SEMI-ALGEBRAIC NEIGHBORHOODS OF CLOSED SEMI-ALGEBRAIC SETS

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ABSTRACT. Given a closed (not necessarily compact) semi-algebraic set \( X \) in \( \mathbb{R}^n \), we construct a nonnegative semi-algebraic \( C^2 \) function \( f \) such that \( X = f^{-1}(0) \) and such that for \( \delta > 0 \) sufficiently small, the inclusion \( X \subset f^{-1}([0, \delta]) \) is a retraction. As a corollary, we obtain several formulas for \( \chi(X) \).

1. Introduction

Let \( X \) be a compact algebraic set in \( \mathbb{R}^n \). The set \( X \) is the set of zeros of a nonnegative polynomial function \( f \). This function \( f \) may not be proper as it is explained by the following example due to H. King: let \( f(x, y) = (x^2 + y^2)((y(x^2 + 1) - 1)^2 + y^2) \), then \( f^{-1}(0) = \{0\} \) but \( f(x, (1 + x^2)^{-1}) \to 0 \) as \( \|x\| \to +\infty \).

In [Dur], Durfee proved that any compact algebraic set \( X \) can be written as the set of zeros of a proper nonnegative polynomial function \( g \). Following’s Thom terminology, he called such a function a rug function for \( X \). Then he defined the notion of algebraic neighborhoods: a subset \( T \) with \( X \subset T \subset \mathbb{R}^n \) is an algebraic neighborhood of \( X \) in \( \mathbb{R}^n \) if \( T = g^{-1}([0, \delta]) \), where \( g \) is a rug function for \( X \) and \( \delta \) is a positive real smaller than all nonzero critical values of \( g \). Using the gradient vector field of \( g \), he showed that the inclusion \( X \subset T \) is a homotopy equivalence. Thanks to Lojasiewicz’s work [Lo1,Lo2] on the trajectories of a gradient vector field, it is not difficult to see that this homotopy equivalence is actually a retraction. Durfee also proved that two algebraic neighborhoods of a compact algebraic set are isotopic. Here also, this uniqueness result is obtained integrating appropriate gradient vector fields.

If \( X \) is a non-compact algebraic set in \( \mathbb{R}^n \) and \( f \) is a nonnegative polynomial such that \( X = f^{-1}(0) \), then \( X \) is not in general a deformation retract of \( f^{-1}([0, \delta]) \), where \( \delta \) is a small regular value of \( f \). Let \( f(x, y) = |y(xy - 1)|^2 \) (\( f \) is the square of the Broughton polynomial [Br] and let \( X = f^{-1}(0) \). For \( \delta \) a sufficiently small positive regular value of \( f \), \( f^{-1}([0, \delta]) \) has one connected components whereas \( X \) has three.

Our aim is to extend Durfee’s results to the case of closed (not necessarily compact) semi-algebraic sets. More precisely, we consider a closed semi-algebraic set \( X \) in \( \mathbb{R}^n \) and an open semi-algebraic neighborhood \( U \) of \( X \).

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in $\mathbb{R}^n$. We say that $f : U \to \mathbb{R}$ is an approaching function for $X$ in $U$ (Definition 2.2) if:

1. $f$ is semi-algebraic, $C^2$, nonnegative,
2. $X = f^{-1}(0)$,
3. there exists $\delta > 0$ such that $f^{-1}([0, \delta])$ is closed in $\overline{U}$.

However, the notion of approaching function is not enough to get a deformation retract as it is suggested by the Broughton example above. Let $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a proper $C^2$ semi-algebraic function, let $f : U \to \mathbb{R}$ be a $C^2$ nonnegative semi-algebraic function such that $X = f^{-1}(0)$ and let $\Gamma_{f,\rho}$ be the set of points $x$ in $U \setminus X$ where $\nabla f(x)$ and $\nabla \rho(x)$ are colinear (here $\nabla f$ denotes the gradient vector field of $f$). We say that $f$ is $\rho$-quasiregular (Definition 2.5) if there does not exist any sequence $(x_k)_{k \in \mathbb{N}}$ of points in $\Gamma_{f,\rho}$ such that $\|x_k\| \to +\infty$ and $f(x_k) \to 0$. A $\rho$-quasiregular approaching semi-algebraic neighborhood of $X$ in $U$ (Definition 3.1) is defined as a set $T = f^{-1}([0, \delta])$ such that:

1. $f$ is a $\rho$-quasiregular approaching function for $X$ in $U$,
2. $\delta$ is a positive number smaller than all nonzero critical values of $f$,
3. $f^{-1}([0, \delta])$ is closed in $\overline{U}$,
4. $\Gamma_{f,\rho}$ does not intersect $f^{-1}([0, \delta])$ outside a compact subset $K$ of $\mathbb{R}^n$.

We prove that $\rho$-quasiregular approaching semi-algebraic neighborhoods always exist (Corollary 2.8) and that if $T = f^{-1}([0, \delta])$ is a $\rho$-quasiregular approaching semi-algebraic neighborhood of $X$ in $U$ then $X$ is a strong deformation retract of $T$ (Theorem 3.2). In order to construct this retraction, we study a vector field $w$ that is equal to the gradient of $f$ inside a compact subset of $\mathbb{R}^n$ and to the orthogonal projection of the gradient of $f$ onto the levels of $\rho$ outside a compact set. Using the Lojasiewicz inequality with parameters due to Fekak [Fe] and the usual Lojasiewicz gradient we establish an inequality of “Lojasiewicz’s type” for the norm of $w$. The retraction is then achieved “pushing” $T = f^{-1}([0, \delta])$ along the trajectories of $w$.

After we show that two $\rho$-quasiregular approaching semi-algebraic neighborhoods of $X$ are isotopic (Theorem 4.4). As above, the isotopy is obtained integrating a vector field which is equal to a gradient vector field on a compact set of $\mathbb{R}^n$ and to the projection of this gradient vector field onto the levels of $\rho$ at infinity.

As a corollary, this enables us to prove that when $X$ is smooth of class $C^3$, every approaching semi-algebraic neighborhood of $X$ is isotopic to a tubular neighborhood of $X$ (Theorem 5.7).

We end the paper with degree formulas for the Euler-Poincaré characteristic of any closed semi-algebraic set obtained thanks to the machinery developed before (Theorem 6.3, Corollary 6.4 and Corollary 6.5), and with a Petrovskii-Oleinik inequality for the Euler-Poincaré characteristic of any real algebraic set (Proposition 6.8).

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2. $\rho$-QUASIREGULAR APPROACHING FUNCTIONS

In this section, we define the notion of a $\rho$-quasiregular approaching function for a closed semi-algebraic set, which generalizes the notion of a rug function introduced by Durfee [Dur].

Let us consider a closed semi-algebraic set $X$ in $\mathbb{R}^n$. Let $U$ be an open semi-algebraic neighborhood of $X$. We know that $X$ is the zero set in $U$ of a continuous nonnegative semi-algebraic function $f : U \to \mathbb{R}$ (for example one can take for $f$ the restriction to $U$ of the distance function to $X$). For any $\delta > 0$, the set $f^{-1}([0, \delta])$ is closed in $U$ for the induced topology. However, even if $\delta$ is very small, it is not necessarily closed in $\overline{U}$, as it is shown in the following examples.

Example 1. The set $X = \{0\}$ is a closed semi-algebraic set in $\mathbb{R}$, the set $U = [-1, +\infty[$ is an open semi-algebraic neighborhood of $X$ in $\mathbb{R}$. Let $f : U \to \mathbb{R}$ be defined by $f(x) = x^2(x + 1)$. It is clear that for any $\delta > 0$, the set $f^{-1}([0, \delta])$ is not closed in $\overline{U} = [-1, +\infty[$.

Example 2. The set $X = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ is a closed semi-algebraic set in $\mathbb{R}^2$, the set $U = \{(x, y) \in \mathbb{R}^2 \mid x^2y^2 < 1\}$ is an open semi-algebraic neighborhood of $X$ in $\mathbb{R}^2$. Let $f : U \to \mathbb{R}$ be defined by $f(x, y) = y^2$. For any $\delta > 0$, the set $f^{-1}([0, \delta])$ is not closed in $\overline{U} = \{(x, y) \in \mathbb{R}^2 \mid x^2y^2 \leq 1\}$.

We would like to avoid this situation. For this we need to put a condition on the tuple $(X, U, f)$.

Definition 2.1. Let $X$ be a closed semi-algebraic set in $\mathbb{R}^n$, let $U$ be an open neighborhood of $X$ and let $f : U \to \mathbb{R}$ be a nonnegative continuous semi-algebraic function such that $X = f^{-1}(0)$. We say that $(X, U, f)$ satisfies the condition $(A)$ if there does not exist any sequence $(x_k)_{k \in \mathbb{N}}$ of points in $U$ such that $\lim_{k \to +\infty} f(x_k) = 0$ and such that $\lim_{k \to +\infty} x_k$ exists and belongs to $\text{Bd}(U) = \overline{U} \setminus U$.

It is clear that this condition is satisfied when $U = \mathbb{R}^n$. Let us remark that for any couple $(X, U)$, $X$ being a closed semi-algebraic set in $\mathbb{R}^n$ and $U$ an open semi-algebraic neighborhood of $X$, there exists a function $f$ such that $(X, U, f)$ satisfies the condition $(A)$. Let $d_1 : \mathbb{R}^n \to \mathbb{R}$ be the distance function to $X$ and let $d_2 : \mathbb{R}^n \to \mathbb{R}$ be the distance function to $\text{Bd}(U)$. Let $d : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$d(x) = \frac{d_1(x)}{d_1(x) + d_2(x)}.$$

The tuple $(X, U, d_U)$ satisfies the condition $(A)$.

We will explain how to construct from a function $f$ such that $(X, U, f)$ satisfies the condition $(A)$, a nonnegative continuous semi-algebraic function $g$ such that $X = g^{-1}(0)$ and $g^{-1}([0, \delta])$ is closed in $\overline{U}$ for $\delta$ small enough. Actually we will prove a stronger result.

Let us fix a proper $C^2$ semi-algebraic function $\rho : \mathbb{R}^n \to [0, \infty[$. We will denote by $\Sigma_\rho$ the set $\rho^{-1}(r)$, by $D_\rho$ the set $\rho^{-1}([0, r])$ and by $E_\rho$ the set
that for $r$ sufficiently big, $\Sigma_r$ is a non-empty compact $C^2$-submanifold of $\mathbb{R}^n$. We will call such a $\rho$ a control function.

**Lemma 2.2.** Let $X$ be a closed semi-algebraic set in $\mathbb{R}^n$, let $U$ be an open semi-algebraic neighborhood of $X$ and let $f : U \to \mathbb{R}$ be continuous nonnegative semi-algebraic function such that $X = f^{-1}(0)$ and $(X, U, f)$ satisfies the condition (A). For every integer $q \geq 0$, let $f_q : U \to \mathbb{R}$ be defined by $f_q(x) = (1 + \rho(x))^q f(x)$. Let $V \subset U$ be an open semi-algebraic neighborhood of $X$. There exists an integer $q_0$ such that for every integer $q \geq q_0$, there exists $\delta_q > 0$ such that $f_q^{-1}([0, \delta_q])$ is included in $V$ and closed in $\overline{V}$. Furthermore, if $X$ is compact then one can choose $q_0$ such that for every integer $q \geq q_0$, $f_q^{-1}([0, \delta_q])$ is compact in $\overline{V}$.

**Proof.** Let $Z$ be the following closed semi-algebraic set : $Z = \overline{U \setminus V}$. Let $d : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative semi-algebraic function such that $X = d^{-1}(0)$ and $Z = d^{-1}(1)$. Let $U_1$ be the open semi-algebraic neighborhood of $X$ in $\mathbb{R}^n$ defined by $U_1 = d^{-1}([0, \frac{1}{4}])$ and let $V_1$ be the open semi-algebraic neighborhood of $X$ in $U$ defined by $V_1 = U_1 \cap U$. It is straightforward to see that $\overline{V_1} \subset V$.

Let us study first the case when $U$ is bounded. There exists $\delta > 0$ such that $f^{-1}([0, \delta]) \subset V_1$. Otherwise, we would be able to construct a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $\overline{U \setminus V_1}$ such that $\lim_{k \to +\infty} f(x_k) = 0$. By compactness of $\overline{U \setminus V_1}$, there would exist a subsequence of points $(x_{\varphi(k)})_{k \in \mathbb{N}}$ in $\overline{U \setminus V_1}$ such that $f(x_{\varphi(k)})$ tends to 0 and $x_{\varphi(k)}$ tends to a point $y$ in $\overline{U \setminus V_1}$. If $y$ belongs to $U$ then $f(y) = 0$, which is impossible. So $y$ belongs to $\overline{U \setminus V_1}$, which is also impossible by the condition (A). Since $\overline{V_1}$ is included in $V$ and $\overline{V_1}$ is bounded, the set $f^{-1}([0, \delta])$ is compact in $\overline{V}$.

If $U$ is not bounded and $X$ is not compact, then the following semi-algebraic set $F = U \setminus V_1$ is unbounded as well. There exists $r_0$ such that for every $r \geq r_0$, $\Sigma_r \cap F$ is not empty (the set $\{r \in \mathbb{R} \mid \Sigma_r \cap F \neq \emptyset\}$ is an unbounded semi-algebraic set of $\mathbb{R}$). Let $\alpha : [r_0, +\infty[ \to \mathbb{R}$ be defined by

$$\alpha(r) = \inf\{f(x) \mid x \in \Sigma_r \cap F\}.$$ 

The function $\alpha$ is a semi-algebraic function. Let us show that it is positive. If $\alpha(r) = 0$ then there exists a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $F \cap \Sigma_r$ such that $f(x_k)$ tends to 0. By compactness of $\Sigma_r$, we can extract a subsequence $(x_{\varphi(k)})_{k \in \mathbb{N}}$ such that $f(x_{\varphi(k)})$ tends to 0 and $x_{\varphi(k)}$ tends to a point $y$ in $\Sigma_r \cap F$, which is included in $\Sigma_r \cap \overline{U}$. If $y$ belongs to $U$ then $f(y) = 0$ and so $y$ belongs to $X$, which is impossible for $d(y) \geq \frac{1}{2}$. Hence $y$ is in $\text{Bd}(U)$.

This is impossible by the condition (A). The function $\alpha^{-1}$ is semi-algebraic. From Proposition 2.11 in [Co1] (see also Proposition 2.6.1 in [BCR]), there exists $r_1 \geq r_0$ and an integer $q_0$ such that $\alpha(r)^{-1} < r^q$ for every $r \geq r_1$ and every integer $q \geq q_0$. This implies that for every $x \in F \cap E_{r_1}$ and for $q \geq q_0$, $f_q(x) = (1 + \rho(x))^q f(x) > 1$. It is clear that $(X, U, f_q)$ satisfies the condition (A). The same argument as in the case $U$ bounded shows
that there exists $\epsilon_q$ such that $f_q^{-1}([0, \epsilon_q]) \cap D_{r_1}$ is included in $V_1 \cap D_{r_1}$. We take for $\delta_q$ the minimum of 1 and $\epsilon_q$. Since $\overline{V_1} \subset V$, it is easy to see that $f_q^{-1}([0, \delta_q])$ is closed in $\overline{V}$.

It remains to study the case $U$ unbounded but $X$ compact. There exists $r_2 > 0$ such that $X \cap E_{r_2}$ is empty. Let $\beta : [r_2, +\infty[ \to \mathbb{R}$ be defined by

$$\beta(r) = \inf\{f(x) \mid x \in U \cap \Sigma_r\}.$$ 

Thanks to condition (A), we can prove that it is a positive semi-algebraic function. There exists $r_3 \geq r_2$ and an integer $q_1$ such that $\beta(r)^{-1} < r^q$ for every $r \geq r_3$ and every integer $q \geq q_1$. Hence for $x \in U \cap E_{r_3}$ and for $q \geq q_1$, $f_q(x) = (1 + \rho(x))^q.f(x) > 1$. The tuple $(X, U, f_q)$ satisfies the condition (A). As in the previous cases, there exists $\epsilon_q > 0$ such that $f_q^{-1}([0, \epsilon_q]) \cap D_{r_3}$ is included in $V_1 \cap D_{r_3}$. We take for $\delta_q$ the minimum of 1 and $\epsilon_q$. The set $f_q^{-1}([0, \delta_q])$ is compact in $\overline{V_1}$ because it is compact in $\mathbb{R}^n$.

**Definition 2.3.** Let $X$ be a closed semi-algebraic set in $\mathbb{R}^n$ and let $U$ be an open semi-algebraic neighborhood of $X$ in $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called an approaching function for $X$ in $U$ if:

1. $f$ is semi-algebraic, $C^2$, nonnegative,
2. $X = f^{-1}(0)$, 
3. there exists $\delta > 0$ such that $f^{-1}([0, \delta])$ is closed in $\overline{U}$. Furthermore, if $X$ is compact then $f^{-1}([0, \delta])$ is compact in $\overline{U}$.

**Proposition 2.4.** Let $X$ be a closed semi-algebraic set in $\mathbb{R}^n$ and let $U$ be an open semi-algebraic neighborhood of $X$ in $\mathbb{R}^n$. There exist approaching functions for $X$ in $U$.

**Proof.** From [DM] Corollary C.12, it is possible to find a $C^2$ semi-algebraic function $\phi : \mathbb{R}^n \to [0, 1]$ such that $X = \phi^{-1}(0)$ and $\text{Bd}(U) = \phi^{-1}(1)$. Let $f$ be the restriction of $\phi$ to $U$. The tuple $(X, U, f)$ satisfies the condition (A). Applying Lemma 2.2 to $f$ and $U$, we can construct approaching functions for $X$ in $U$. \qed

In his study of regularity at infinity of polynomial functions, Tibar introduced the notion of $\rho$-regularity [Ti]. Let us recall this definition in the real setting.

**Definition 2.5.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. We say that the fibre $f^{-1}(t_0)$ is $\rho$-regular at infinity if for any sequence $(x_k) \subset \mathbb{R}^n$ such that $\|x_k\| \to +\infty$ and $f(x_k) \to t_0$, there exists $k_0$ such that if $k \geq k_0$, then $f^{-1}(f(x_k))$ is transverse to $\rho^{-1}(\rho(x_k))$ at $x_k$.

Tibar proved that if the fibre $f^{-1}(t_0)$ is $\rho$-regular at infinity then $f$ is topologically trivial at infinity at $t_0$. As a consequence, if an interval $[a, b]$ does not contain any critical value of $f$ and is such that every fibre $f^{-1}(t)$, $t \in [a, b]$, is $\rho$-regular at infinity, then $f$ is topologically trivial over $[a, b]$.

We will need a slight modification of this definition. For every open semi-algebraic set $U$ and for every $C^2$ semi-algebraic function $g : U \to \mathbb{R}$, let $\Gamma_{g, \rho}$
be the semi-algebraic set defined by:
\[ \Gamma_{q, \rho} = \{ x \in U \mid \nabla g(x) \text{ and } \nabla \rho(x) \text{ are colinear and } g(x) \neq 0 \} . \]

**Definition 2.6.** Let \( g : U \to \mathbb{R} \) be a \( C^2 \) semi-algebraic function. We say that \( g \) is \( \rho \)-quasiregular if there does not exist any sequence \((x_k)_{k \in \mathbb{N}}\) in \( \Gamma_{f, \rho} \) such that \( \|x_k\| \) tends to infinity and \( |g(x_k)| \) tends to 0.

Note that our definition does not imply that \( g^{-1}(0) \) has only isolated singularities, unlike Tibar’s definition.

**Proposition 2.7.** Let \( X \) be a closed semi-algebraic set in \( \mathbb{R}^n \) and let \( U \) be an open semi-algebraic neighborhood of \( X \). Let \( f : U \to \mathbb{R} \) be a \( C^2 \) semi-algebraic nonnegative function such that \( X = f^{-1}(0) \). For every integer \( q \), let \( f_q : U \to \mathbb{R} \) be defined by \( f_q = (1 + \rho(x))^q \cdot f(x) \). There exists an integer \( q_0 \) such that for every integer \( q \geq q_0 \), the function \( f_q \) is \( \rho \)-quasiregular.

Proof. Let \( r_0 \) be the greatest critical value of \( \rho \) and let \( \beta : [r_0, +\infty) \to \mathbb{R} \) be defined by
\[ \beta(r) = \inf\{ f(x) \mid x \in \Sigma_r \cap \Gamma_{f, \rho} \}. \]
The function \( \beta \) is semi-algebraic. It is positive since for \( r > r_0 \), the function \( f|_{\Sigma_r \cap U} \) admits a finite number of critical values. As in Lemma 2.2, this implies that there exists \( r_1 > r_0 \) and an integer \( q_0 \) such that for \( x \in \Gamma_{f, \rho} \cap E_{r_1} \) and for \( q \geq q_0 \), \( (1 + \rho(x))^q \cdot f(x) > 1 \). Since \( \Gamma_{f, \rho} = \Gamma_{f_q, \rho} \), every function \( f_q \) is \( \rho \)-quasiregular for \( q \geq q_0 \). \qed

**Corollary 2.8.** Let \( X \) be a closed semi-algebraic set in \( \mathbb{R}^n \) and let \( U \) be an open semi-algebraic neighborhood of \( X \). Let \( f : U \to \mathbb{R} \) be a \( C^2 \) semi-algebraic nonnegative function such that \( X = f^{-1}(0) \). Assume that \( (X, U, f) \) satisfies the condition (A). For every integer \( q \geq 0 \), let \( f_q : \mathbb{R}^n \to \mathbb{R} \) be defined by \( f_q(x) = (1 + \rho(x))^q \cdot f(x) \). There exists an integer \( q_0 \) such that for every \( q \geq q_0 \), the function is a \( \rho \)-quasiregular approaching function for \( X \) in \( U \).

If \( X \) is an algebraic set, it is the zero set of a nonnegative polynomial \( f \). Choosing for \( \rho \) a proper nonnegative polynomial and applying the above process, we obtain \( \rho \)-quasiregular approaching functions for \( X \) that are nonnegative polynomials.

Let us compare our notion of \( \rho \)-quasiregular approaching function with the notion of rug function due to Durfee [Dur]. If \( X \) is a compact algebraic set of \( \mathbb{R}^n \), a rug function for \( X \) is a proper nonnegative polynomial \( f \) such that \( X = f^{-1}(0) \). It is clear that such a function is a \( \rho \)-quasiregualr approaching function for \( X \) in \( \mathbb{R}^n \).

3. Retraction on a closed semi-algebraic set

In this section, we prove that any closed semi-algebraic set is a strong deformation retract of certain closed semi-algebraic neighborhoods of it. First let us specify the closed semi-algebraic neighborhoods that we will consider.
Definition 3.1. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set, let $\rho$ be a control function and let $U$ be an open semi-algebraic neighborhood of $X$. A subset $T$ with $X \subset T \subset U$ is a $\rho$-quasiregular approaching semi-algebraic neighborhood of $X$ in $U$ if $T = f^{-1}([0, \delta])$ where:

1. $f$ is a $\rho$-quasiregular approaching function for $X$ in $U$,
2. $\delta$ is a positive number smaller than all nonzero critical values of $f$,
3. $f^{-1}([0, \delta])$ is closed in $\overline{U}$ and compact in $U$ if $X$ is compact,
4. if $\Gamma_f(\rho)$ is the following polar set:
   \[ \Gamma_f(\rho) = \{ x \in U \setminus X \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are colinear} \}, \]
   then $\Gamma_f(\rho)$ does not intersect $f^{-1}([0, \delta])$ outside a compact subset $K$ of $\mathbb{R}^n$.

For short, we will say that such a $T$ is an approaching semi-algebraic neighborhood. By the results of the previous section, it is clear that approaching semi-algebraic neighborhoods always exist.

Theorem 3.2. Let $X$ be a closed semi-algebraic set and let $T$ be an approaching semi-algebraic neighborhood of $X$. Then $X$ is a strong deformation retract of $T$.

Proof. If $X$ is compact, this is already proved by Durfee [Du] and Lojasiewicz [Lo1,Lo2]. So let us assume that $X$ is not compact.

Let us fix $f$, $U$, $\delta$, $\rho$ and $K$ which satisfy the conditions of the above definition and such that $T = f^{-1}([0, \delta])$. Furthermore let us assume that $\delta < 1$. We will focus first on the behaviour of $f$ at infinity.

Let $r_0 > 0$ be such that $K \cap E_{r_0}$ is empty and such that $\Sigma_r$ is a $C^2$ sub manifold for $r \geq r_0$. Let $A = T \cap E_{r_0}$. The set $A$ is a closed semi-algebraic set of $\mathbb{R}^n$ and $A \cap \Gamma_f(\rho)$ is empty. Let us consider the following closed semi-algebraic set $Y$ of $\mathbb{R}^{n+1}$:
\[ Y = \{ (x, t) \in \mathbb{R}^{n+1} \mid x \in A \text{ and } \rho(x) = t \}. \]

We will denote by $Y_t$ the fibre $\{ x \in A \mid (x, t) \in Y \}$. Observe that $Y_t = A \cap \Sigma_t$.

Let $F : A \rightarrow \mathbb{R}$ be defined by:
\[ F(x) = \| \nabla f(x) - \langle \nabla f(x), \nabla \rho(x) \| \frac{\nabla \rho(x)}{\| \nabla \rho(x) \|} \|. \]

The function $F$ is just the norm of the orthogonal projection of $\nabla f(x)$ on the manifold $\Sigma_{\rho(x)}$. Moreover it is a continuous semi-algebraic function on $A$.

Let $\tilde{f}$ and $\tilde{F}$ be the semi-algebraic functions defined on $Y$ by $\tilde{f}(x, t) = f(x)$ and $\tilde{F}(x, t) = F(x)$. They are continuous in $x$ and verify $\tilde{F}^{-1}(0) \subset \tilde{f}^{-1}(0)$. This inclusion is easy to check since $F(x) = 0$ if and only if $\nabla f(x)$ and $\nabla \rho(x)$ are colinear on $A$, this can occur only if $x$ belongs to $X$.

We can apply Lojasiewicz's inequality with parameters due to Fekak (see [Fe], p128). We need some notations: for every $t$, $\tilde{f}_t$ and $\tilde{F}_t$ are the functions on $Y_t$ defined by $\tilde{f}_t(x) = \tilde{f}(x, t)$ and $\tilde{F}_t(x) = \tilde{F}(x, t)$; for every $S \subset \mathbb{R}$, $Y_S$ denotes the set $Y \cap (\mathbb{R}^n \times S)$. Fekak's theorem states that there exists a finite
partition into semi-algebraic subsets of $\mathbb{R} = \cup S_i$, continuous semi-algebraic functions $h_i : Y_i | S_i \to \mathbb{R}$ and rational numbers $p_i / q_i$ such that:

i) $|f(x,t)|^{p_i/q_i} \leq h_i(x,t)|F(x,t)|$ on $Y_i | S_i$ for $t \in S_i$,

ii) $p_i / q_i$ is the Lojasiewicz exponent of $f_i$ with respect to $F_i$ for $t \in S_i$.

Since $\cup S_i$ is a finite semi-algebraic partition of $\mathbb{R}$, there exist $t_0 \in \mathbb{R}$ and $t_0$ such that $S_{t_0} = [t_0, +\infty[$. Then for every $t \geq t_0$, we have:

i) $|\tilde{f}(x,t)|^{p_i/q_i} \leq h_i(x,t)|\tilde{F}(x,t)|$ for $x \in Y_t$,

ii) $p_i / q_i$ is the Lojasiewicz exponent of $\tilde{f}_i$ with respect to $\tilde{F}_i$.

We know that $\tilde{f}_i = f_{|Y_i}$ and $\tilde{F}_i = \|\nabla f_{|Y_i}\|$. By the Lojasiewicz gradient inequality applied to $f_{|Y_i}$, we get $p_i/q_i \leq 1$. Let $\alpha = p_i/q_i$ and let $B = T \cap E_{t_0}$. We have proved that there exist $0 \leq \alpha < 1$ and a continuous semi-algebraic function $h : B \times [t_0, +\infty[ \to \mathbb{R}$ such that for every $x \in B$:

$$|f(x)|^{\alpha} \leq h(x, \rho(x))F(x),$$

where $F(x)$ is the norm of the vector field

$$v(x) = \nabla f(x) - \langle \nabla f(x), \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|} \rangle \cdot \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|}.$$

Let $C$ be the compact semi-algebraic set defined by $C = T \cap D_{2t_0}$. By the Lojasiewicz gradient inequality, there exits $d > 0$ and $0 \leq \beta < 1$ such that on $C$:

$$|f(x)|^{\beta} \leq d\|\nabla f(x)\|.$$

Here we have applied the Kurdyka-Parusinski version of the Lojasiewicz gradient inequality [KP].

We will glue the two vector fields $v$ and $\nabla f$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a $C^\infty$-function such that:

- $\varphi(x) = 1$ if $\rho(x) \leq 1, 3t_0$,
- $\varphi(x) = 0$ if $\rho(x) \geq 1, 7t_0$,
- $0 < \varphi(x) < 1$ if $1, 3t_0 < \rho(x) < 1, 7t_0$.

Let $w$ be the following vector field on $T$:

$$w(x) = (1 - \varphi(x))v(x) + \varphi(x)\nabla f(x).$$

We want to find an inequality of “Lojasiewicz’s type” for $\|w\|$. First observe that $\|w(x)\| \geq \|v(x)\|$, for

$$w(x) = v(x) + \varphi(x) \cdot \langle \nabla f(x), \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|} \rangle \cdot \frac{\nabla \rho(x)}{\|\nabla \rho(x)\|}.$$

Let $M$ be defined by :

$$M = \max\{h(x, \rho(x)) \mid x \in T \text{ and } 1, 2t_0 \leq \rho(x) \leq 1, 8t_0\}.$$

We have $|f(x)|^{\alpha} \leq M\|w(x)\|$ for $x \in T \cap \{x \mid 1, 2t_0 \leq \rho(x) \leq 1, 8t_0\}$. For $x \in T \cap D_{1,3t_0}$, $|f(x)|^{\beta} \leq d\|\nabla f(x)\|$ and $\nabla f(x) = w(x)$. Calling $\gamma$ the
maximum of $\alpha$ and $\beta$ and $N$ the maximum of $M$ and $d$ and since $\delta < 1$, we get that for $x \in T \cap D_{1,8t_0}$:

$$|f(x)|^\gamma \leq N.\|w(x)\|. \quad (1)$$

Now for $x \in T \cap E_{1,7t_0}$, $w(x) = v(x)$ and then:

$$|f(x)|^\gamma \leq h(x, \rho(x))).\|w(x)\|. \quad (2)$$

On one hand, we have:

$$\langle \nabla f(x), w(x) \rangle = (1 - \varphi(x))\langle \nabla f(x), v(x) \rangle + \varphi(x)\langle \nabla f(x), \nabla f(x) \rangle,$$

hence

$$\langle \nabla f(x), w(x) \rangle = (1 - \varphi(x))\langle v(x), v(x) \rangle + \varphi(x)\langle \nabla f(x), \nabla f(x) \rangle,$$

since $\langle v(x), \nabla f(x) \rangle = \langle v(x), v(x) \rangle$. On the other hand, 

$$\langle w(x), w(x) \rangle = (1 - \varphi(x)^2)\langle v(x), v(x) \rangle + \varphi(x)^2\langle \nabla f(x), \nabla f(x) \rangle.$$

Using the fact that $0 \leq \varphi(x) \leq 1$, it is easy to see that

$$\langle \nabla f(x), w(x) \rangle \geq \langle w(x), w(x) \rangle \leftrightarrow \langle \nabla f(x), \nabla f(x) \rangle \geq \langle v(x), v(x) \rangle.$$

Since the inequality on the right hand side is verified, we have proved:

$$\langle \nabla f(x), w(x) \rangle \geq \langle w(x), w(x) \rangle. \quad (3)$$

We are going to integrate the vector field $-\frac{w}{\|w\|}$. Let $\phi_t$ be the flow associated with the differential equation:

$$\dot{x} = -\frac{w}{\|w\|}.$$

For every $x \in T$, let $b(x) = \sup\{t \mid f(\phi_t(x)) \geq 0\}$ and $\omega(x) = \lim_{t \to b(x)} \phi_t(x)$. We write $\phi_x(t)$ the trajectory that passes through $x$. The following facts are proved using inequalities (1), (2) and (3) and adapting to our situation the techniques of Lojasiewicz (see [Lo1], [Lo2], [Ku], [KMP] or [NS] for details).

**Fact 1** For all $x \in T$, $\{\phi_x(t) \mid 0 \leq t \leq b(x)\} \subset T$.

**Fact 2** For all $x \in T \cap E_{1,7t_0}$, for all $t$ such that $0 \leq t \leq b(x)$, $\|\phi_x(t)\| = \|x\|$.

**Fact 3** For all $x \in T \cap D_{1,8t_0}$, for all $t$ such that $0 \leq t \leq b(x)$, $\|\phi_x(t)\| \leq 1,8t_0$.

**Fact 4** For all $x \in T$, $b(x) < +\infty$.

**Fact 5** For all $x \in T$, $f(\omega(x)) = 0$.

**Fact 6** The mapping $\omega : T \to X, x \mapsto \omega(x)$ is continuous.

**Fact 7** The mapping $b : T \to X, x \mapsto b(x)$ is continuous.
Now we can end the proof of Theorem 3.2. The retraction is given by the following mapping:

\[ G : [0,1] \times T \rightarrow T \]
\[ (t, x) \mapsto G(t, x) = \phi(tb(x), x). \]

If \( \delta \geq 1 \), we can push \( f^{-1}([0, \delta]) \) onto \( f^{-1}([0, \delta']) \), \( \delta' < 1 \), along the trajectories of \( V \).

We end this section with a remark. Using the same method, one can prove the following result. Let \( X \subseteq \mathbb{R}^n \) be a closed semi-algebraic set and let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a nonnegative semi-algebraic function such that \( X = f^{-1}(0) \). Let \( \Gamma_{f, \rho} \) be the set defined by

\[ \Gamma_{f, \rho} = \{ x \in \mathbb{R}^n \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are not colinear and } f(x) \neq 0 \}. \]

Let \( r \) be a regular value of \( \rho \). Assume that the following assumption is satisfied: there is no sequence of points \( (x_k) \) in \( \Gamma_{f, \rho} \cap D_r \) such that \( \rho(x_k) \rightarrow r \) and \( f(x_k) \rightarrow 0 \). Then for \( \delta > 0 \) sufficiently small, the inclusion \( X \cap D_r \subset f^{-1}([0, \delta]) \cap D_r \) is a deformation retract.

For example, this result can be applied if \( f \) has only isolated critical points on its zero level and \( X \) intersects \( \Sigma_r \) transversally.

4. Uniqueness of \( \rho \)-quasiregular approaching neighborhood

In this section, we prove that two \( \rho \)-quasiregular approaching semi-algebraic neighborhoods of a closed non-compact semi-algebraic set are isotopic. Let us recall what Durfee proved in the case of a compact semi-algebraic set.

**Theorem 4.1.** Let \( X \) be a compact semi-algebraic set, let \( f_1 \) and \( f_2 \) be two rug functions for \( X \). Let \( \delta_1 \) (resp. \( \delta_2 \)) be a positive regular value of \( f_1 \) (resp. \( f_2 \)) smaller than all nonzero critical values of \( f_1 \) (resp. \( f_2 \)). Let \( T_1 = f_1^{-1}([0, \delta_1]) \) and \( T_2 = f_2^{-1}([0, \delta_2]) \). There is a continuous family of homeomorphisms \( h_t : \mathbb{R}^n \rightarrow \mathbb{R}^n, 0 \leq t \leq 1, \) such that:

1. \( h_0 \) is the identity,
2. for all \( t \), \( h_t|_{X} \) is the identity,
3. \( h_1(T_1) = T_2 \) and \( h_1 \) is a smooth diffeomorphism of \( T_1 \setminus X \) onto \( T_2 \setminus X \).

**Proof.** See [Dur], Proposition 1.7. Actually Durfee considers the case of a compact algebraic set but his proof works in our case.

We will prove the following theorem.

**Theorem 4.2.** Let \( X \) be a closed non-compact semi-algebraic set and let \( \rho \) be a control function. If \( T_1 \) and \( T_2 \) are two \( \rho \)-quasiregular approaching semi-algebraic neighborhoods of \( X \) in \( U \) then there is a continuous family of homeomorphisms \( h_t : \mathbb{R}^n \rightarrow \mathbb{R}^n, 0 \leq t \leq 1, \) such that:

1. \( h_0 \) is the identity,
2. for all \( t \), \( h_t|_{X} \) is the identity,
3. \( h_1(T_1) = T_2 \) and \( h_1 \) is a smooth diffeomorphism of \( T_1 \setminus X \) onto \( T_2 \setminus X \).
Proof. Let us write \( T_i = f_i^{-1}([0, \delta_i]) \) where \( f_i \) is a \( \rho \)-quasiregular approaching function for \( X \) in \( U_i \), \( i = 1, 2 \). We will prove our result adapting the ideas of Durfee [Dur]. There are three steps.

Let us first consider the case \( f_1 = f_2 = f \) and \( U_1 = U_2 = U \). We can assume without loss of generality that \( \delta_1 < \delta_2 \). Thanks to condition (4) in Definition 3.1, we see that \( f^{-1}(\delta) \) is \( \rho \)-regular at infinity for every \( \delta \) in \( [\delta_1, \delta_2] \). Since \([\delta_1, \delta_2] \) does not contain any critical value of \( f \), Tibar’s work implies that \( T_1 \) and \( T_2 \) are diffeomorphic. Let us be more precise and explain how the family \( h_t \) is obtained. As we did in the proof of Theorem 3.2, we can construct a vector field \( v \) on \( f^{-1}([\delta_1, \delta_2]) \) which is equal to the orthogonal projection of \( \nabla f \) on the levels of \( \rho \) outside a set \( D_R \), and equal to \( \nabla f \) inside a set \( D_R' \), \( R' < R \). Then we extend \( v \) to a complete vector field \( \tilde{w} \) defined on \( \mathbb{R}^n \) using a smooth function equal to 1 on the closed set \( f^{-1}([\delta_1, \delta_2]) \) and to 0 on the closed set \( X \cup (\mathbb{R}^n \setminus U) \). Integrating this vector field gives the required family \( h_t \).

The second case is when \( f_2 = (1 + \rho)^q f_1 \) and \( U_1 = U_2 = U \). Let \( \delta \) be the minimum of \( \delta_1 \) and \( \delta_2 \). Let \( v_1 \) (resp. \( v_2 \)) be the orthogonal projection of \( \nabla f_1 \) on the levels of \( \rho \). By condition (4) in Definition 3.1, there exists \( R > 0 \) such that \( v_1 \) and \( v_2 \) do not vanish in \( f_1^{-1}([0, \delta]) \cap E_R \). It is clear that on this set, they do not point in opposite direction. There exists a neighborhood \( U' \) of \( X \cap D_{2R} \) in \( D_{2R} \) such that \( \nabla f_1 \) and \( \nabla f_2 \) are nonzero and do not point in opposite direction on \( U' \setminus X \). This fact is proved in the same way as Lemma 1.8 in [Dur]. Hence there exists \( \delta' \) such that \( \nabla f_1 \) and \( \nabla f_2 \) are nonzero and do not point in opposite direction on \( f_1^{-1}([0, \delta']) \cap D_{2R} \). Let \( \delta'' \) be the minimum of \( \delta \) and \( \delta' \). By the first case, it is enough to prove that \( f_2([0, \delta'']) \) and \( f_1([0, \delta'']) \) are isotopic. Let \( S \) be the closed set \( f_1^{-1}([0, \delta'']) \setminus f_2^{-1}([0, \delta'']) \) and let \( g : S \to [0, 1] \) be defined by:

\[
g(x) = \frac{f_2(x) - \delta''}{f_2(x) - f_1(x)}.
\]

Note that \( g^{-1}(0) = f_2^{-1}(\delta'') \) and \( g^{-1}(1) = f_1^{-1}(\delta') \). The gradient of \( g \) is

\[
\nabla g(x) = \frac{(f_2(x) - \delta'')\nabla f_1(x) + (\delta'' - f_1(x))\nabla f_2(x)}{(f_2(x) - f_1(x))^2}.
\]

Let \( v \) be its orthogonal projection on the levels of \( \rho \). It is nonzero in \( S \cap E_R \). Moreover, \( \nabla g \) is nonzero in \( S \cap D_{2R} \). Gluing these two vector fields, we obtain a \( C^1 \) vector field \( w \) on \( S \) and we proceed as in the first case.

The third case is the general case. Let \( U = U_1 \cap U_2 \). By Lemma 2.2 and the second case above, we can assume that \( T_1 \subset U \), \( T_2 \subset U \) and \( T_1 \) and \( T_2 \) are closed in \( U \). We need some lemmas.

**Lemma 4.3.** For every integer \( q \geq 0 \), let \( f_{1q} : \mathbb{R}^n \to \mathbb{R} \) be defined by \( f_{1q}(x) = (1 + \rho(x))^q f_1(x) \). Let \( v_{1q} \) (resp. \( v_2 \)) be the orthogonal projection of \( \nabla f_{1q} \) (resp \( \nabla f_2 \)) on the levels of \( \rho \). There exist \( q_0 \in \mathbb{N} \) and \( R > 0 \) such that for all \( q \geq q_0 \) the vector fields \( v_{1q} \) and \( v_2 \) are nonzero and do not point in...
opposite direction in $f^{-1}_q([0, \delta_q]) \cap E_R$, where $\delta_q$ is a small regular value of $f_{1q}$ such that $f^{-1}_q([0, \delta_q]) \subset U$ and $f^{-1}_q([0, \delta_q])$ is closed in $\overline{U}$.

Proof. We known that there exists $R' > 0$ and $U' \subset U$ such that $v_1$ and $v_2$ do not vanish in $U' \cap E_{R'}$ since $f_1$ and $f_2$ are $\rho$-quasiregular. Let $\Gamma_{f_1, \rho}$, $\Gamma_{f_2, \rho}$ and $\Gamma_{f_1, f_2, \rho}$ be the following semi-algebraic sets:

\[
\Gamma_{f_1, \rho} = \{ x \in U \setminus X \mid v_1(x) = 0 \}, \quad \Gamma_{f_2, \rho} = \{ x \in U \setminus X \mid v_2(x) = 0 \},
\]

\[
\Gamma_{f_1, f_2, \rho} = \{ x \in U \setminus X \mid v_1(x) \text{ and } v_2(x) \text{ point in opposite direction} \},
\]

and let $\Gamma$ be the union of these three sets. Let $r_0$ be the greatest critical value of $\rho$ and let $l : ]r_0, +\infty[ \to \mathbb{R}$ be defined by $l(r) = \inf \{ f_1(x) \mid x \in \Sigma_r \cap \Gamma \}$. Then $l$ is a positive semi-algebraic function. To see that it is positive, it is enough to apply Lemma 1.8 of [Dur] to the semi-algebraic subset $X \cap \Sigma_r$ of the smooth semi-algebraic set $\Sigma_r$. As in Lemma 2.2, this implies that there exists $R > r_0$ and an integer $q_0$ such that for $x \in \Gamma \cap E_R$ and for $q \geq q_0$, $(1 + \rho(x))^q f_1(x) > 1$. Since $v_{1q} = (1 + \rho)^q v_1$, we see that $\Gamma_{f_1, f_2, \rho} = \Gamma_{f_{1q}, f_{2q}, \rho}$. We take $\delta_q$ to be the minimum of $\delta_1$ and 1. This ends the proof of Lemma 4.3.

\[\square\]

Lemma 4.4. For every integer $q \geq 0$, let $f_{1q} : \mathbb{R}^n \to \mathbb{R}$ (resp. $f_{2q} : \mathbb{R}^n \to \mathbb{R}$) be defined by $f_{1q}(x) = (1 + \rho(x))^q f_1(x)$ (resp. $f_{2q}(x) = (1 + \rho(x))^q f_2(x)$). Let $v_{1q}$ (resp. $v_{2q}$) be the orthogonal projection of $\nabla f_{1q}$ (resp. $\nabla f_{2q}$) on the levels of $\rho$. There exist $q_0 \in \mathbb{N}$ and $R > 0$ such that for all $q \geq q_0$ and for all $l \in \mathbb{N}$ the vector fields $v_{1q}$ and $v_{2q}$ are nonzero and do not point in opposite direction in $f^{-1}_q([0, \delta_q]) \cap E_R$, where $\delta_q$ is a small regular value of $f_{1q}$ such that $f^{-1}_q([0, \delta_q]) \subset U$ and $f^{-1}_q([0, \delta_q])$ is closed in $\overline{U}$.

Proof. It is clear because $v_{2q} = (1 + \rho)^l v_2$ and $\Gamma_{f_{1q}, f_{2q}, \rho} = \Gamma_{f_{1q}, f_{2q}, \rho}$. This ends the proof of Lemma 4.4. \[\square\]

Let us fix $q$ and $\delta_q$ which satisfy the conclusion of Lemma 4.3. Applying Lemma 2.2 to the open semi-algebraic neighborhood $f^{-1}_q([0, \delta_q])$ of $X$ and the approaching function $f_2$, we can find $l$ such that $f^{-1}_{2q}([0, \epsilon_l]) \subset f^{-1}_{1q}([0, \delta_q])$, where $\epsilon_l$ is a small regular value of $f_{2q}$. Thanks to Lemma 4.4, we can proceed as we did for the second case, namely we consider the closed set $S' = f^{-1}_q([0, \delta_q]) \setminus f^{-1}_{2q}([0, \epsilon_l])$ and the function $h : S' \to [0, 1]$ defined by:

\[
h(x) = \frac{f_{2q}(x) - \epsilon_l}{(f_{2q}(x) - \epsilon_l) - (\delta_q - f_{1q}(x))}.
\]

This ends the proof of Theorem 4.2. \[\square\]

We end this section with this question: if $X$ is a closed non-compact semi-algebraic set in $\mathbb{R}^n$, if $\rho_1$ and $\rho_2$ are two distinct control functions, are a $\rho_1$-quasiregular approaching semi-algebraic neighborhood and a $\rho_2$-quasiregular approaching semi-algebraic neighborhood of $X$ isotopic?
5. The smooth case

In this section, we assume that $X$ is a closed non-compact semi-algebraic set in $\mathbb{R}^n$ and also a $C^3$ submanifold of dimension $k < n$. We also assume that $\rho$ is a control function of class $C^3$. We show that any $\rho$-quasiregular approaching semi-algebraic neighborhood of $X$ is isotopic to a tubular neighborhood of $X$. For this, we construct a kind of distance function to $X$ which is $C^2$ in a semi-algebraic neighborhood of $X$ and $\rho$-quasiregular.

Let us fix $X$ and $\rho$ satisfying the above assumptions. Let $r_0 > 0$ be such that for all $r \geq r_0$, $\Sigma_r$ is a $C^3$ submanifold that intersects $X$ transversally. Let $F$ be the following set:

$$F = \{(x, v) \in X \times \mathbb{R}^n \mid \rho(x) > r_0, \langle v, \nabla \rho(x) \rangle = 0$$

$$\text{and } \langle v, w \rangle = 0 \text{ for all } w \in T_x(X \cap \Sigma_{\rho(x)})\}.$$  

It is a $C^2$-vector bundle over $X \cap \{x \mid \rho(x) > r_0\}$ whose fibers are $n-k$-dimensional. Moreover it is semi-algebraic. We will denote the fiber over $x$ by $F_x$. Observe that $F_x$ is the normal space of $X \cap \Sigma_{\rho(x)}$ in $\Sigma_{\rho(x)}$.

Let $N$ be the normal bundle over $X \cap \{x \mid \rho(x) < 2r_0\}$:

$$N = \{(x, v) \in X \times \mathbb{R}^n \mid \rho(x) < 2r_0 \text{ and } v \perp T_xX\}.$$  

Is is also a $C^2$ semi-algebraic vector bundle. We denote the fiber over $x$ by $N_x$.

We will glue these two bundles. By [DM] Corollary C.12, it is possible to find a $C^2$ semi-algebraic function $\phi : X \mapsto [0, 1]$ such that $X \cap E_{7/4r_0} = \phi^{-1}(1)$ and $X \cap D_{3/4r_0} = \phi^{-1}(0)$. For each $x$ such that $r_0 < \rho(x) < 2r_0$, let $P_x$ be restriction to $F_x$ of the orthogonal projection to $N_x$.

We can define a bundle $H \subset X \times \mathbb{R}^n$ over $X$ in the following way:

- if $\rho(x) < 5/4r_0$ then $H_x = N_x$,
- if $r_0 < \rho(x) < 2r_0$ then $H_x = \{v \in \mathbb{R}^n \mid \exists w \in F_x \text{ such that } v = \phi(x)w + (1 - \phi(x))P_x(w)\}$,
- if $\rho(x) > 7/4r_0$ then $H_x = F_x$.

It is an exercise of linear algebra to prove that $H$ is a vector bundle whose fibres are $n-k$-dimensional planes. Furthermore, it is $C^2$ semi-algebraic because $F$ and $N$ are $C^2$ semi-algebraic bundles and because $\phi$ is a $C^2$ semi-algebraic function. This bundle $H$ will enables us to construct the desired “distance” function to $X$. Let $\varphi$ be the following $C^2$ semi-algebraic mapping:

$$\varphi : H \to \mathbb{R}^n$$

$$(x, v) \mapsto x + v.$$  

Then there exists a semi-algebraic open neighborhood $U$ of the zero-section $X \times \{0\}$ in $H$ such that the restriction $\varphi|_U$ is a $C^2$ diffeomorphism onto a neighborhood $V$ of $X$. Moreover, we can take $U$ of the form:

$$U = \{(x, v) \mid \|v\| < \varepsilon(x)\},$$  

where $\varepsilon : X \to \mathbb{R}$ is a control function of class $C^3$. The function $\varepsilon$ is also a control function of class $C^3$. Observe that $\rho$ and $\varphi$ is a $C^2$ semi-algebraic function. We will glue these two bundles. By [DM] Corollary C.12, it is possible to find a $C^2$ semi-algebraic function $\phi : X \mapsto [0, 1]$ such that $X \cap E_{7/4r_0} = \phi^{-1}(1)$ and $X \cap D_{3/4r_0} = \phi^{-1}(0)$. For each $x$ such that $r_0 < \rho(x) < 2r_0$, let $P_x$ be restriction to $F_x$ of the orthogonal projection to $N_x$.

We can define a bundle $H \subset X \times \mathbb{R}^n$ over $X$ in the following way:

- if $\rho(x) < 5/4r_0$ then $H_x = N_x$,
- if $r_0 < \rho(x) < 2r_0$ then $H_x = \{v \in \mathbb{R}^n \mid \exists w \in F_x \text{ such that } v = \phi(x)w + (1 - \phi(x))P_x(w)\}$,
- if $\rho(x) > 7/4r_0$ then $H_x = F_x$.

It is an exercise of linear algebra to prove that $H$ is a vector bundle whose fibres are $n-k$-dimensional planes. Furthermore, it is $C^2$ semi-algebraic because $F$ and $N$ are $C^2$ semi-algebraic bundles and because $\phi$ is a $C^2$ semi-algebraic function. This bundle $H$ will enables us to construct the desired “distance” function to $X$. Let $\varphi$ be the following $C^2$ semi-algebraic mapping:

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$$U = \{(x, v) \mid \|v\| < \varepsilon(x)\},$$  

where $\varepsilon : X \to \mathbb{R}$ is a control function of class $C^3$. Observe that $\rho$ and $\varphi$ is a $C^2$ semi-algebraic function.
where \( \varepsilon \) is a positive \( C^2 \) semi-algebraic function on \( X \). The proof of this result is given in [Co2], Lemma 6.15 for the normal bundle. This proof actually holds in our case. This provides us with a \( C^2 \) semi-algebraic retraction \( \pi : V \to X \) and a \( C^2 \) semi-algebraic distance function \( d' : V \to X \) defined by \( \pi(\varphi(x, v)) = x \) and \( d'(\varphi(x, v)) = \|v\|^2 \).

**Lemma 5.1.** There exists an open semi-algebraic neighborhood \( W \) of \( X \) in \( V \) such that for every \( y \in W \), \( \rho(y) \leq 1, \rho(\pi(y)) \). Furthermore, one can choose \( W \) of the form:

\[
W = \{ y \in V \mid d'(y) < \varepsilon'(\pi(y)) \},
\]

where \( \varepsilon' : X \to \mathbb{R} \) is a positive \( C^2 \) semi-algebraic function.

**Proof.** Let \( A \) be the following semi-algebraic set:

\[
A = \{ y \in V \mid \rho(y) > 1, \rho(\pi(y)) \}
\]

Let \( \alpha : \pi(A) \to \mathbb{R} \) be the function defined as follows:

\[
\alpha(x) = \inf \{ d'(y) \mid y \in \pi^{-1}(x) \cap A \}.
\]

This is a semi-algebraic function on \( \pi(A) \). Let us prove that it is positive. The continuity of \( \rho \circ \varphi \) implies that for every \( x \) in \( \pi(A) \), there exists \( \delta_x \) with \( 0 < \delta_x < \varepsilon(x) \), such that for every \( v \) in \( H_x \) with \( \|v\| \leq \delta_x \), \( \rho(\varphi(x, v)) \leq 1, \rho(\varphi(x, 0)) \). Since \( \|v\|^2 = d'(y) \) if \( y = \varphi(x, v) \), this proves that \( \alpha(x) \geq \delta_x > 0 \). Let us show that \( \alpha \) is locally bounded from below by positive constants, i.e. for every \( x \in \pi(A) \), there exist \( c > 0 \) and a neighborhood \( \Omega \) of \( x \) in \( \pi(A) \) such that \( \alpha > c \) on \( \Omega \). If it is not the case, we can find a sequence of points \( x_n \) in \( \pi(A) \) tending to \( x \) such that \( \alpha(x_n) \) tends to 0. Hence there exists a sequence of points \( y_n = \varphi(x_n, v_n) \) such that \( v_n \) tends to 0, \( x_n \) tends to \( x \) and \( \rho(\varphi(x_n, v_n)) > 1, \rho(\varphi(x_n, 0)) \). By continuity, we obtain \( \rho(\varphi(x, 0)) \geq 1, \rho(\varphi(x, 0)) \), which is impossible. Let \( \tilde{\alpha} : X \to \mathbb{R} \) be defined by \( \tilde{\alpha}(x) = \alpha(x) \) if \( x \in \pi(A) \) and \( \tilde{\alpha}(x) = \varepsilon(x) \) if \( x \notin \pi(A) \). The function \( \tilde{\alpha} \) is semi-algebraic, positive and locally bounded from below by positive constants. Applying Lemma 6.12 of [Co2], we can find a positive semi-algebraic \( C^2 \) function \( \varepsilon' : X \to \mathbb{R} \) such that \( \varepsilon' < \tilde{\alpha} \) on \( X \).

Let us study the function \( d' : W \to \mathbb{R} \) more precisely. Let \( B \) be the following semi-algebraic set:

\[
B = \left\{ y \in W \cap E_{2n} \mid \frac{\langle \nabla \rho(y), \nabla \rho(\pi(y)) \rangle}{\|\nabla \rho(y)\| \cdot \|\nabla \rho(\pi(y))\|} < 0, 9 \right\}.
\]

Let \( \beta : \pi(B) \to \mathbb{R} \) be the function defined as follows:

\[
\beta(x) = \inf \{ d'(y) \mid y \in \pi^{-1}(x) \cap B \}.
\]

This is a semi-algebraic function on \( \pi(B) \) and for every \( x \in \pi(B) \), \( \beta(x) \leq \varepsilon'(x) \). The same argument as in the above lemma shows that \( \beta \) is positive and locally bounded from below by positive constants. Let \( \tilde{\beta} : X \to \mathbb{R} \) be defined by \( \tilde{\beta}(x) = \beta(x) \) if \( x \in \pi(B) \) and \( \tilde{\beta}(x) = \varepsilon'(x) \) if \( x \notin \pi(B) \). The function \( \tilde{\beta} \) is semi-algebraic, positive and locally bounded from below.
by positive constants. We can find a positive semi-algebraic \( C^2 \) function \( \varepsilon'' : X \to \mathbb{R} \) such that \( \varepsilon'' < \beta \) on \( X \).

Let \( W' \) be defined by:
\[
W' = \{ y \in V \mid d'(y) < \varepsilon''(\pi(y)) \}.
\]
Note that \( W' \) is included in \( W \). For every \( y \) in \( W' \cap E_{2r_0} \), we have:
\[
\frac{\langle \nabla \rho(y), \nabla \rho(\pi(y)) \rangle}{\| \nabla \rho(y) \| \cdot \| \nabla \rho(\pi(y)) \|} \geq 0, 9.
\]
Since \( \nabla d'(y) \) belongs to \( [\nabla \rho(\pi(y))]^\perp \), this can be reformulated in the following way: for every \( y \) in \( W' \cap E_{2r_0} \), we have:
\[
\frac{\langle \nabla \rho(y), \nabla d'(y) \rangle}{\| \nabla \rho(y) \| \cdot \| \nabla d'(y) \|} \leq \sqrt{0, 19}.
\]

**Lemma 5.2.** There exist \( q_0 \in \mathbb{N} \) and \( r_0' > 0 \) such that for every \( q \geq q_0 \) and for every \( x \in X \cap E_{r_0'} \),
\[
\frac{1}{(1 + \rho(x))^q} \leq \varepsilon''(x).
\]

**Proof.** Let \( h : [0, +\infty[ \to \mathbb{R} \) be defined by:
\[
h(r) = \min \{ \varepsilon''(x) \mid x \in X \cap \Sigma_r \}.
\]
Since \( h \) is a positive semi-algebraic function, there exists an integer \( q_0 \) and a real \( r_0' > 0 \) such that \( \frac{1}{h(r)} < r^{q_0} \) for every \( r \geq r_0' \). Hence for every \( q \geq q_0 \) and every \( x \in X \cap E_{r_0'} \), we have:
\[
\frac{1}{(1 + \rho(x))^q} \leq \varepsilon''(x).
\]
\[\square\]

**Corollary 5.3.** There exist \( q_0 \in \mathbb{N} \) and \( r_0'' > 0 \) such that for every \( q \geq q_0 \) and for every \( y \in W' \cap E_{r_0''} \),
\[
\frac{1}{(1 + \rho(\pi(y)))^q} \leq \varepsilon''(\pi(y)).
\]

**Proof.** By Lemma 5.1, we can find \( r_0'' > 0 \) such that \( \pi(y) \) belongs to \( X \cap E_{r_0'} \) if \( y \) belongs to \( W' \cap E_{r_0''} \). \[\square\]

**Lemma 5.4.** There exist \( q_1 \in \mathbb{N} \) and \( r_1' > 0 \) such that for every \( q \geq q_1 \) and for every \( x \in X \cap E_{r_1'} \), \( \| \nabla \rho(x) \| \leq (1 + \rho(x))^q \).

**Proof.** Let \( c > 0 \) be such that \( [c, +\infty[ \) does not contain any critical value of \( \rho \). Let \( l : [c, +\infty[ \to \mathbb{R} \) be defined by:
\[
l(r) = \max \{ \| \nabla \rho(x) \| \mid x \in X \cap \Sigma_r \}.
\]
Since \( l \) is a positive semi-algebraic function, there exits an integer \( q_1 \) and a real \( r_1' > 0 \) such that \( l(r) < r^{q_1} \) for every \( r \geq r_1' \). Hence for every \( q \geq q_1 \) and every \( x \in X \cap E_{r_1'} \), we have \( \| \nabla \rho(x) \| \leq (1 + \rho(x))^q \).
\[\square\]
Corollary 5.5. There exist $q_1 \in \mathbb{N}$ and $r_1'' > 0$ such that for every $q \geq q_1$ and for every $y \in W' \cap E_{r_1''}$, $\|\nabla \rho(\pi(y))\| \leq (1 + \rho(\pi(y)))^q$.

Proof. The proof is the same as Corollary 5.3. □

Proposition 5.6. There exists an integer $q_2$ such that for every $q \geq q_2$, the function $d_q' : W' \rightarrow \mathbb{R}$ defined by $d_q'(y) = (1 + \rho(\pi(y)))^q d'(y)$ is a $\rho$-quasiregular approaching function for $X$ in $W'$.

Proof. Since $W' = \{ y \in V \mid d'(y) < \varepsilon''(\pi(y)) \}$ and $\varepsilon''$ is a positive function, $(X, W', d')$ satisfies the condition (A). Let $W_1 = \{ y \in V \mid d'(y) < \frac{1}{2} \varepsilon''(\pi(y)) \}$.

We have $\overline{W_1} \subset W'$. By Corollary 5.3, for every $q \geq q_0$, the set $E_{r_0''} \cap d_q^{-1}([0, \frac{1}{19}])$ is included in $W_1$. The tuple $(X, W', d_q')$ satisfies the condition (A). As it has been already explained in Lemma 2.2, there exists $\epsilon_q > 0$ such that $d_q^{-1}([0, \epsilon_q]) \cap D_{r_0} \subset W_1 \cap D_{r_0''}$. Let $\delta_q$ be the minimum of $\frac{1}{2}$ and $\epsilon_q$. The set $d_q^{-1}([0, \delta_q])$ is included in $W_1$, hence closed in $\overline{W_1}$ and in $\overline{W'}$. This proves that $d_q'$ is an approaching function for $X$ in $W'$.

Let us show that it is $\rho$-quasiregular. Let us fix $r$ greater than $r_0''$, $r_1''$ and $2r_0$ and let us fix $q_2$ greater than $q_0$ and $q_1$. For every $y$ in $W \cap E_r$, let $P_y$ be the orthogonal projection onto the space $\nabla \rho(\pi(y))$. We have:

$$\nabla d_q'(y) = (1 + \rho(\pi(y)))^{q-1} \cdot (1 + \rho(\pi(y))) \nabla d'(y) + q d'(y) \nabla \rho(\pi(y)),$$

hence,

$$\frac{P_y(\nabla d_q'(y))}{(1 + \rho(\pi(y)))^{q-1}} = (1 + \rho(\pi(y))) P_y(\nabla d'(y)) + q d'(y) P_y(\nabla \rho(\pi(y))).$$

Let us prove that, for $q \geq q_2$ and $R \geq r$ sufficiently big, $T(y)$ can not vanish if $y$ belongs to $d_q^{-1}([0, 1]) \cap E_R$, where

$$T(y) = (1 + \rho(\pi(y))) P_y(\nabla d'(y)) + q d'(y) P_y(\nabla \rho(\pi(y))).$$

First observe that if $y$ lies in $d_q^{-1}([0, 1]) \cap E_R$, $q \geq q_2$ and $R \geq r$, then :

$$\frac{\langle \nabla \rho(y), \nabla \rho(\pi(y)) \rangle}{\| \nabla \rho(y) \| \cdot \| \nabla \rho(\pi(y)) \|} \geq 0.9,$$

and

$$\frac{\langle \nabla \rho(y), \nabla d'(y) \rangle}{\| \nabla \rho(y) \| \cdot \| \nabla d'(y) \|} \leq \sqrt{0.919}.$$ 

This implies that

$$\| P_y(\nabla \rho(\pi(y))) \| \leq \sqrt{0.19} \| \nabla \rho(\pi(y)) \|$$

and

$$\| P_y(\nabla d'(y)) \| \geq 0.9 \| \nabla d'(y) \|.$$ 

Therefore, we have :

$$\| q d'(y) P_y(\nabla \rho(\pi(y))) \| \leq \sqrt{0.19} q d'(y) \| \nabla \rho(\pi(y)) \|.$$
and,
\[ \| (1 + \rho(\pi(y))) P_y(\nabla d'(y)) \| \geq 0.9 (1 + \rho(\pi(y))) \cdot \| \nabla d'(y) \|, \]
that is to say:
\[ \| (1 + \rho(\pi(y))) P_y(\nabla d'(y)) \| \geq 0.9 (1 + \rho(\pi(y))) \cdot 2 \sqrt{d'(y)}. \]

In order to prove that \( T(y) \) does not vanish if \( y \in d_q^{-1}([0,1]) \cap E_R \) for \( q \geq q_2 \) and \( R \geq r \) sufficiently big, it is enough to prove that:
\[ \frac{1,8}{\sqrt{0,19}} > \frac{q \sqrt{d'(y)} \| \nabla \rho(\pi(y)) \|}{1 + \rho(\pi(y))}. \]

But if \( y \in d_q^{-1}([0,1]) \cap E_R \) where \( q \geq q_2 \) and \( R \geq r \) then we have:
\[ \sqrt{d'(y)} \leq \frac{1}{(1 + \rho(\pi(y)))^{q_2}}. \]

So, if we show that
\[ \frac{1,8}{\sqrt{0,19}} > \frac{q \| \nabla \rho(\pi(y)) \|}{1 + \rho(\pi(y)))^{q_2 + 1}}, \]
then the required result is established. Let \( q \) be such that \( \frac{q}{2} + 1 > q_1 \). By Corollary 5.5, we have:
\[ \frac{q \| \nabla \rho(\pi(y)) \|}{(1 + \rho(\pi(y)))^{q_2 + 1}} \leq \frac{q}{(1 + \rho(\pi(y)))^{q_2 + 1 - q_1}}, \]
for \( y \in d_q^{-1}([0,1]) \cap E_R, R \geq r \). Lemma 5.1 implies that there exists \( R_q \geq r \) such that if \( y \) belongs to \( d_q^{-1}([0,1]) \cap E_R, \) with \( R \geq R_q \), then we have:
\[ \frac{q}{(1 + \rho(\pi(y)))^{q_2 + 1 - q_1}} < \frac{1,8}{\sqrt{0,19}}. \]

This proves the proposition. \( \square \)

We can state the main result of this section.

**Theorem 5.7.** Let \( X \) be a closed non-compact semi-algebraic set in \( \mathbb{R}^n \) which is a \( C^3 \) submanifold. Let \( \rho \) be a control function of class \( C^3 \). Any \( \rho \)-quasiregular approaching semi-algebraic neighborhood of \( X \) is isotopic to a tubular neighborhood of \( X \).

**Proof.** By the previous proposition, we known that there exist \( \rho \)-quasiregular approaching functions \( d'_q \) for \( X \) in \( W \) of the form \( d'_q(y) = (1 + \rho(\pi(y)))^q \cdot d'(y) \). But for \( \nu > 0 \) sufficiently small the set \( d_q^{-1}([0,\nu]) \) is a tubular neighborhood of \( X \). It is enough to use Theorem 4.2 to conclude. \( \square \)
6. Degree formulas for the Euler-Poincaré characteristic of a closed semi-algebraic set

In this section, we give degree formulas for the Euler-Poincaré characteristic of a closed semi-algebraic set $X$ included in $\mathbb{R}^n$. When $X$ is algebraic, we deduce from these formulas a Petrovskii-Oleinik inequality for $|1 - \chi(X)|$.

Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative $C^2$ semi-algebraic function such that $X = f^{-1}(0)$, i.e. $f$ is an approaching function for $X$ in $\mathbb{R}^n$. Let $\rho$ be a control function. For every $q \in \mathbb{N}$, we will denote by $f_q$ the function defined by $f_q(x) = (1 + \rho(x))^q \cdot f(x)$. We will also denote by $\Gamma_{f,\rho}$ (resp. $\Gamma_{f_q,\rho}$) the following polar set:

\[ \Gamma_{f,\rho} = \{ x \in \mathbb{R}^n \setminus X \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are colinear} \} . \]

Note that $\Gamma_{f,\rho} = \Gamma_{f_q,\rho}$ for each $q \in \mathbb{N}$.

**Proposition 6.1.** There exists an integer $q_0$ such that for every $q \geq q_0$, the following property holds: for any sequence $(x_k)_{k \in \mathbb{N}} \subset \Gamma_{f_q,\rho}$ such that $\lim_{k \to +\infty} \|x_k\| = +\infty$, we have $\lim_{k \to +\infty} f_q(x_k) = +\infty$.

Let us fix an integer $q$ satisfying the property of the previous proposition. Let $\Sigma(f_q)$ be the set of critical points of $f_q$ and let $\Sigma^*(f_q)$ be the set of critical points of $f_q$ lying in $\mathbb{R}^n \setminus X$.

**Corollary 6.2.** The set $\Sigma^*(f_q)$ is compact.

**Proof.** It is clearly closed as the set of preimages of the nonzero critical values of $f_q$. If it is not bounded, there exists a sequence of points $(x_k)_{k \in \mathbb{N}}$ such that $x_k \notin X$, $\nabla f_q(x_k) = 0$ and $\lim_{k \to +\infty} \|x_k\| = +\infty$. Since for each $k \in \mathbb{N}$, $x_k$ also belongs to $\Gamma_{f_q,\rho}$, this gives a contradiction. □

Let us decompose $\Sigma^*(f_q)$ into the finite union of its connected components $K_1^q, \ldots, K_{m_q}^q$:

\[ \Sigma^*(f_q) = \bigcup_{i=1}^{m_q} K_i^q. \]

Before stating the main results of this section, we need to introduce some notations. For each $i \in \{1, \ldots, m_q\}$, let $U_i$ be a relatively compact neighborhood of $K_i^q$ such that $\partial U_i$ is a smooth hypersurface and $U_i \cap \Sigma^*(f_q) = K_i^q$. For any mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{-1}(0) \cap U_i = K_i^q$ or $F^{-1}(0) \cap U_i$ is empty, we will denote by $\deg_{K_i^q} F$ the topological degree of the following mapping:

\[ \frac{F}{\|F\|} : \partial U_i \to S^{n-1}, \quad x \mapsto \frac{F(x)}{\|F(x)\|}. \]

It is well-known that this topological degree does not depend on the choice of the relatively compact neighborhood $U_i$. 

Theorem 6.3. The Euler-Poincaré characteristic of $X$ is related to $\nabla f_q$ by the following formula:

$$\chi(X) = 1 - \sum_{i=1}^{m_q} \deg_{K^q_i} \nabla f_q.$$ 

Proof. By Proposition 6.1, $f_q$ is a $\rho$-quasiregular approaching function for $X$ in $\mathbb{R}^n$. Theorem 3.2 implies that for $\varepsilon > 0$ sufficiently small:

$$\chi(X) = \chi(\{f_q \leq \varepsilon\}) + \chi(\{f_q \geq \varepsilon\}).$$

By the Mayer-Vietoris sequence, we have:

$$1 = \chi(\{f_q \leq \varepsilon\}) + \chi(\{f_q \geq \varepsilon\}) - \chi(\{f_q = \varepsilon\}).$$

We will apply Morse theory to the manifold with boundary $D_R$ and to the function $f_q$. We will follow the terminology of [Dut], Section 2, p.46-47. Let us first show that $f_q$ does not admit any inward critical point on $\Sigma_R \cap \{f_q \geq \varepsilon\}$ for $R$ sufficiently big and $\varepsilon$ sufficiently small (an inward critical point $p$ is a point critical point $q$ of $f_q:\Sigma_R$ such that $\nabla f_q(p)$ is a negative multiple of $\nabla \rho(p)$). If it is not the case, then we can find a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $\Gamma_{f_q,\rho}$ such that $\nabla f_q(x_k)$ is a negative multiple of $\nabla \rho(x_k)$. Using the version at infinity of the Curve Selection Lemma (see [NZ], Lemma 2), we obtain that $\lim_{k \to +\infty} f_q(x_k)$ exists and belongs to $[0, +\infty[$, which contradicts the property of Proposition 6.1.

Let us fix $R$ sufficiently big and $\varepsilon$ sufficiently small so that $\Sigma^*(f_q) \subset D_R$, $f_q$ does not have inward critical points in $\Sigma_R \cap \{f_q \geq \varepsilon\}$ and

$$\chi(\{f_q \geq \varepsilon\}) = \chi(\{f_q \geq \varepsilon\} \cap D_R) \text{ and } \chi(\{f_q = \varepsilon\}) = \chi(\{f_q = \varepsilon\} \cap D_R).$$

Since $f_q$ does not have inward critical points in $\Sigma_R \cap \{f_q \geq \varepsilon\}$, Morse theory for manifolds with boundary implies that

$$\chi(\{f_q \geq \varepsilon\} \cap D_r) - \chi(\{f_q = \varepsilon\} \cap D_r) = \sum_{i=1}^{m_q} \deg_{K^q_i} \nabla f_q. \quad (2)$$

The final result is just a combination of equality (1) and equality (2). \qed

Let $F_q: \mathbb{R}^n \to \mathbb{R}^n$ be the mapping defined by $F_q(x) = q \cdot f \cdot \nabla \rho + (1 + \rho) \cdot \nabla f$. Note that for every $x \in \mathbb{R}^n$, $\nabla f_q(x) = (1 + \rho(x))^{q-1} F_q(x)$. Hence $\nabla f_q$ and $F_q$ admit the same zeros in $\mathbb{R}^n$.

Corollary 6.4. The Euler-Poincaré characteristic of $X$ is related to $F_q$ by the following formula:

$$\chi(X) = 1 - \sum_{i=1}^{m_q} \deg_{K^q_i} F_q.$$ 

Proof. It is enough to prove that for every $i \in \{1, \ldots, m_q\}$, $\deg_{K^q_i} F_q = \deg_{K^q_i} \nabla f_q$. Let us choose a relatively compact neighborhood $U_i$ of $K^q_i$ such that $\partial U_i$ is a smooth manifold, $F_q^{-1}(0) \cap U_i = K^q_i = \nabla f_q^{-1}(0) \cap U_i$. The result is clear since on $\partial U_i$, we have $\frac{\nabla f}{\|\nabla f\|} = \frac{F_q}{\|F_q\|}$. \qed
Corollary 6.5. Let \( G_q : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) be the mapping defined by \( G_q(\lambda; x) = (f(x)\lambda - 1, F_q(x)) \). The set \( G_q^{-1}(0) \) is compact and if \( R > 0 \) is such that \( G_q^{-1}(0) \subset B_R^{n+1} \), then
\[
\chi(X) = 1 - \deg_{S^n_R} G_q.
\]
Here \( B_R^{n+1} \) and \( S^n_R \) are the ball and the sphere of radius \( R \) in \( \mathbb{R}^{n+1} \).

Proof. Since \( G_q(\lambda; x) = 0 \) if and only if \( F_q(x) = 0 \), \( f(x) \neq 0 \) and \( \lambda = \frac{1}{f(x)} \), it is straightforward to see that \( G_q^{-1}(0) \) is compact. The rest of the proof is easy. □

These formulas are global versions of a result due to Khimshiashvili [Khi] on the Euler characteristic of the real Milnor fibre. It states that, if \( g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) is an analytic function-germ with an isolated critical point at the origin, then
\[
\chi(g^{-1}(\delta) \cap B^n_{\varepsilon}) = 1 - \text{sign} (-\delta)^n \deg_0 \nabla g,
\]
for any regular value \( \delta \) of \( g \), \( 0 < |\delta| \ll \varepsilon \ll 1 \). Here \( \deg_0 \nabla g \) is the topological degree of \( \frac{\nabla g}{||\nabla g||} : S^{n-1} \to S^{n-1} \).

In their fundamental paper [OP], Petrovskii and Oleinik estimated the Euler characteristic of some real projective algebraic sets. More precisely they gave an upper bound for the following quantities:

- \(|\chi(Y) - 1|\) where \( Y \) is a real projective hypersurface of even dimension,
- \(|2\chi(Z_-) - 1|\) where \( Z_- \) is the subset of \( \mathbb{R}P^n \) that is bounded by a real projective hypersurface \( Y \) of odd dimension and even degree and corresponds to the negative values of the polynomial that determines \( Y \).

These results were generalized by Kharlamov [Kha1,Kha2]. In [Ar], Arnol’d found a new proof based, on Khimshiashvili’s formula, and an equivalent formulation of the original Petrovskii-Oleinik inequalities. Let us state Arnol’d’s version of these inequalities. We need some notations. With every \( n \)-tuples of positive integers \( \mathbf{m} = (m_1, \ldots, m_n) \) and with every positive integer \( m_0 \), we will associate the following objects:

- \( \Delta_n(\mathbf{m}) \) is the parallelepiped in \( \mathbb{R}^n \) defined by the inequalities
  \[
  0 \leq x_1 \leq m_1 - 1, \ldots, 0 \leq x_n \leq m_n - 1,
  \]
- \( \mu = m_1 \cdots m_n \) is the number of integral points in the parallelepiped \( \Delta_n(\mathbf{m}) \),
- \( \nu = \frac{1}{2}(m_1 + \cdots + m_n - n) \) is the mean value of the sum of the coordinates of the points in \( \Delta_n(\mathbf{m}) \),
- \( \Pi_n(\mathbf{m}) \) is the number of integral points on the central section \( x_1 + \cdots + x_n = \nu \) of the parallelepiped \( \Delta_n(\mathbf{m}) \),
- \( \Pi_n(\mathbf{m}, m_0) \) is the number of integral points in \( \Delta_n(\mathbf{m}) \) that lie in the strip
  \[
  \nu - \frac{1}{2} m_0 \leq x_1 + \cdots + x_n \leq \nu + \frac{1}{2} m_0,
  \]
Oₙ(m, m₀) is the number of integral points in Δₙ(m) that satisfy the inequalities
\[ \nu - \frac{1}{2} m₀ \leq x₁ + \cdots + xₙ \leq \nu. \]

Arnol’d [Ar] proved the following theorem.

**Theorem 6.6.** Let \( f \) be a homogeneous polynomial of degree \( d \) in \( \mathbb{R}^n \) defining a non-singular hypersurface \( Y \) in \( \mathbb{R}^{n-1} \). If \( n \) is even, we have:
\[ |1 - \chi(Y)| \leq \Piₙ(d - 1), \]
where \( d - 1 = (d - 1, \ldots, d - 1) \) in \( \mathbb{N}^n \).

If \( n \) is odd and \( d \) is even, let \( Z_- \) be the subset of \( \mathbb{R}^n \) that is bounded by \( Y \) and corresponds to the negative values of the polynomial \( f \). We have:
\[ |1 - 2\chi(Z^-)| \leq \Piₙ(d - 1). \]

In [Kho1] (see also [Kho2]), Khovanskii gave an affine version of this theorem.

**Proposition 6.7.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a polynomial of degree \( d \) such that the surface \( \{f = 0\} \) is nonsingular and the domains \( \{f \leq c\} \) are compact for every \( c \in \mathbb{R} \). Then the Euler-Poincaré of the domain \( \{f \leq 0\} \) satisfies the inequality
\[ |1 - 2\chi(\{f \leq 0\})| \leq \Piₙ(d - 1, d - 1), \]
where \( d - 1 = (d - 1, \ldots, d - 1) \) in \( \mathbb{N}^n \).

Our aim is to give a Petroskii-Oleinik inequality for the Euler-Poincaré characteristic of any algebraic set in \( \mathbb{R}^n \). Let \( X \) be an algebraic set in \( \mathbb{R}^n \) defined as the zero set of the polynomials \( f₁, \ldots, fₖ \), each \( fᵢ \) having degree \( dᵢ \). Hence \( X = \{x \in \mathbb{R}^n \mid f(x) = 0\} \) where \( f = f₁² + \cdots + fₖ² \). The degree of the polynomial \( f \) is \( d = 2\text{Max}\{d₁, \ldots, dₖ\} \). The following proposition gives an upper bound for \( |1 - \chi(X)| \) in terms of \( d \).

**Proposition 6.8.** Let \( X \) be an algebraic set in \( \mathbb{R}^n \) defined as the set of zeros of a nonnegative polynomial \( f \) of even degree \( d \). We have:
\[ |1 - \chi(X)| \leq Oₙ₊₁(d + 1, 2), \]
where \( d + 1 = (d + 1, \ldots, d + 1) \) in \( \mathbb{N}^{n+1} \).

**Proof.** Let \( \omega : \mathbb{R}^n \to \mathbb{R} \) be defined by \( \omega(x) = x₁² + \cdots + xₙ² \). Applying the argument described above to the functions \( f \) and \( \omega \), we find that there exists an integer \( q \) sufficiently big and a real \( R > 0 \) sufficiently big such that:
\[ \chi(X) = 1 - \deg_{Sₙ⁻} G_q. \]

Let \( \delta \) be a small positive regular value of \( G_q \) and let \( \{p₁, \ldots, p_l\} \) be the set of preimages of \( \delta \) by \( G_q \) lying in \( Sₙ⁻ \). We have:
\[ 1 - \chi(X) = \deg_{Sₙ⁻}(G_q - \delta) = \sum_{j=1}^{l} \deg_{p_j}(G_q - \delta). \]
Since each component of $G_q - \delta$ has a degree not exceeding $d+1$, the square of the euclidian distance function in $\mathbb{R}^{n+1}$ has degree 2 and $2 + (n+1)(d+1) \equiv n + 1 \mod 2$, Theorem 2 of [Kho1] applied to the vector field $G_q - \delta$ and the function $R - (x_1^2 + \cdots + x_n^2 + \lambda^2)$ gives:

$$|\sum_{j=1}^l \deg_{p_j}(G_q - \delta)| \leq O_{n+1}(d+1,2),$$

where $d+1 = (d+1, \ldots, d+1)$ in $\mathbb{N}^{n+1}$.

\[ \square \]

**References**


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