}]\right)$ over $\mathbb{A}^1_k \setminus \{0\}$. So 3) implies 1). To show that 1) implies 2) we use the following fact, which is a consequence of a result due to M.H. Gizatullin [13] : if a nonsingular affine surface S defined over an algebraically closed field k admits an $\mathbb{A}^1$-fibration $q : S \to \mathbb{A}^1_k$, then this fibration is unique up to automorphisms of the base if and only if S does not admit a completion by a nonsingular projective surface $\bar{S}$ for which the boundary divisor $\bar{S}\setminus S$ is a *zigzag*, that is, a chain of nonsingular proper rational curves. In [8], the first author constructed canonical completions $\bar{S}$ of a Danielewski surface $S(\gamma)$ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$ for which the dual graph Γ' of the boundary divisor $\bar{S}\setminus S(\gamma)$ is isomorphic to the tree obtained from Γ be deleting its leaves and replacing its root by a chain with two elements. Clearly, $\bar{S}\setminus S(\gamma)$ is a zigzag if and only if Γ is a comb. The construction given in *loc. cit.* only depends on the existence of an $\mathbb{A}^1$-bundle structure $\rho : S(\gamma) \to X(r)$ on a Danielewski surface $S(\gamma)$. So it remains valid over an arbitrary base field k. Now let $S = S(\gamma)$ be a Danielewski surface admitting two distinct $\mathbb{A}^1$-fibrations. Given an algebraic closure $\bar{k}$ of k, the surface $S_{\bar{k}} = S \times_{\mathrm{Spec}(k)} \mathrm{Spec}\left(\bar{k}\right)$ is a Danielewski surface isomorphic to the one defined by the tree γ consider as a fine $\bar{k}$-weighted tree via the inclusion $k \subset \bar{k}$. Since every $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_k$ lifts to an $\mathbb{A}^1$-fibration $\pi_{\bar{k}} : S_{\bar{k}} \to \mathbb{A}^1_{\bar{k}}$ it follows that $S_{\bar{k}}$ admits two $\mathbb{A}^1$-fibrations with distinct general fibers. So we deduce from Gizatullin's criterion above that γ is a comb. Thus 1) implies 2).

It remains to show that every Danielewski surface $\pi : S = S(\gamma) \to \mathbb{A}^1_k$ defined by a fine k-weighted comb γ of height $h \geq 1$ admits a closed embedding in an affine space as a surface $S_{P_0,\ldots,P_{h-1}}$. This follows from a general construction described in Â§4.6 of [9] that can be simplified in our more restrictive context. For the convenience of the reader, we indicate below the main steps of the proof. If γ is a chain, then $S(\gamma)$ is isomorphic to the affine plane $\mathbb{A}^2_k$ which embeds in $\mathbb{A}^{h+2}_k$ as a surface $S_{P_0,\ldots,P_{h-1}}$ for which all the polynomials P_i, $i = 0,\ldots,h-1$ have degree one. We assume from now on that γ has at least two elements at level 1 (see Remark 2.9 above). We denote by $e_{0,0} < e_{1,0} < \cdots < e_{h-1,0}$ the elements of the sub-chain $C = \Gamma \setminus L(\Gamma)$ of Γ consisting of elements of Γ which are not leaves of Γ. For every $l = 1,\ldots,h$, the elements of Γ at level l distinct from $e_{l,0}$ are denoted by $e_{l,1},\ldots,e_{l,r_l}$ provided that they exist. Since γ is a comb, it follows from 2.5 above that S is isomorphic to the surface associated with a certain fine k-weighted tree with the same underlying tree Γ as γ and equipped with a weight function w such that $w\left(\overleftarrow{e_{i,0}e_{i+1,0}}\right) = 0$ for every index $i = 0,\ldots,h-2$ and such that $w\left(\overleftarrow{e_{h-1,0}e_{h-1,1}}\right) = 0$. We consider S as an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ and we denote by $S_i = \mathrm{Spec}\left(k[x][u_i]\right)$ the trivialising open subsets of S over $X(r)$. For every $l = 0,\ldots,h-1$ and every $i = 1,\ldots,s_l$, we let $\tau_{l,i}(x,u_i) = xu_i + w\left(\overleftarrow{e_{l-1,0}e_{l,i}}\right) \in k[x][u_i]$. With this notation, the canonical function ψ on S restricts on an open subset S_i corresponding to a leaf $e_{l,i}$ of Γ at level l to the polynomial $x^{l-1}\tau_{l,i}(x,u_i) \in k[x][u_i]$. Therefore, $y_{-1} = \psi$ is constant with the value $a_{0,i} = w\left(\overleftarrow{e_{0,0}e_{1,i}}\right) \in k^*$ on the irreducible component $\pi^{-1}(o)$ corresponding to a leaf $e_{1,i}$, $i = 1,\ldots,r_1$, at level 1. It vanishes identically on every irreducible component of $\pi^{-1}(o)$ corresponding to a leaf of γ at level $l \geq 2$. More generally, direct computations show that there exists a unique datum consisting of regular functions $y_{-1},\ldots,y_{h-2}$ and y_{h-1} on S and polynomials $P_i \in k[t]$, $i = 0,\ldots,h-1$ satisfying the following conditions :

a) For every $l = 0,\ldots,h-1$, and every $l \leq m \leq h$, y_{l-1} restricts on an open subset S_i corresponding to a leaf $e_{m,i}$ of γ at level m to a polynomial $y_{l-1,i} \in k[x][u_i]$ such that

$$y_{l-1,i} = \begin{cases} L_{l,i}(u_i) & \mod x \quad \text{if } m = l \\ a_{l,i} + xL_{l+1,i}(u_i) & \mod x^2 \quad \text{if } m = l+1 \\ \xi_m x^{m-l-1}\tau_{m,i}(x,u_i) + \nu_{m,i}x^{m-l} & \mod x^{m-l+1} \quad \text{if } m > l+1, \end{cases}$$

where $L_{l,i}(u_i), L_{l+1,i}(u_i) \in k[u_i]$ are polynomials of degree 1, $a_{l,i}, \xi_m \in k^*$ and $\nu_{m,i} \in k$. Furthermore $a_{l,i} \neq a_{l,j}$ for every pair of distinct indices i and j.

b) For every $l = 0,\ldots,h-1$, P_l is the unique monic polynomial with simple roots $a_{l,1},\ldots,a_{l,r_l}$ such that $x^{-1}y_{l-1}\prod_{i=0}^{l-1}P_i(y_{i-1})P_l(y_{l-1})$ is a regular function on S.

By construction, these functions $y_{-1},\ldots,y_{h-2},y_{h-1}=z$ distinguish the irreducible components of the fiber $\pi^{-1}(o)$ and induce coordinate functions on them. It follows that the morphism $i=(\pi,y_{-1},\ldots,y_{h-1},z):S\hookrightarrow\mathbb{A}_k^{h+2}$ is an embedding. The same argument as in the proof of Lemma 3.6 in [9] shows that i is actually a closed embedding whose image is contained in the surface $S_{P_0,\ldots,P_{h-1}}\subset\mathbb{A}_k^{h+2}$ defined in Theorem 2.11 above. One checks that the induced morphism $\phi:S\to S_{P_0,\ldots,P_{h-1}}$ defines a bijection between the sets of closed points of S and $S_{P_0,\ldots,P_{h-1}}$. Furthermore, ϕ is also birational as y_{-1} induces an isomorphism $\pi^{-1}\left(\mathbb{A}^1\setminus\{o\}\right)\xrightarrow{\sim}\operatorname{Spec}\left(k\left[x,x^{-1}\right][y_{-1}]\right)$. Since $S_{P_0,\ldots,P_{h-1}}$ is nonsingular, we conclude that ϕ an isomorphism by virtue of Zariski Main Theorem (see e.g., 4.4.9 in [14]). □

2.3. **Special Danielewski surfaces.**
It follows from Adjunction Formula that every Danielewski surface S in $\mathbb{A}_k^3$ has a trivial canonical sheaf $\omega_{S/k}=\Lambda^2\Omega^1_{S/k}$. More generally, a Danielewski surface $\pi:S\to\mathbb{A}_k^1$ with a trivial canonical sheaf, or equivalently with a trivial sheaf of relative differential forms $\Omega^1_{S/\mathbb{A}_k^1}$, will be called *special.*

2.12. These surfaces correspond to a distinguished class of weighted trees γ. Indeed, it follows from the gluing construction given in 2.5 above that a Danielewski surface $S(\gamma)$ admits a nowhere vanishing differential 2-form if and only if all the leaves of γ are at the same level. In turn, this means that these surfaces S are the total space of $\mathbb{A}^1$-bundles $\rho:S\to X(r)$ over $X(r)$ defined by means of transition isomorphisms

$$\tau_{ij}:k\left[x,x^{-1}\right][u_i]\to k\left[x,x^{-1}\right][u_j],\quad u_i\mapsto u_j+g_{ij}(x),\quad i,j=1,\ldots,r,$$

where $g=\{g_{ij}\}_{i,j}\in C^1\left(X(r),\mathcal{O}_{X(r)}\right)\simeq\mathbb{C}\left[x,x^{-1}\right]^{2r}$ is a Čech cocycle with values in the sheaf $\mathcal{O}_{X(r)}$ for the canonical open covering $\mathcal{U}_r$. So they can be equivalently characterised among Danielewski surfaces by the fact that the underlying $\mathbb{A}^1$-bundle $\rho:S\to X(r)$ is actually the structural morphism of a principal homogeneous $\mathbb{G}_a$-bundle.

2.13. To determine isomorphism classes of special Danielewski surfaces, we can exploit the fact that the group $\operatorname{Aut}(X(r))\simeq\operatorname{Aut}\left(\mathbb{A}_k^1\setminus\{o\}\right)\times\mathfrak{S}_r$ acts on the set $\mathbb{P}H^1\left(X(r),\mathcal{O}_{X(r)}\right)$ of isomorphism classes of $\mathbb{A}^1$-bundles as above. Indeed, for every $\phi\in\operatorname{Aut}(X(r))$, the image $\phi\cdot[g]$ of a class $[g]\in\mathbb{P}H^1\left(X(r),\mathcal{O}_{X(r)}\right)$ represented by a bundle $\rho:S\to X(r)$ is the isomorphism class of the fiber product bundle $\mathrm{pr}_2:\phi^*S=S\times_{X(r)}X(r)\to X(r)$. The following criterion generalises a result of J. Wilkens [19].

Theorem 2.14. *Two special Danielewski surfaces $\pi_1:S_1\to\mathbb{A}_k^1$ and $\pi_2:S_2\to\mathbb{A}_k^1$ with underlying $\mathbb{A}^1$-bundle structures $\rho_1:S_1\to X(r_1)$ and $\rho_2:S_2\to X(r_2)$ are isomorphic as abstract surfaces if and only if $r_1=r_2=r$ and their isomorphism classes in $\mathbb{P}H^1\left(X(r),\mathcal{O}_{X(r)}\right)$ belongs to the same orbit under the action of $Aut(X(r))$.*

Proof. The condition guarantees that S_1 and S_2 are isomorphic. Suppose conversely that there exists an isomorphism $\Phi:S_1\xrightarrow{\sim}S_2$. The divisor class group of a special Danielewski surface $\pi:S\to\mathbb{A}_k^1$ is generated by the classes of the connected components $C_1,\ldots,C_r$ of $\pi^{-1}(o)$ modulo the relation $C_1+\cdots+C_r=\pi^{-1}(o)\sim 0$, whence is isomorphic to $\mathbb{Z}^{r-1}$. Therefore, $r_1=r_2=r$ for a certain $r\geq 1$. If one of the S_i's, say S_1 is isomorphic to a surface $S_{P,1}\subset\mathbb{A}_k^3$ defined by the equation $xz-P(y)=0$, then the result follows from [17]. Otherwise, we deduce from Theorem 2.11 that the $\mathbb{A}^1$-fibrations $\pi_1:S_1\to\mathbb{A}_k^1$ and $\pi_2:S_2\to\mathbb{A}_k^1$ are unique up to automorphisms of the base. In turn, this implies that Φ induces an isomorphism $\phi:X(r)\xrightarrow{\sim}X(r)$ such that $\phi\circ\rho_1=\rho_2\circ\Phi$. Therefore, $\Phi:S_1\xrightarrow{\sim}S_2$ factors through an isomorphism of $\mathbb{A}^1$-bundles $\tilde{\phi}:S_1\xrightarrow{\sim}\phi^*S_2$, where ϕ^*S_2 denotes the the fiber product $\mathbb{A}^1$-bundle $\mathrm{pr}_2:\phi^*S_2=S_2\times_{X(r)}X(r)\to X(r)$. This completes the proof as $\phi^*S_2\simeq S_2$. □

3. Danielewski surfaces in $\mathbb{A}_k^3$ defined by an equation of the form $x^hz-Q(x,y)=0$ and their automorphisms

In this section, we study Danielewski surfaces $\pi:S\to\mathbb{A}_k^1$ non isomorphic to $\mathbb{A}_k^2$ admitting a closed embedding $i:S\hookrightarrow\mathbb{A}_k^3$ in the affine 3-space as a surface $S_{Q,h}$ defined by the equation

$x^h z - Q(x,y) = 0$. We show that a same abstract Danielewski surface may admit many such closed embeddings. In particular, we establish that S can be embedded as a surface $S_{\sigma,h}$ defined by an equation of the form $x^h z - \prod_{i=1}^{r}(y - \sigma_i(x)) = 0$ for a suitable collection of polynomials $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$. Next we study the automorphism groups of the above surfaces S. We show that, in a closed embedding as a surface $S_{\sigma,h}$, every automorphism of S explicitly arises as the restriction of an automorphism of the ambient space. We will show on the contrary in the next section that it is not true for a general embedding as a surface $S_{Q,h}$.

3.1. **Danielewski surfaces $S_{Q,h}$.**

A surface $S = S_{Q,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$ is a Danielewski surface $\pi = \mathrm{pr}_x\,|_S \colon S \to \mathbb{A}^1_k$ if and only if the polynomial $Q(0,y)$ splits with simple roots $y_1,\ldots,y_r \in k$, where $r = \deg_y(Q(0,y))$. If $r = 1$, then $\pi^{-1}(o) \simeq \mathbb{A}^1_k$ and $\pi : S \to \mathbb{A}^1_k$ is isomorphic to a trivial $\mathbb{A}^1$-bundle. Thus S is isomorphic to the affine plane. Otherwise, if $r \geq 2$, then S is not isomorphic to $\mathbb{A}^2_k$, as follows for instance from the fact that the divisor class group $\mathrm{Div}(S)$ of S is isomorphic to $\mathbb{Z}^{r-1}$, generated by the classes of the connected components $C_1,\ldots,C_r$ of $\pi^{-1}(o)$, with a unique relation $C_1 + \ldots + C_r = \mathrm{div}(\pi^* x) \sim 0$.

The above class of Danielewski surfaces contains affine surfaces $S_{P,h}$ in $\mathbb{A}^3_k$ defined by an equation of the form $x^h z - P(y) = 0$, where $P(y)$ is a polynomial which splits with simple roots $y_1,\ldots,y_r$ in k. Replacing the constants $y_i \in k$ by suitable polynomials $\sigma_i(x) \in k[x]$ leads to the following more general class of examples.

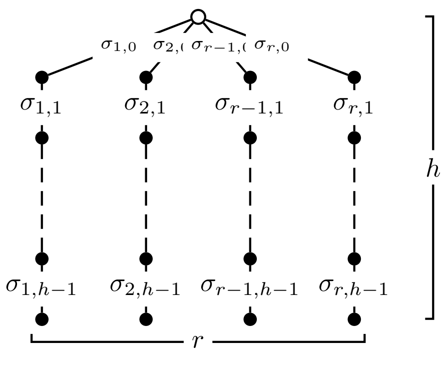
Example 3.1. Let $h \geq 1$ be an integer and let $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ be a collection of $r \geq 2$ polynomials $\sigma_i(x) = \sum_{j=0}^{h-1} \sigma_{i,j} x^j \in k[x]$ such that $\sigma_i(0) \neq \sigma_j(0)$ for every $i \neq j$. The surface $S = S_{\sigma,h}$ in $\mathbb{A}^3_k = \mathrm{Spec}(k[x,y,z])$ defined by the equation

$$x^h z - \prod_{i=1}^{r}(y - \sigma_i(x)) = 0$$

is a Danielewski surface $\pi = \mathrm{pr}_x\,|_S \colon S \to \mathbb{A}^1_k$. The fiber $\pi^{-1}(o)$ consists of r copies C_i of the affine line defined by the equations $\{x = 0, y = \sigma_i(0)\}_{i=1,\ldots,r}$ respectively. For every index $i = 1,\ldots,r$, the open subset $S_i = S \setminus \bigcup_{j \neq i} C_i$ of S is isomorphic to the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(k[x,u_i])$, where u_i denotes the regular function on S_i induced by the rational function

$$u_i = x^{-h}(y - \sigma_i(x)) = z \prod_{j \neq i}(y - \sigma_j(x))^{-1} \in k(S)$$

on S. It follows that $\pi : S \to \mathbb{A}^1_k$ factors through an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ isomorphic to the one with transition pairs $(f_{ij}, g_{ij}) = \left(1, x^{-h}(\sigma_j(x) - \sigma_i(x))\right)$, $i,j = 1,\ldots,r$. The collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ is exactly the one associated with the following fine k-weighted tree $\gamma = (\Gamma, w)$.



So S is isomorphic to the corresponding Danielewski surface $\pi_\gamma : S(\gamma) \to \mathbb{A}^1_k$. By definition (see 2.6 above), the canonical function ψ_γ on $S(\gamma)$ is the unique regular function restricting to the polynomial function $\psi_{\gamma,i} = x^h u_i + \sigma_i(x) \in k[x,u_i]$ on the trivialising open subsets $S_i \simeq \mathbb{A}^2_k$, $i = 1,\ldots,r$ of $S(\gamma)$. So it coincides with the restriction of y on S under the above isomorphism. In the setting of Theorem 2.8, this means that γ corresponds to the Danielewski surface S equipped with the birational morphism $\mathrm{pr}_{x,y} : S \to \mathbb{A}^2_k$.

It turns out that up to isomorphisms, the above class of Danielewski surfaces $S_{\sigma,h}$ contains all possible Danielewski surfaces $S_{Q,h}$, as shown by the following result.

Theorem 3.2. *Let $S_{Q,h}$ be a Danielewski surface in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$, where $Q(x,y) \in k[x,y]$ is a polynomial such that $Q(0,y)$ splits with $r \geq 2$ simples roots in k. Then there exists a collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ of polynomials of degrees $\deg(\sigma_i(x)) < h$ such that $S_{Q,h}$ is isomorphic to the surface $S_{\sigma,h}$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$.*

Proof. Since $Q(0,y)$ splits with simple roots $y_1, \ldots, y_r$ in k, a variant of the classical Hensel Lemma (see e.g., Theorem 7.18 p. 208 in [10]) guarantees that the polynomial $Q(x,y)$ can be written in a unique way as

$$Q(x,y) = R_1(x,y) \prod_{i=1}^{r} (y - \sigma_i(x)) + x^h R_2(x,y),$$

where $R_1(x,y) \in k[x,y] \setminus (x^h k[x,y])$ is a polynomial such that $R_1(0,y)$ is a nonzero constant and where $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ is a collection of polynomials of degree strictly lower than h such that $\sigma_i(0) = y_i$ for every index $i = 1, \ldots, r$. Since $y_i \neq y_j$ for every $i \neq j$ and $R_1(0,y)$ is a nonzero constant, it follows that for every index $i = 1, \ldots, r$, the rational function

$$u_i = x^{-h}(y - \sigma_i(x)) = \prod_{j \neq i} (y - \sigma_j(x))^{-1} R_1(x,y)^{-1} (z - R_2(x,y))$$

on $S_{Q,h}$ restricts to a regular function on the complement S_i in $S_{Q,h}$ of the irreducible components of the fiber $\mathrm{pr}_x^{-1}(0)$ defined by the equations $\{x = 0, y = y_j\}_{j \neq i}$ and induces an isomorphism $S_i \simeq \mathrm{Spec}(k[x, u_i])$. Therefore, the collection $\sigma = \{\sigma_i(x)\}_{i=1,\ldots,r}$ is precisely the one associated with the fine k-weighted rake $\gamma = (\Gamma, w)$ with all its leaves at a same level h corresponding to the Danielewski surface $\mathrm{pr}_x : S_{Q,h} \to \mathbb{A}^1_k$ equipped with the birational morphism $\psi = \mathrm{pr}_{x,y} : S_{Q,h} \to \mathbb{A}^2_k$ (see 2.8 and 2.12 above). In turn, we deduce from example 3.1 that the Danielewski surface $S(\gamma)$ associated with γ embeds as the surface $S_{\sigma,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^h z - \prod_{i=1}^r (y - \sigma_i(x)) = 0$. This completes the proof. □

Definition 3.3. Given a Danielewski surface S isomorphic to a certain surface $S_{Q,h}$ in $\mathbb{A}^3_k$, a closed embedding $i_s : S \hookrightarrow \mathbb{A}^3_k$ of S in $\mathbb{A}^3_k$ as a surface $S_{\sigma,h}$ defined by the equation

$$x^h z - \prod_{i=1}^{r} (y - \sigma_i(x)) = 0$$

is called a *standard embedding of* S. We say that $S_{\sigma,h}$ is a *standard form of* S *in* $\mathbb{A}^3_k$.

3.4. It follows from the above discussion that every Danielewski surface S isomorphic to a certain surface $S_{Q,h}$ in $\mathbb{A}^3_k$ admits a standard embedding in $\mathbb{A}^3_k$. Following the proof of Theorem 3.2, we can in fact construct explicitly the isomorphisms between a Danielewski surface $S_{Q,h}$ and one of its standard forms $S_{\sigma,h}$. Let $Q(x,y) = R_1(x,y) \prod_{i=1}^r (y - \sigma_i(x)) + x^h R_2(x,y)$ be as in the proof of Theorem 3.2. Then, the endomorphism Φ^s of $\mathbb{A}^3_k$ defined by $(x,y,z) \mapsto (x, y, R_1(x,y) z + R_2(x,y))$ induces an isomorphism ϕ^s between $S_{\sigma,h}$ and $S_{Q,h}$. One checks conversely that for every pair (f,g) of polynomials such that $R_1(x,y) f(x,y) + x^h g(x,y) = 1$, the endomorphism Φ_s of $\mathbb{A}^3_k$ defined by

$$(x,y,z) \mapsto \left(x, y, f(x,y) z + g(x,y) \prod_{i=1}^{r} (y - \sigma_i(x)) - f(x,y) R_2(x,y) \right)$$

induces an isomorphism ϕ_s between $S_{Q,h}$ and $S_{\sigma,h}$ such that $\phi^s \circ \phi_s = \mathrm{id}_{S_{Q,h}}$ and $\phi_s \circ \phi^s = \mathrm{id}_{S_{\sigma,h}}$. Note that since $R_1(0,y)$ is a nonzero constant, the regular function $\xi = x^{-h}(R_1 \prod_{i=1}^r (y - \sigma_i(x))) + R_2$ on $S_{\sigma,h}$ still induces a coordinate function on every irreducible component of the fiber $\pi^{-1}(o)$ of the morphism $\pi = \mathrm{pr}_x : S_{\sigma,h} \to \mathbb{A}^1_k$, and the regular functions π, y and ξ define a new closed embedding of $S_{\sigma,h}$ in $\mathbb{A}^3_k$ inducing an isomorphism between $S_{\sigma,h}$ and the surface $S_{Q,h}$. This can be interpreted by saying that a closed embedding $i_{Q,h}: S \hookrightarrow \mathbb{A}_k^3$ of a Danielewski surface S in $\mathbb{A}_k^3$ as a surface $S_{Q,h}$ is a twisted form of a standard embedding of S obtained by modifying the function inducing a coordinate on every irreducible component of the fiber $\pi^{-1}(o)$.

3.5. Using standard forms makes the study of isomorphism classes of Danielewski surfaces $S_{Q,h}$ simpler. For instance, we have the following characterisation which generalises a result due to L. Makar-Limanov [17] for complex surfaces $S_{P,h}$ defined by the equations $x^h z - P(y) = 0$.

Proposition 3.6. *Two Danielewski surfaces S_{σ_1,h_1} and S_{σ_2,h_2} in $\mathbb{A}_k^3$ defined by the equations*

$$x^{h_1} z = P_1(x,y) = \prod_{i=1}^{r_1} (y - \sigma_{1,i}(x)) \quad \text{and} \quad x^{h_2} z = P_2(x,y) = \prod_{i=1}^{r_2} (y - \sigma_{2,i}(x))$$

are isomorphic if and only if $h_1 = h_2 = h$, $r_1 = r_2 = r$ and there exists a triple $(a, \mu, \tau(x)) \in k^* \times k^* \times k[x]$ such that $P_2(ax, y) = \mu^r P_1(x, \mu^{-1} y + \tau(x))$.

Proof. The condition is sufficient. Indeed, one checks that the automorphism

$$(x,y,z) \mapsto \left(ax, \mu(y - \tau(x)), \mu^r a^{-2} z\right)$$

of $\mathbb{A}_k^3$ induces an isomorphism between $S_{\sigma_1,h}$ and $S_{\sigma_2,h}$. Conversely, suppose that $S_1 = S_{\sigma_1,h_1}$ and $S_2 = S_{\sigma_2,h_2}$ are isomorphic. Then $h_1 = h_2 = h$ and $r_1 = r_2 = r$ by virtue of Theorem 2.14 above. If $h = 1$ then the result follows from [17]. Otherwise, if $h \geq 2$ then it follows from Theorem 2.11 and example 3.1 above that the underlying $\mathbb{A}^1$-bundle structures $\rho_1 : S_1 \to X(r)$ and $\rho_2 : S_2 \to X(r)$ corresponding to the transition functions

$$\left\{g_{1,ij} = x^{-h}(\sigma_{1,j}(x) - \sigma_{1,i}(x))\right\}_{i,j=1,\ldots,r} \text{ and } \left\{g_{2,ij} = x^{-h}(\sigma_{2,j}(x) - \sigma_{2,i}(x))\right\}_{i,j=1,\ldots,r}$$

respectively are unique such structures on S_1 and S_2 up to automorphisms of the base $X(r)$. Therefore, every isomorphism $\Phi : S_1 \xrightarrow{\sim} S_2$ induces an automorphism ϕ of $X(r)$ such that $\rho_2 \circ \Phi = \phi \circ \rho_1$. Consequently, every such isomorphism Φ is determined by a collection of local isomorphisms $\Phi_i : S_{1,i} \xrightarrow{\sim} S_{2,\alpha(i)}$ where $\alpha \in \mathfrak{S}_r$, defined by k-algebra isomorphisms

$$\Phi_i^* : k[x]\left[u_{2,\alpha(i)}\right] \longrightarrow k[x][u_{1,i}], \quad x \mapsto a_i x, \quad u_{2,\alpha(i)} \mapsto \lambda_i u_{1,i} + b_i(x), \quad i = 1, \ldots, r$$

where $a_i, \lambda_i \in k^*$ and where $b_i \in k[x]$. These local isomorphisms glue to a global one if and only if $a_i = a$ and $\lambda_i = \lambda$ for every index $i = 1, \ldots, r$, and the relation $\lambda g_{1,ij}(x) + b_i(x) = g_{2,\alpha(i)\alpha(j)}(ax) + b_j(x)$ holds in $k\left[x, x^{-1}\right]$ for every indices $i, j = 1, \ldots, r$. Since the $\sigma_{1,i}$'s and $\sigma_{2,i}$'s have degrees strictly lower than h, we conclude that the latter condition is equivalent to the fact that $b_i(x) = b(x)$ for every $i = 1, \ldots, r$ and that the polynomial $c(x) = \sigma_{2,\alpha(i)}(ax) - \lambda a^h \sigma_{1,i}(x)$ does not depend on the index i. Letting $\mu = \lambda a^h$ and $\tau(x) = \mu^{-1} c(x)$, this means exactly that $P_2(ax, y) = \mu^r P_1(x, \mu^{-1} y + \tau(x))$. ☐

3.7. The proof above implies in particular that all standard embeddings of a same Danielewski surface are algebraically equivalent. It is natural to ask if a closed embedding $i_{Q,h} : S \hookrightarrow \mathbb{A}_k^3$ of Danielewski surface S as a surface $S_{Q,h}$ is algebraically equivalent to a standard one.If so, then we say that the embedding $i_{Q,h}$ is *rectifiable*. The fact that the endomorphisms Φ^s and Φ_s of $\mathbb{A}_k^3$ constructed in 3.4 are not invertible in general may lead one to suspect that there exists non-rectifiable embeddings of Danielewski surfaces nonisomorphic to the affine plane. This is actually the case, and the first known examples have been recently discovered by G. Freudenburg and L. Moser-Jauslin [12]. For instance, they established that the surface S_1 in $\mathbb{A}_{\mathbb{C}}^3$ defined by the equation $f_1 = x^2 z - (1-x)\left(y^2 - 1\right) = 0$ is a non-rectifiable embedding of a Danielewski surface. Indeed, a standard form for S_1 would be the Danielewski surface S_0 defined by the equation $f_0 = x^2 z - \left(y^2 - 1\right) = 0$. We observe that the level surface $f_0^{-1}(1)$ of f_0 is a singular surface. On the other hand, all the level surfaces of f_1 are nonsingular as follows for instance from the Jacobian Criterion. Therefore, condition 3) in Definition 1.5 cannot be satisfied and so, it is impossible to find an automorphism of $\mathbb{A}_{\mathbb{C}}^3$ mapping S_1 isomorphically onto S_0.

The classification of these embeddings up to algebraic equivalence is a difficult problem in general (see [18] for the case $h = r = 2$). However, if $k = \mathbb{C}$, the following result shows that things become simpler if one works in the holomorphic category.

Theorem 3.8. *The embeddings $i_{Q,h}:S\hookrightarrow\mathbb{A}^3_{\mathbb{C}}$ of a Danielewski surface S as a surface defined by the equation $x^hz-Q(x,y)=0$ are all* analytically *equivalent.*

Proof. It suffices to show that every embedding $i_{Q,h}$ is analytically equivalent to a standard one $i_{\sigma,h}$. In view of the proof of Theorem 3.2, we can let $Q(x,y)=R_1(x,y)\prod_{i=1}^r(y-\sigma_i(x))+x^hR_2(x,y)$. It is enough to construct an holomorphic automorphism Ψ of $\mathbb{A}^3_{\mathbb{C}}$ such that

$$\Psi^*\left(x^hz-\prod_{i=1}^r(y-\sigma_i(x))\right)=\alpha\left(x^hz-Q(x,y)\right)$$

for a suitable invertible holomorphic function α on $\mathbb{A}^3_{\mathbb{C}}$. We let $R_1(0,y)=\lambda\in\mathbb{C}^*$ and we let $f(x,y)\in\mathbb{C}[x,y]$ be a polynomial such that $\lambda\exp(xf(x,y))\equiv R_1(x,y)\bmod x^h$. Now the result follows from the fact that the holomorphic automorphism Ψ of $\mathbb{A}^3_{\mathbb{C}}$ defined by

$$\Psi(x,y,z)=\left(x,y,\lambda\exp(xf(x,y))z-x^{-h}[\lambda\exp(xf(x,y))-R_1(x,y)]\prod_{i=1}^r(y-\sigma_i(x))+R_2(x,y)\right)$$

satisfies $\Psi^*\left(x^hz-Q(x,y)\right)=\lambda\exp(xf(x,y))\left(x^hz-\prod_{i=1}^r(y-\sigma_i(x))\right)$. □

Example 3.9. We observed in 3.7 that the surfaces S_0 and S_1 defined by the equations $f_0=x^2z-\left(y^2-1\right)=0$ and $f_1=x^2z-(1-x)\left(y^2-1\right)=0$ are algebraically inequivalent embeddings of a same surface S. However, they are analytically equivalent via the automorphism $(x,y,z)\mapsto\left(x,y,e^{-x}z-x^{-2}\left(e^{-x}-1+x\right)(y^2-1)\right)$ of $\mathbb{A}^3_{\mathbb{C}}$.

3.2. Automorphisms of Danielewski surfaces $S_{Q,h}$ in $\mathbb{A}^3_k$.

In [16] and [17], Makar-Limanov computed the automorphism groups of surfaces in $\mathbb{A}^3$ defined by the equation $x^hz-P(y)=0$, where $h\geq 1$ and where $P(y)$ is an arbitrary polynomial. In particular, he established that every automorphism of such a surface is induced by the restriction of an automorphism of the ambient space. Recently, A. Crachiola [3] established that this also holds for surfaces defined by the equations $x^hz-y^2-r(x)y=0$, where $h\geq 1$ and where $r(x)$ is an arbitrary polynomial such that $r(0)\neq 0$. This subsection is devoted to the proof of the more general structure Theorem 3.15 below. We begin with the case of Danielewski surfaces in standard form.

Theorem 3.10. *The automorphism group of a Danielewski surface $S_{\sigma,h}$ defined by the equation*

$$x^hz-P(x,y)=0,\qquad \textit{where}\qquad P(x,y)=\prod_{i=1}^r(y-\sigma_i(x))$$

is induced by the restriction of an automorphism of $\mathbb{A}^3_k$ belonging to the subgroup $G_{\sigma,h}$ of $\mathrm{Aut}\left(\mathbb{A}^3_k\right)$ generated by the following automorphisms:

(a) $\Delta_b(x,y,z)=\left(x,y+x^hb(x),z+x^{-h}\left(P\left(x,y+x^hb(x)\right)-P(x,y)\right)\right)$*, where $b(x)\in k[x]$.*

(b) *If there exists a polynomial $\tau(x)$ such that $P(x,y+\tau(x))=\tilde{P}(y)$ then the automorphisms $H_a(x,y,z)=\left(ax,y+\tau(ax)-\tau(x),a^{-h}z\right)$, where $a\in k^*$ should be added.*

(c) *If there exists a polynomial $\tau(x)$ such that $P(x,y+\tau(x))=\tilde{P}(x^{q_0},y)$, then the cyclic automorphisms $\tilde{H}_a(x,y,z)=\left(ax,y+\tau(ax)-\tau(x),a^{-h}z\right)$, where $a\in k^*$ and $a^{q_0}=1$ should be added.*

(d) *If there exists a polynomial $\tau(x)$ such that $P(x,y+\tau(x))=y^i\tilde{P}(x,y^s)$, where $i=0,1$ and $s\geq 2$, then the cyclic automorphisms $S_\mu(x,y,z)=\left(x,\mu y+(1-\mu)\tau(x),\mu^iz\right)$, where $\mu\in k^*$ and $\mu^s=1$ should be added.*

(e) *If $char(k)=s>0$ and $P(x,y)=\tilde{P}\left(y^s-c(x)^{s-1}y\right)$ for a certain polynomial $c(x)\in k[x]$ such that $c(0)\neq 0$, then the automorphism $T_c(x,y,z)=(x,y+c(x),z)$ should be added.*

(f) *If $h=1$, then the involution $I(x,y,z)=(z,y,x)$ should be added.*

Remark 3.11. Automorphisms of type a) in Theorem 3.10 correspond to algebraic actions of the additive group $\mathbb{G}_a$ on the surface $S_{\sigma,h}$. More precisely, for every polynomial $b\in k[x]$, the subgroup $\left\{\Delta_{tb(x)},t\in k\right\}$ of $\mathrm{Aut}(S_{\sigma,h})$ is isomorphic to $\mathbb{G}_a$, the corresponding $\mathbb{G}_a$-action on $S_{\sigma,h}$

being defined by $t\star(x,y,z)=\Delta_{tb(x)}(x,y,z)$. Similarly, automorphisms of type b) correspond to algebraic actions of the multiplicative group $\mathbb{G}_m$.

Proof. It is clear that every automorphism of $\mathbb{A}^3_k$ of types (a)-(f) above leaves $S_{\sigma,h}$ invariant, whence induces an automorphism of $S_{\sigma,h}$. If $h=1$, then the converse follows from [16]. Otherwise, if $h\geq 2$, then the same argument as the one used in the proof of Proposition 3.6 above show that every automorphism of $S_{\sigma,h}$ is determined by a datum $\mathcal{A}_\Phi=(\alpha,\mu,a,b(x))$ such that that the polynomial $c(x)=\sigma_{\alpha(i)}(ax)-\mu\sigma_i(x)+x^hb(x)$ does not depend on the index $i=1,\ldots,r$. Furthermore, it follows from the construction of the closed embedding of $S_{\sigma,h}$ in $\mathbb{A}^3_k$ given in Example 3.1 that every such collection correspond to an automorphism of $S_{\sigma,h}$ induced by the restriction of the following automorphism Ψ of $\mathbb{A}^3_k$:

$$\Psi(x,y,z)=\left(ax,\mu y+c(x),a^{-h}\mu^r z+(ax)^{-h}(\prod_{i=1}^{r}(\mu y+c(x)-\sigma_i(ax))-\mu^r\prod_{i=1}^{r}(y-\sigma_i(x)))\right).$$

One checks easily using this description that the composition of two automorphisms Φ_1 and Φ_2 of $S_{\sigma,h}$ defined by data $\mathcal{A}_{\Phi_1}=(\alpha_1,\mu_1,a_1,b_1)$ and $\mathcal{A}_{\Phi_2}=(\alpha_2,\mu_2,a_2,b_2)$ is the automorphism with corresponding datum $\mathcal{A}_\Phi=\left(\alpha_2\circ\alpha_1,\mu_2\mu_1,a_2a_1,a_2^{-h}\mu_2b_1(x)+b_2(a_1x)\right)$.

Clearly, automorphisms of type (a) coincide with the ones determined by data $\mathcal{A}=(\mathrm{Id},1,1,b(x))$, where $b(x)\in k[x]$. In view of the composition rule above, it suffices to consider from now on automorphisms corresponding to data $\mathcal{A}=(\alpha,\mu,a,0)$.

1Â°) If α is trivial, then $\mu=1$ by virtue of Lemma 3.13 below, and so $\mathcal{A}=(\mathrm{Id},1,a,0)$. Then, the relation $c(x)=\sigma_i(ax)-\sigma_i(x)$ holds for every $i=1,\ldots,r$.

1Â°a) If $a^q\neq 1$ for every $q=1,\ldots,h-1$, then there exists a polynomial $\tau(x)\in k[x]$ such that $\sigma_i(x)=\sigma_i(0)+\tau(x)$ for every $i=1,\ldots,r$. Thus $c(x)=\tau(ax)-\tau(x)$ and $P(x,y+\tau(x))=\tilde{P}(y)=\prod_{i=1}^r(y-\sigma_i(0))$ and the corresponding automorphism is of type (b).

1Â°b) If $a\neq 1$ but $a^{q_0}=1$ for a minimal $q_0=2,\ldots,h-1$, then there exists polynomials $\tau(x)$ and $\tilde{\sigma}_i(x)$, $i=1,\ldots,r$, such that $\sigma_i(x)=\tilde{\sigma}_i(x^{q_0})+\tau(x)$ for every $i=1,\ldots,r$. So there exists a polynomial $\tilde{P}$ such that $P(x,y+\tau(x))=\tilde{P}(x^{q_0},y)$. Moreover, $c(x)=\tau(ax)-\tau(x)$ and the corresponding automorphism is of type (c).

2Â°) If α is not trivial then $\mu^s=1$. Since $\Phi=\Phi_2\circ\Phi_1$, where $\Phi_1$and Φ_2 denote the automorphisms with data $\mathcal{A}_{\Phi_1}=(\mathrm{Id},1,a,0)$ and $\mathcal{A}_{\Phi_2}=(\alpha,\mu,1,0)$ respectively, it suffices to consider the situation that Φ is determined by a datum $\mathcal{A}_\Phi=(\alpha,\mu,1,0)$, where $\mu\in k^*$ and $\mu^s=1$. So the relation $\sigma_{\alpha(i)}(x)=\mu\sigma_i(x)+c(x)$ holds for every $i=1,\ldots,r$.

2Â°a) If $\mu^s=1$ but $\mu^{s'}\neq 1$ for every $s'=1,\ldots,s-1$, then, letting $\tau(x)=(1-\mu)^{-1}c(x)$ and $\tilde{\sigma}_i(x)=\sigma_i(x)-\tau(x)$ for every $i=1,\ldots,r$, we arrive at the relation $\tilde{\sigma}_{\alpha(i)}(x)=\mu\tilde{\sigma}_i(x)$ for every $i=1,\ldots,r$. Furthermore, if i_0 is a unique fixed point of α then $\tilde{\sigma}_{i_0}(x)=0$ as $\sigma_{i_0}(x)=\tau(x)$. So we conclude that $P(x,y+\tau(x))=y^i\tilde{P}(x,y^s)$ where $i=0,1$ and where s denotes the length of the nontrivial cycles in α. The corresponding automorphism is of type (d).

2Â°b) If $\mu=1$ then α is fixed point free by virtue of Lemma 3.13 and $\mathrm{char}(k)=s$, where s denotes the common length's of the cycles occurring in α. Moreover, $s'\cdot c(0)\neq 0$ for every $s'=1,\ldots,s-1$ and $\sigma_{i_m}(x)=\sigma_{i_1}(x)+(m-1)\cdot c(x)$ for every index i_m occurring in a cycle $(i_1,\ldots,i_s)$ of length s in α. Letting $r=ds$, we may suppose up to a reordering that α decomposes as the product of the standard cycles $(is+1,is+2\ldots,(i+1)s)$, where $i=0,\ldots,d-1$. Letting $R(x,y)=\prod_{m=1}^s(y-m\cdot c(x))=y^s-c(x)^{s-1}y$, we conclude that

$$P(x,y)=\prod_{i=0}^{d-1}R(x,y-\sigma_{is}(x))=\tilde{P}\left(x,y^s-c(x)^{s-1}y\right)$$

for a suitable polynomial $\tilde{P}(x,y)\in k[x,y]$. The corresponding automorphism is of type (e).

□

3.12. In the proof of Theorem 3.10 above, we used the fact that every automorphism Φ of a Danielewski surface $S=S_{\sigma,h}$, where $h\geq 2$, is determined by a certain datum $\mathcal{A}_\Phi=(\alpha,\mu,a,b(x))\in$

$\mathfrak{S}_r \times k^* \times k^* \times k[x]$ for which the polynomial $\tilde{c}(x) = \sigma_{\alpha(i)}(ax) - \mu\sigma_i(x) \in k[x]$ does not depend on the index i. Actually, we needed the following more precise result.

Lemma 3.13. *The elements in a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x))$ corresponding to an automorphism Φ of S satisfy the following additional properties*

1) The permutation α is either trivial or has at most a unique fixed point. If it is nontrivial then all nontrivial cycles with disjoint support occurring in a decomposition of α have the same length $s \geq 2$.

2) If α is trivial then $\mu = 1$ and the converse also holds provided that char$(k) \neq s$. Otherwise, if α is nontrivial and char$(k) \neq s$ then $\mu^s = 1$ but $\mu^{s'} \neq 1$ for every $1 \leq s' < s$.

Proof. To simplify the notation, we let $y_i = \sigma_i(0)$ for every $i = 1, \ldots, r$. Note that by hypothesis, $y_i \neq y_j$ for every $i \neq j$. If $\alpha \in \mathfrak{S}_r$ has at least two fixed points, say i_0 and i_1, then $y_{i_0}(1-\mu) = y_{i_1}(1-\mu) = \tilde{c}(0)$, and so, $\mu = 1$ and $\tilde{c}(0) = 0$ as $y_{i_0} \neq y_{i_1}$. In turn, this implies that α is trivial. Indeed, otherwise there would exist an index i such that $\alpha(i) \neq i$ but $y_{\alpha(i)} = y_i$, in contradiction with our hypothesis. Suppose from now that α is nontrivial and let $s \geq 2$ be the infimum of the length's of the nontrivial cycles occurring in decomposition of α into a product of cycles with disjoint supports. We deduce that $y_i(1-\mu^s) = y_j(1-\mu^s)$ for every pair of distinct indices i and j in the support of a same cycle of length s. Thus $\mu^s = 1$ as $y_i \neq y_j$ for every $i \neq j$. If $\mu = 1$ then $s' \cdot \tilde{c}(0) \neq 0$ for every $s' = 1, \ldots, s-1$. Indeed, otherwise we would have $y_{\alpha^{s'}(i)} = y_i + s' \cdot \tilde{c}(0) = y_i$ for every index $i = 1, \ldots, r$ which is impossible since α is nontrivial. In particular, α is fixed-point free. On the other hand $s \cdot \tilde{c}(0) = 0$ as $y_i = y_{\alpha^s(i)} = y_i + s \cdot \tilde{c}(0)$ for every index i in the support of a cycle of length s in α. This is possible if and only if the characteristic of the base field k is exactly s. We also conclude that every cycle in α have length s for otherwise there would exist an index i such that $\alpha^s(i) \neq i$ but $y_{\alpha^s(i)} = y_i + s \cdot \tilde{c}(0) = y_i$ in contradiction with our hypothesis.

If $\mu \neq 1$ then $\mu^{s'} \neq 1$ for every $s' < s$. Indeed, otherwise there would exist an index i such that $\alpha^{s'}(i) \neq i$ but $y_{\alpha^{s'}(i)} = \mu^{s'} y_i + \tilde{c}(0) \sum_{p=0}^{s'-1} \mu^p = y_i$, which is impossible. The same argument also implies that all the nontrivial cycles in α have length s. □

3.14. By combining Theorems 3.2 and 3.10, we obtain the following description of the automorphisms groups of Danielewski surfaces $S_{Q,h}$.

Theorem 3.15. *Let $S_{Q,h}$ be the Danielewski surface in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$ and let $S_{\sigma,h}$ be one of its standard forms. Then, every automorphism of $S_{Q,h}$ is of the form $\Phi^s \circ \psi \circ \Phi_s$, where ψ belongs to the subgroup $G_{\sigma,h}$ of the automorphisms group of $\mathbb{A}^3_k$ defined in Theorem 3.10 and Φ^s and Φ_s are the endomorphisms of $\mathbb{A}^3_k$ defined in 3.4.*

3.16. We have seen in 3.7 that the embeddings $i_{Q,h}$ are not rectifiable in general and so that the isomorphisms ϕ^s and ϕ_s do not extend to algebraic automorphisms of $\mathbb{A}^3_k$. Therefore, in contrast with the case of standard embeddings i_s for which every automorphisms of a Danielewski surface $S \simeq S_{\sigma,h}$ arises as the restriction of an automorphism of the ambient space $\mathbb{A}^3_k$, the above result may lead one to suspect that for a general embedding $i_{Q,h}$ of S as a surface $S_{Q,h}$, certain automorphisms of S do not extend to algebraic automorphisms $\mathbb{A}^3_k$. In the next section we give examples of embeddings for which this phenomenon occurs. However, if $k = \mathbb{C}$, Theorem 3.8 leads on the contrary the following result.

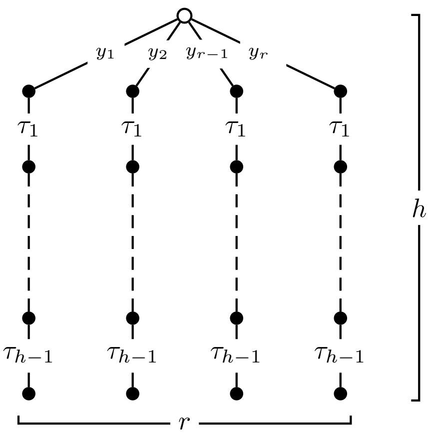
Corollary 3.17. *Every algebraic automorphism of a Danielewski surface $S_{Q,h}$ in $\mathbb{A}^3_{\mathbb{C}}$ is extendable to a* holomorphic *automorphism of $\mathbb{A}^3_{\mathbb{C}}$.*

4. Special Danielewski surfaces and multiplicative group actions

In this section, we fix a base field k of characteristic zero and we consider special Danielewski surfaces S admitting a nontrivial action of the multiplicative group $\mathbb{G}_m = \mathbb{G}_{m,k}$. We establish that every such surface is isomorphic to a Danielewski surface $S_{Q,h}$ which admits a standard embedding in $\mathbb{A}^3_k$ as a surface defined by an equation of the form $x^h z - P(y) = 0$ for a suitable polynomial $P(y)\in k[y]$. In this embedding, every multiplicative group action on S arises as the restriction of a linear $\mathbb{G}_m$-action on $\mathbb{A}^3_k$. We show on the contrary that this is not the case for a general embedding of S as a surface $S_{Q,h}$.

4.1. **Multiplicative group actions on special Danielewski surfaces.**
Every Danielewski surface isomorphic to a surface $S_{P,h}$ in $\mathbb{A}^3_k$ defined by an equation of the form $x^hz-P(y)=0$ for a certain polynomial $P(y)$ admits an nontrivial action of the multiplicative group $\mathbb{G}_m$ which arises as the restriction of the $\mathbb{G}_m$-action Ψ on $\mathbb{A}^3_k$ defined by $\Psi(a;x,y,z)=H_a(x,y,z)=\left(ax,y,a^{-h}z\right)$. In the setting of Lemma 3.13 above, the automorphisms H_a correspond to data $\mathcal{A}_{\phi_a}=(1,1,a,0)$, where $a\in k^*$. Here we establish that Danielewski surfaces isomorphic to a surface $S_{P,h}$ in $\mathbb{A}^3_k$ are characterised by the fact that they admit such a nontrivial $\mathbb{G}_m$-action.

4.1. By virtue of example 3.1 above, the collection of polynomials $\sigma_i(x)$, $i=1,\ldots,r$, corresponding to a Danielewski surface $S_{P,h}\subset\mathbb{A}^3_k$ is given by $\sigma_i(x)=y_i$ for every $i=1,\ldots,r$, where $y_1,\ldots,y_r$ denote the roots of the polynomial P. In turn, we deduce from Theorem 2.14 and Proposition 3.6 above that a Danielewski surface $S_{Q,h}$ with a standard form $S_{\sigma,h}$ defined by a datum $\left(r,h,\sigma=\{\sigma_i(x)\}_{i=1,\ldots,r}\right)$ is isomorphic to a surface $S_{P,h}$ as above if and only if there exists a polynomial $\tau(x)\in k[x]$ such that $\sigma_i(x)=\sigma_i(0)+\tau(x)$ for every $i=1,\ldots,r$. So we conclude that every such surface correspond to a fine k-weighted rake γ of the following type.



4.2. One can easily deduce from the description of the automorphism group of a Danielewski surface $S_{\sigma,h}$ given Theorem 3.10 above that such a surface admits a nontrivial $\mathbb{G}_m$-action if and only if it is isomorphic to a surface $S_{P,h}$. More generally, we have the following result.

Theorem 4.3. *A special Danielewski surface S admits a nontrivial action of the multiplicative group $\mathbb{G}_m$ if and only if it is isomorphic to a surface $S_{P,h}$ in $\mathbb{A}^3_k$ defined by the equation $x^hz-P(y)=0$.*

Proof. We may suppose that $S=S(\gamma)$ is the Danielewski surface associated with a fine k-weighted tree $\gamma=(\Gamma,w)$ with $r\geq 2$ elements at level 1 and with all its leaves at level $h\geq 1$. We denote by $\sigma=\{\sigma_i(x)\}_{i=1,\ldots,r}$ the collection of polynomial associated with γ (see 2.5). By virtue of Theorem 2.14 above, the collection $\tilde{\sigma}$ defined by

$$\tilde{\sigma}_i(x)=\sigma_i(x)-\frac{1}{r}\sum_{i=1}^{r}\sigma_i(x)\quad i=1,\ldots,r$$

leads to a Danielewski surface isomorphic to S. So we may suppose from the beginning that $\sigma_1(x)+\cdots+\sigma_r(x)=0$. If $h=1$ then it follows that S is isomorphic to a surface in $\mathbb{A}^3_k$ defined by an equation of the form $xz-P(y)=0$, and so, the assertion follows from the above discussion. Otherwise, if $h\geq 2$ then it follows from Theorem 2.11 that the structural $\mathbb{A}^1$-fibration

$\pi = \pi_\gamma : S = S(\gamma) \to \mathbb{A}^1_k$ is unique up to automorphisms of the base. We consider S as an $\mathbb{A}^1$-bundle $\rho : S \to X(r)$ defined by the transition cocycle

$$g = \left\{g_{ij} = x^{-h}\left(\sigma_j(x) - \sigma_i(x)\right)\right\}_{i,j=1,\ldots,r}.$$

The same argument as in the proof of Theorem 3.6 implies that every automorphism Φ of S is determined by a datum $\mathcal{A}_\Phi = (\alpha, \mu, a, b(x)) \in \mathfrak{S}_r \times k^* \times k^* \times k[x]$ for which the polynomial $\sigma_{\alpha(i)}(ax) - \mu\sigma_i(x) \in k[x]$ does not depend on the index i. In view of the composition rule given in the same proof, we deduce that an automorphism Φ of S may belong to a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathbb{G}_m$ only if its associated datum is of the form $\mathcal{A}_\Phi = (\alpha, \mu, a, 0)$. Suppose that there exists a nontrivial automorphism Φ determined by such a datum $\mathcal{A}_\Phi$. Then, since $\alpha \in \mathfrak{S}_r$, there exists an integer $N \geq 1$ such that the polynomial $c(x) = \sigma_i\left(a^N x\right) - \mu^N\sigma_i(x)$ does not depend on the index $i = 1, \ldots, r$. Since $\sigma_1(x) + \cdots + \sigma_r(x) = 0$ by hypothesis, we conclude that the identity $\sigma_i\left(a^N x\right) = \mu^N\sigma_i(x)$ holds for every index $i = 1, \ldots, r$. In particular, it follows that $\sigma_i(0) = \mu^N\sigma_i(0)$ for every index $i = 1, \ldots, r$. Thus $\mu^N = 1$ since γ is a fine k-weighted tree with at least two elements at level 1. Suppose that one of the polynomials σ_i is not constant. Then the above identity implies that $a^{Np} = 1$ for a certain integer p. Therefore, every automorphism Φ of S with associated datum $(\alpha, \mu, a, 0)$ is cyclic and $\mathrm{Aut}(S)$ can not contain a subgroup isomorphic to $\mathbb{G}_m$. So, S admits a nontrivial $\mathbb{G}_m$-action only if the polynomials σ_i, $i = 1, \ldots, r$ are constant. This completes the proof since these fine k-weighted trees correspond to Danielewski surfaces $S_{P,h}$ by virtue of 4.1 above. □

4.2. **Extensions of multiplicative group actions on a Danielewski surface.** It follows from Theorem 3.10 that every special Danielewski surface S equipped with a nontrivial $\mathbb{G}_m$-action admits an equivariant embedding in $\mathbb{A}^3_k$ as a surface $S_{P,h}$ defined by an equation of the form $x^h z - P(y) = 0$. In this embedding, the $\mathbb{G}_m$-action on S even arises as the restriction of a linear $\mathbb{G}_m$-action on $\mathbb{A}^3_k$ corresponding to automorphisms of type b) in 3.10. On the other hand, a surface S isomorphic to a surface $S_{P,h}$ admits closed embeddings $i_{Q,h} : S \hookrightarrow \mathbb{A}^3_k$ in $\mathbb{A}^3_k$ as surfaces $S_{Q,h}$ defined by equations of the form $x^h z - R(x,y)P(y) = 0$ (see Theorem 3.2). It is natural to ask if there always exists $\mathbb{G}_m$-actions on $\mathbb{A}^3_k$ making these general embeddings equivariant. Clearly, this holds if the embedding $i_{Q,h}$ is algebraically equivalent to a standard embedding of S as a surface $S_{P,h}$. The following result shows that there exists non rectifiable closed embeddings $i_{Q,h}$ of S for which no nontrivial $\mathbb{G}_m$-action on S can be extended to an action on the ambient space.

Theorem 4.4. *Every Danielewski surface $S \subset \mathbb{A}^3_k$ defined by the equation $x^h z - (1-x)P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{G}_m$-action $\tilde{\theta} : \mathbb{G}_m \times S \to S$ which is not algebraically extendable to $\mathbb{A}^3_k$. More precisely, for every $a \in k \setminus \{0,1\}$ the automorphism $\tilde{\theta}_a = \tilde{\theta}(a, \cdot)$ of S do not extend to an algebraic automorphism of $\mathbb{A}^3_k$.*

Proof. The endomorphisms Φ^s and Φ_s of $\mathbb{A}^3_k$ defined by $\Phi^s(x,y,z) = (x, y, (1-x)z)$ and $\Phi_s(x,y,z) = \left(x, y, (\sum_{i=0}^{h-1} x^i)z + P(y)\right)$ induce isomorphisms ϕ^s and ϕ_s between S and the surface $S_{P,h}$ defined by the equation $x^h z - P(y) = 0$ (see 3.4). The latter admits an action $\theta : \mathbb{G}_m \times S_{P,h} \to S_{P,h}$ of the multiplicative group $\mathbb{G}_m$ defined by $\theta(a,x,y,z) = H_a(x,y,z) = \left(ax, y, a^{-h}z\right)$ for every $a \in k^*$. The corresponding action $\tilde{\theta}$ on S is therefore defined by $\tilde{\theta}(a,x,y,z) = \tilde{\theta}_a(x,y,z) = \phi^s \circ H_a(x,y,z)\mid_{S_{P,h}} \circ \phi_s$. Since by construction, $\tilde{\theta}_a^*(x) = ax$ for every $a \in k^*$, the assertion is a consequence of the following Lemma which guarantees that the automorphisms $\tilde{\theta}_a$ of S are not algebraically extendable to an automorphism of $\mathbb{A}^3_k$ for every $a \in k^* \setminus \{1\}$. □

Lemma 4.5. *If Φ is an algebraic automorphism of $\mathbb{A}^3_k$ extending an automorphism of S, then $\Phi^*(x) = x$.*

Proof. Our proof is similar to the one of Theorem 2.1 in [18]. We let Φ be an automorphism of $\mathbb{A}^3_k$ extending an arbitrary automorphism of S. Since $f_1 = x^h z - (1-x)P(y)$ is an irreducible polynomial, there exists $\mu \in k^*$ such that $\Phi^*(f_1) = \mu f_1$. Therefore, for every $t \in k$, the automorphism Φ induces an isomorphism between the level surfaces $f_1^{-1}(t)$ and $f_1^{-1}\left(\mu^{-1}t\right)$ of f_1.

There exists an open subset $U \subset \mathbb{A}^1_k$ such that for every $t \in U$, $f_1^{-1}(t)$ is a special Danielewski surfaces isomorphic to a one defined by a fine k-weighted rake γ whose underlying tree Γ is isomorphic to the one associated with S. Since Γ is not a comb, it follows from Theorem 2.11 that for every $t \in U$, the projection $\mathrm{pr}_x : f_1^{-1}(t) \to \mathbb{A}^1_{\mathbb{C}}$ is a unique $\mathbb{A}^1$-fibration on $f_1^{-1}(t)$ up to automorphisms of the base. Furthermore, $\mathrm{pr}_x : f_1^{-1}(t) \to \mathbb{A}^1_k$ has a unique degenerate fiber, namely $\mathrm{pr}_x^{-1}(0)$. Therefore, for every $t \in U$, the image of the ideal $(x, f_1 - t)$ of $k[x,y,z]$ by Φ^* is contained in the ideal $(x, \mu f_1 - t) = \left(x, P(y) + \mu^{-1}t\right)$, and so $\Phi^*(x) \in \bigcap_{t \in U}\left(x, P(y) + \mu^{-1}t\right) = (x)$. Since Φ is an automorphism of $\mathbb{A}^3_k$, we conclude that there exists $c \in k^*$ such that $\Phi^*(x) = cx$. In turn, this implies that for every $t, u \in k$, Φ induces an isomorphism between the surfaces $S_{t,u}$ and $\tilde{S}_{t,u}$ defined by the equations $f_1 + tx + u = x^h z - (1-x) P(y) + tx + u = 0$ and $f_1 + \mu^{-1}ctx + \mu^{-1}u = x^h z - (1-x) P(y) + \mu^{-1}ctx + \mu^{-1}u = 0$ respectively. Since $\deg(P) \geq 2$ there exists $y_0 \in k$ such that $P'(y_0) = 0$. Note that y_0 is not a root of P as these ones are simple. We let $t = -u = -P(y_0)$. Since $h \geq 2$, it follows from the Jacobian Criterion that $S_{t,u}$ is singular, and even non normal along the nonreduced component of the fiber $\mathrm{pr}_x^{-1}(0)$ defined by the equation $\{x = 0; y = y_0\}$. Therefore $\tilde{S}_{t,u}$ must be singular along a multiple component of the fiber $\mathrm{pr}_x^{-1}(0)$. This the case if and only if the polynomial $P(y) - \mu^{-1}cP(y_0)$ has a multiple root, say y_1, such that $P(y_1) - \mu^{-1}P(y_0) = 0$. Since $P(y_0) \neq 0$ this condition is satisfied if and only if $c = 1$. This completes the proof. □

Example 4.6. In particular, even the involution of the surface S defined by the equation $x^2 z - (1-x) P(y) = 0$ induced by the endomorphism $J(x,y,z) = (-x, y, (1+x)((1+x)z + P(y)))$ of $\mathbb{A}^3_k$ does not extend to an algebraic automorphism of $\mathbb{A}^3_k$.

It turns out that this kind of phenomenon does not occur with additive group actions. More precisely, we have the following result.

Proposition 4.7. *Let $S_{Q,h}$ be the Danielewski surface in $\mathbb{A}^3_k$ defined by the equation $x^h z - Q(x,y) = 0$. Then, every $\mathbb{G}_a$-action on $S_{Q,h}$ arrises as the restriction of a $\mathbb{G}_a$-action on $\mathbb{A}^3_k$ defined by $\tilde{\Delta}(t,x,y,z) = \left(x, y + x^h b(x)t, z + x^{-h}(Q(x, y + x^h b(x)t) - Q(x,y))\right)$, for a certain polynomial $b(x) \in k[x]$.*

Proof. With the notation of Remark 3.11, it follows from Theorem 3.15 that every additive group action on $S_{Q,h}$ is induced by the restriction to $S_{Q,h}$ of a collection of endomorphisms of $\mathbb{A}^3_k$ of the form $\delta_{t,b} = \Phi^s \circ \Delta_{tb(x)} \circ \Phi_s$, where $b \in k[x]$. One checks that

$$\delta_{t,b}(x,y,z) = (x, y + x^h b(x)t, z + x^{-h}(Q(x, y + x^h b(x)t) - Q(x,y)) + \alpha(x,y)(x^h z - Q(x,y))),$$

for a certain polynomial $\alpha(x,y) \in k[x,y]$. Note that if $\alpha(x,y) \neq 0$, these endomorphisms $\delta_{t,b}$ do not define a $\mathbb{G}_a$-action on $\mathbb{A}^3_k$. However, they induce an action on $S_{Q,h}$ which coincides with the one induced by the $\mathbb{G}_a$-action $\tilde{\Delta}$ above. □

4.8. If $k = \mathbb{C}$, Corollary 3.17 implies in particular that every automorphism of S extends to an holomorphic automorphism of $\mathbb{A}^3_{\mathbb{C}}$. This leads the following result which contrasts with an example, given by H. Derksen, F. Kutzschebauch and J. Winkelmann in [6], of a non-extendable $\mathbb{C}_+$-action on an hypersurface in $\mathbb{A}^5_{\mathbb{C}}$ which is even holomorphically inextendable .

Proposition 4.9. *Every surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^h z - (1-x) P(y) = 0$, where $h \geq 2$ and where $P(y)$ has $r \geq 2$ simple roots, admits a nontrivial $\mathbb{C}^*$-action which is algebraically inextendable but holomorphically extendable to $\mathbb{A}^3_{\mathbb{C}}$.*

Proof. We let $\tilde{\theta} : \mathbb{C}^* \times S \to S$ be the $\mathbb{C}^*$-action on the surface $S \subset \mathbb{A}^3_{\mathbb{C}}$ defined by the equation $x^2 z - (1-x) P(y) = 0$ constructed in the proof of Theorem 4.4. For every $a \in \mathbb{C}^*$, the automorphism $\tilde{\theta}(a, \cdot)$ of S maps a closed point $(x,y,z) \in S$ to the point $\tilde{\theta}(a,x,y,z) = \left(ax, y, a^{-2}(1-ax)((1+x)z + P(y))\right)$. One checks that the holomorphic automorphism Φ_a of $\mathbb{A}^3_{\mathbb{C}}$ such that $\Phi_a|_S = \tilde{\theta}(a, \cdot)$ is the following one:

$$\Phi_a(x,y,z) = \left(ax, y, a^{-2} e^{(1-a)x} z + (ax)^{-2} P(y) \left(e^{(1-a)x}(x-1) - ax + 1\right)\right).$$

Clearly, the holomorphic map $\Phi : \mathbb{C}^* \times \mathbb{A}^3_{\mathbb{C}} \to \mathbb{A}^3_{\mathbb{C}}$, $(a, (x, y, z)) \mapsto \Phi_a(x, y, z)$ defines a $\mathbb{C}^*$-action on $\mathbb{A}^3_{\mathbb{C}}$ extending the one $\tilde{\theta}$ on S. □

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