



# Higher order peak algebras

Daniel Krob, Jean-Yves Thibon

► **To cite this version:**

Daniel Krob, Jean-Yves Thibon. Higher order peak algebras. Article à paraître dans 'Annals of Combinatorics'. 2006. <hal-00018554>

**HAL Id: hal-00018554**

**<https://hal.archives-ouvertes.fr/hal-00018554>**

Submitted on 7 Feb 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# HIGHER ORDER PEAK ALGEBRAS

DANIEL KROB AND JEAN-YVES THIBON

ABSTRACT. Using the theory of noncommutative symmetric functions, we introduce the *higher order peak algebras*  $(\mathbf{Sym}(N))_{N \geq 1}$ , a sequence of graded Hopf algebras which contain the descent algebra and the usual peak algebra as initial cases ( $N = 1$  and  $N = 2$ ). We compute their Hilbert series, introduce and study several combinatorial bases, and establish various algebraic identities related to the multisection of formal power series with noncommutative coefficients.

## 1. INTRODUCTION

The peak algebra of the symmetric group  $\mathfrak{S}_n$  is a subalgebra of its descent algebra  $\Sigma_n$ , spanned by sums of permutations having the same peak set, a certain subset of the descent set. The direct sum of these peak algebras turns out to be a Hopf subalgebra of the direct sum of all descent algebras, which can itself be identified with  $\mathbf{Sym}$ , the Hopf algebra of noncommutative symmetric functions.

The peak Hopf algebra  $\Pi$  has been introduced (somewhat implicitly, and independently) in two papers published in 1997. In [17], Stembridge introduced the dual Hopf algebra as a subalgebra of quasi-symmetric functions, while in [7, Sec. 5.6.4], Prop. 5.41 implied the existence of  $\Pi$  and many of its properties (see [2]). Since then, a larger peak algebra, related to descent algebras of type  $B$  and  $D$ , has been discovered [1].

Actually, the results of [7] implied more than the classical peak algebra. The peak algebra was obtained as the image of  $\mathbf{Sym}$  under a certain endomorphism  $\theta_q$  for  $q = -1$ . For generic values of  $q$ ,  $\theta_q$  is an automorphism, but it degenerates at (non trivial) roots of unity. In this way, we obtain an infinite family of Hopf subalgebras of  $\mathbf{Sym}$ , denoted by  $\mathbf{Sym}(N) = \theta_q(\mathbf{Sym})$  when  $q$  is a primitive  $N$ th root of unity. Then  $\mathbf{Sym}$  and  $\Pi$  correspond to  $N = 1$  and  $N = 2$ , respectively.

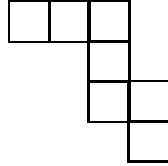
In the following, we shall investigate the Hopf algebras  $\mathbf{Sym}(N)$  for arbitrary  $N$ . Our main results consist in the introduction of several linear bases, in which the various natural operations of  $\mathbf{Sym}(N)$  admit interesting combinatorial expressions. These bases allow us to obtain algebraic identities generalizing those involving the noncommutative tangent in [5, 2].

Finally, let us mention that the commutative images (in ordinary symmetric functions) of the first two algebras are known to be the Grothendieck rings of Hecke algebras  $H_n(q)$  at  $q = 1$  (symmetric groups, same as generic case) and  $q = -1$ . In general, the commutative image of the  $N$ -th peak algebra is the Grothendieck ring of the tower  $H_n(\zeta)$ , where  $\zeta$  is a primitive  $N$ -th root of unity (symmetric functions not depending on power sums  $p_m$  with  $N|m$ , cf. [10]). It would be interesting to find an interpretation of the higher order peak algebras in this context.

## 2. PRELIMINARIES

Our notations will be those of [5, 7, 4, 8, 9, 3]. We recall here the most important ones in order to make the paper self-contained.

**2.1. Compositions and permutations.** A *composition* of  $n$  is a sequence  $I = (i_1, i_2, \dots, i_r)$  of positive integers summing to  $n$ . The *length* of  $I$  is  $\ell(I) = r$ . A composition  $I$  can be represented by a skew Ferrers diagram called a *ribbon diagram* of shape  $I$ , e.g.,



is a ribbon diagram of shape  $I = (3, 1, 2, 1)$ . The *conjugate composition*  $I^\sim$  is associated with the conjugate ribbon diagram in the sense of skew Ferrers diagrams. On our example,  $I^\sim = (2, 3, 1, 1)$ .

One associates with a composition  $I = (i_1, i_2, \dots, i_r)$  of  $n$  the subset  $D(I)$  of  $[1, n-1]$  defined by  $D(I) = \{i_1, i_1 + i_2, \dots, i_1 + i_2 + \dots + i_{r-1}\}$ . Compositions of  $n$  are ordered by *reverse refinement*, denoted by  $\preceq$ , and defined by  $I \preceq J$  iff  $D(I) \subseteq D(J)$ .

One associates with a permutation  $\sigma$  of  $\mathfrak{S}_n$  two compositions  $D(\sigma)$  and  $P(\sigma)$  respectively called the *descent composition* and the *peak composition* of  $\sigma$ . The descent composition  $D(\sigma)$  of the permutation  $\sigma$  encodes its descent set  $\mathcal{D}(\sigma) = \{i \in [1, n-1], \sigma(i) > \sigma(i+1)\}$ , i.e., is characterized by  $D(D(\sigma)) = \mathcal{D}(\sigma)$ . The peak composition  $P(\sigma)$  of  $\sigma$  encodes its peak set  $\mathcal{P}(\sigma) = \{i \in [2, n-1], \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$ , that is,  $D(P(\sigma)) = \mathcal{P}(\sigma)$ . For example  $D(265341) = (2, 1, 2, 1)$ ,  $\mathcal{D}(265341) = \{2, 3, 5\}$ ,  $P(265341) = (2, 3, 1)$  and  $\mathcal{P}(265341) = \{2, 5\}$ .

While each subset of  $[1, n-1]$  is obviously the descent set of some permutation  $\sigma \in \mathfrak{S}_n$ , a subset  $P$  of  $[1, n-1]$  is the peak set of a permutation  $\sigma \in \mathfrak{S}_n$  if and only if

$$(1) \quad \begin{cases} 1 \notin P, \\ \forall i \in [2, n-1], i \in P \Rightarrow i-1 \notin P. \end{cases}$$

The corresponding compositions are called *peak compositions* of  $n$ . A peak composition of  $n$  is one whose only entry allowed to be equal to 1 is the last one. It is easy to check that the number of peak compositions of  $n$  is  $f_n$ , where  $f_n$  is the  $n$ -th Fibonacci number, with the convention  $f_0 = f_1 = f_2 = 1$  and  $f_{n+2} = f_{n+1} + f_n$ .

Finally, the peak set  $P(I)$  of a composition  $I = (i_1, i_2, \dots, i_r)$  is defined as follows. If  $D(I) = \{d_1, d_2, \dots, d_{r-1}\}$  with  $d_1 \leq \dots \leq d_{r-1}$ , then  $P(I)$  is obtained from  $D(I)$  by removing each  $d_i$  such that  $d_i - d_{i-1} = 1$  (with the convention  $d_0 = 0$ ). For example, if  $I = (1, 3, 1, 2)$ ,  $D(I) = \{1, 4, 5\}$  and  $P(I) = \{4\}$ .

**2.2. Noncommutative symmetric functions.** The Hopf algebra of noncommutative symmetric functions, introduced in [5], is the free associative  $\mathbb{C}$ -algebra  $\mathbf{Sym} =$

$\mathbb{C}\langle S_1, S_2, \dots \rangle$  over an infinite sequence of noncommuting indeterminates  $(S_k)_{k \geq 1}$  called the *noncommutative complete symmetric functions*. It is graded by  $\deg S_n = n$ , and endowed with the coproduct

$$(2) \quad \Delta(S_n) = \sum_{k=0}^n S_k \otimes S_{n-k} .$$

Its graded dual is Gessel's algebra of quasi-symmetric functions. Various applications are discussed in the series [5, 7, 4, 8, 9, 3].

Let  $\sigma(t)$  be the generating series of the  $(S_k)_{k \geq 0}$ , with the convention  $S_0 = 1$ ,

$$(3) \quad \sigma(t) = \sum_{k \geq 0} S_k t^k .$$

The *noncommutative power sums of the first kind*, denoted by  $(\Psi_k)_{k \geq 1}$ , are the coefficients of the series

$$(4) \quad \psi(t) = \sum_{k \geq 1} \Psi_k t^{k-1} = \sigma(t)^{-1} \frac{d}{dt} \left( \sigma(t) \right) .$$

The *noncommutative ribbon Schur functions*  $R_I$  are characterized by the equivalent relations

$$(5) \quad S^I = \sum_{J \preceq I} R_J , \quad R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} S^J$$

where  $S^I = S_{i_1} S_{i_2} \cdots S_{i_r}$ .

The *commutative image* of a noncommutative symmetric function  $F$  is the ordinary symmetric function  $\chi(F)$  obtained through the algebra morphism  $\chi(S_n) = h_n$ , the complete symmetric function (notation as in [13]). The commutative image of  $\Psi_n$  is then the power sum  $p_n$ . Similarly, the commutative image of  $R_I$  is the ordinary ribbon Schur function of shape  $I$ . We recall that ribbon Schur functions were introduced by McMahon (see [12], t. 1, p. 200, where they are denoted by  $h_I$ ).

**2.3. Relation with the descent algebra of the symmetric group.** The sum in the group algebra  $\mathbb{C}[\mathfrak{S}_n]$  of all permutations  $\sigma$  whose descent composition  $D(\sigma)$  is equal to  $I$  is denoted by  $D_{=I}$ . The  $D_{=I}$  with  $|I| = n$  form a basis of a subalgebra of  $\mathbb{C}[\mathfrak{S}_n]$ , introduced by Solomon in [15], which is denoted  $\Sigma_n$  and called the *descent algebra* of  $\mathfrak{S}_n$ . One can define an isomorphism  $\alpha$  of graded vector spaces

$$(6) \quad \alpha : \mathbf{Sym} = \bigoplus_{n \geq 0} \mathbf{Sym}_n \longrightarrow \Sigma = \bigoplus_{n \geq 0} \Sigma_n$$

by  $\alpha(D_{=I}) = R_I$ . Then,  $\alpha(S^I) = D_{\preceq I}$ , the sum in  $\mathbb{C}[\mathfrak{S}_n]$  of all permutations  $\sigma$  whose descent composition  $D(\sigma)$  is  $\preceq I$ .

The direct sum  $\Sigma$  can be turned into an algebra by extending the natural products of its components  $\Sigma_n$  by setting  $xy = 0$  for  $x \in \Sigma_p$  and  $y \in \Sigma_q$  with  $p \neq q$ . We can then define the *internal product*, denoted by  $*$ , on  $\mathbf{Sym}$  by requiring that  $\alpha$  be

an anti-isomorphism of algebras. In other words, the internal product  $*$  of  $\mathbf{Sym}$  is defined by

$$(7) \quad F * G = \alpha^{-1}(\alpha(G) \alpha(F)).$$

**2.4. The  $(1-q)$ -transform.** In the commutative case, the  $(1-q)$ -transform is the endomorphism  $\vartheta_q$  of the algebra  $Sym$  of commutative symmetric functions defined on power sums by  $\vartheta_q(p_n) = (1 - q^n) p_n$ . Our terminology comes from the  $\lambda$ -ring notation, which allows to write it as  $\vartheta_q(f(X)) = f((1-q)X)$  (cf. [6, 11]). One just has to point out here the (traditional) abuse of notation in using the same minus sign for scalars and in the  $\lambda$ -ring, though these operations are completely different.

In [7], this  $\lambda$ -ring formalism was extended to the algebra of noncommutative symmetric functions. A consistent definition of  $\vartheta_q(F) = F((1-q)A)$  for  $F \in \mathbf{Sym}$  was proposed and its fundamental properties were obtained. The first step in the construction of this noncommutative version consists in defining the noncommutative complete symmetric functions  $(S_n((1-q)A))_{n \geq 0}$  by their generating series

$$(8) \quad \sigma_t((1-q)A) = \sum_{n \geq 0} S_n((1-q)A) t^n = \sigma_{-qt}(A)^{-1} \sigma_t(A),$$

where  $\sigma_t(A)$  is the generating series of the complete functions  $S_n(A)$  (usually denoted by  $\sigma(t)$  when  $A$  plays no role). The noncommutative  $(1-q)$ -transform is the algebra endomorphism  $\vartheta_q$  of  $\mathbf{Sym}$  defined by  $\vartheta_q(S_n) = S_n((1-q)A)$ . One can show that

$$(9) \quad \vartheta_q(F(A)) = F((1-q)A) = F(A) * \sigma_1((1-q)A)$$

for  $F(A) \in \mathbf{Sym}$ , where  $*$  is the internal product.

The most important property of  $\vartheta_q$  obtained in [7] is its diagonalization. We recall first that a noncommutative symmetric function  $\pi \in \mathbf{Sym}$  is called a *Lie idempotent* of order  $n$  if  $\pi$  is an homogeneous noncommutative symmetric function of  $\mathbf{Sym}_n$  in the primitive Lie algebra  $L(\Psi)$ , generated by  $(\Psi_n)_{n \geq 1}$ , and which is idempotent for the internal product of  $\mathbf{Sym}$ , or equivalently which has a commutative image equal to  $p_n/n$  (see [5]). It is shown in [7] that there is a unique family  $(\pi_n(q))_{n \geq 1}$  of Lie idempotents (with  $\pi_n(q) \in \mathbf{Sym}_n$ ) possessing the characteristic property

$$(10) \quad \vartheta_q(\pi_n(q)) = (1 - q^n) \pi_n(q).$$

These Lie idempotents were used in [7] for describing the structure of  $\vartheta_q$ , which is a semi-simple endomorphism of  $\mathbf{Sym}_n$ . The eigenvalues of  $\vartheta_q$  in  $\mathbf{Sym}_n$  are

$$(11) \quad p_\lambda(1-q) = \prod_{i=1}^r (1 - q^{\lambda_i})$$

where  $\lambda = (\lambda_1, \dots, \lambda_r)$  runs over all partitions of  $n$ . The projector on the eigenspace  $\mathcal{E}_\lambda$  associated with the eigenvalue  $p_\lambda(1-q)$  is the endomorphism  $F \rightarrow F * E_\lambda(\pi)$  of  $\mathbf{Sym}_n$ , where  $E_\lambda(\pi)$  is obtained through a construction introduced in [7], which we briefly recall here. Consider first the decomposition of  $S_n$  in the multiplicative basis

of  $\mathbf{Sym}$  constructed from the  $\pi_n(q)$ , i.e.,

$$(12) \quad S_n = \sum_{|I|=n} c_I(q) \pi^I(q)$$

where  $\pi^I(q) = \pi_{i_1}(q) \dots \pi_{i_r}(q)$ . Then, the projector  $E_\lambda(\pi)$  on  $\mathcal{E}_\lambda$  is given by

$$(13) \quad E_\lambda(\pi) = \sum_{I^\downarrow=\lambda} c_I(q) \pi^I(q)$$

where  $I^\downarrow$  denotes the partition obtained by reordering the components of  $I$ . It can be shown that the eigenspace  $\mathcal{E}_\lambda$  is spanned by the  $\pi^I(q)$  such that  $I^\downarrow = \lambda$  (see the proof of Theorem 5.14 of [7]).

The following formula can be easily deduced from these results.

**Proposition 2.1.** *Let  $q \in \mathbb{C}$  be a complex number. Then we have :*

$$(14) \quad \det \vartheta_q | \mathbf{Sym}_n = \left( \prod_{i=1}^{n-1} (1 - q^i)^{(n-i+3)2^{n-i-2}} \right) (1 - q^n) .$$

*Proof* – Since the determinant of an endomorphism is the product of its eigenvalues, we get

$$\det \vartheta_q | \mathbf{Sym}_n = \prod_{\substack{I=(i_1, \dots, i_r) \\ |I|=n}} \left( \prod_{k=1}^r (1 - q^{i_k}) \right) = \prod_{i=1}^n (1 - q^i)^{c(n,i)}$$

where  $c(n, i)$  is the number of times the integer  $i$  appears as a component of a composition of  $n$ . Formula (14) can therefore be established if we can find a closed expression for this number. To this purpose, let

$$F_i(x, y) = \sum_{k \geq 0} (x + x^2 + \dots + x^{i-1} + x^i y + x^{i+1} + x^{i+2} + \dots)^k = \sum_{k \geq 0} \left( \frac{x}{1-x} - x^i + x^i y \right)^k .$$

Then,

$$\sum_{n \geq 0} c(n, i) x^n = \frac{d}{dy} (F_i(x, y)) |_{y=1} = x^i \left( \sum_{k \geq 0} k \left( \frac{x}{1-x} \right)^{k-1} \right) = x^i \frac{1 - 2x + x^2}{(1 - 2x)^2} .$$

This shows that  $c(n, i) = c(n - i + 1, 1)$ , and that  $c(n, 1)$  is equal to 1 if  $n = 1$  and to  $(n + 2)2^{n-3}$  for  $n \geq 2$ , whence the result.  $\blacksquare$

**Note 2.2.** If  $q$  is a complex number, Proposition 2.1 shows that  $\vartheta_q$  is an automorphism of  $\mathbf{Sym}$  if and only if  $q$  is not a root of unity. The higher order peak algebras  $\mathbf{Sym}(N)$  arise precisely when we specialize  $q$  to a primitive  $N$ -th root of unity.

**2.5. The peak algebra.** The sum in the group algebra  $\mathbb{C}[\mathfrak{S}_n]$  of all permutations  $\sigma$  whose peak composition is equal to  $I$  is denoted by  $P_{=I}$ . It follows from the results of [7] that the family  $(P_{=I})$ , where  $I$  runs over peak compositions of  $n$ , form a basis of a subalgebra of  $\mathbb{C}[\mathfrak{S}_n]$ , first mentioned explicitly by Nyman in [14]. It is denoted by  $\Pi_n$  and called the *peak algebra* of  $\mathfrak{S}_n$ . Note that  $\Pi_n$  is a subalgebra of  $\Sigma_n$ . Indeed, for every peak composition  $I$  of  $n$ , one has

$$P_{=I} = \sum_{P(J)=D(I)} D_{=J} ,$$

where  $J$  runs over all compositions of  $n$ . Using the isomorphism  $\alpha$  defined in Section 2.3, we can therefore define the *peak functions*  $\Pi_I = \alpha(P_{=I})$  in  $\mathbf{Sym}_n$  by

$$\Pi_I = \sum_{P(J)=D(I)} R_J$$

for all peak compositions  $I$ . The space spanned by all the peak classes in  $\mathbf{Sym}_n$  is called  $\mathcal{P}_n$ . It is obviously isomorphic to  $\Pi_n$ , and so  $\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n$  is isomorphic to  $\Pi = \bigoplus_{n \geq 0} \Pi_n$ .

We will now explain the connection between these constructions and the  $(1 - q)$ -transform  $\theta_q$  introduced in Section 2.4 (see also [2]). Let us first recall the following result which expresses  $\theta_q(R_I)$  in the ribbon basis of  $\mathbf{Sym}$  (cf. equation (121) of [7]),

$$(15) \quad R_I((1 - q)A) = \sum_{P(J) \subseteq D(I) \Delta (D(I)+1)} (1 - q)^{hl(J)} (-q)^{b(I,J)} R_J(A) ,$$

Here  $I, J$  are compositions of  $n$ ,  $hl(J) = |P(J)| + 1$  and  $b(i, j)$  is an integer which will not bother us at this point since we set  $q = -1$  from now on. Rewriting this equation in terms of peak classes of  $\mathbf{Sym}_n$  gives

$$\theta_{-1}(R_I) = \sum_{D(J) \subseteq D(I) \Delta (D(I)+1)} 2^{|D(J)|+1} \Pi_J ,$$

where  $J$  runs over peak compositions of  $n$ . Hence, the image of  $\mathbf{Sym}_n$  by  $\theta_{-1}$ , which we will denote by  $\widetilde{\mathbf{Sym}}_n$ , is contained in the subspace  $\mathcal{P}_n$  of  $\mathbf{Sym}_n$  spanned by the peak classes. In fact,

$$\widetilde{\mathbf{Sym}}_n = \mathcal{P}_n ,$$

since  $\widetilde{\mathbf{Sym}}_n$  and  $\mathcal{P}_n$  have the same dimension. Indeed, recall that the dimension of  $\Pi_n$  (and therefore of  $\mathcal{P}_n$ ) is equal to the Fibonacci number  $f_n$  (as shown in Section 2.1). Thanks to equation (10), we know that the elements

$$\pi^I(-1) = \pi_{i_1}(-1) \pi_{i_2}(-1) \cdots \pi_{i_r}(-1)$$

where  $I = (i_1, \dots, i_r)$  runs over all compositions of  $n$  into odd parts, form a basis of  $\widetilde{\mathbf{Sym}}_n$ , since  $\theta_{-1}(\pi_k(-1)) = 0$  if and only if  $k$  is even. But the number of compositions of  $n$  with odd parts is easily seen to be  $f_n$ . Therefore,

$$\mathcal{P} = (\widetilde{\mathbf{Sym}} := \bigoplus_{n \geq 0} \widetilde{\mathbf{Sym}}_n) .$$

3. HIGHER ORDER PEAK ALGEBRAS

**3.1. Definition and first properties.** Let  $N \geq 1$  and let  $\zeta \in \mathbb{C}$  be a primitive  $N$ -th root of unity. We set  $\Theta_q = \frac{\theta_q}{1-q}$ , and we denote by  $\Theta_\zeta$  the endomorphism of **Sym** defined by

$$(16) \quad \Theta_\zeta(S_n) = \left. \frac{S_n((1-q)A)}{1-q} \right|_{q=\zeta}$$

for  $n \geq 1$ . Due to the fact that  $\Theta_\zeta$  is not invertible (cf. Note 2.2), it is of interest to consider the following algebra.

**Definition 3.1.** *The higher order peak algebra, or generalized peak algebra of order  $N$  is the  $\mathbb{C}$ -algebra  $\mathbf{Sym}(N) = \Theta_\zeta(\mathbf{Sym})$ .*

We will see (Theorem 3.11) that these algebras depend only on  $N$  (and not on the particular choice of  $\zeta$ ). Note that  $\mathbf{Sym}(N)$  has obviously a graded algebra structure, inherited from **Sym**

$$\mathbf{Sym}(N) = \bigoplus_{n \geq 0} \mathbf{Sym}_n(N),$$

where  $\mathbf{Sym}_n(N) := \mathbf{Sym}(N) \cap \mathbf{Sym}_n$ .

**Note 3.2.** For  $N = 1$ ,  $\mathbf{Sym}(1)$  reduces to **Sym**. Indeed,

$$(17) \quad \Theta_q(S_n) = \sum_{i=0}^{n-1} (-q)^i R_{1^i, n-i}$$

for every  $n \geq 1$  (cf. Proposition 5.2 of [7]), which immediately implies that  $\Theta_1(S_n) = \Psi_n$  (cf. Corollary 3.14 of [5]).

**Note 3.3.** For  $N = 2$ , due to the results of Section 2.5,  $\mathbf{Sym}(2)$  reduces to  $\mathcal{P}$ , the inverse image under  $\alpha$  of the usual peak algebra  $\Pi$  within **Sym**.

Let us now consider the multiplicative basis  $(\pi^I(q))_I$  of **Sym**, associated with the family of Lie idempotents  $(\pi_n(q))_{n \geq 1}$  defined by Formula (10) as in Section 2.4. Considering the image of  $(\pi^I(q))_I$  in  $\mathbf{Sym}(N)$  by the morphism  $\Theta_\zeta$ , we obtain that the non-zero elements of the family  $(\Theta_\zeta(\pi^I(q)))_I$  form a basis of  $\mathbf{Sym}(N)$ . But according to equation (10), we have that  $\Theta_\zeta(\pi_n(q)) = 0$  if and only if  $n \equiv 0 [N]$ . Hence the family

$$(18) \quad \Theta_\zeta(\pi^I(q)) = \Theta_\zeta(\pi_{i_1}(q)) \Theta_\zeta(\pi_{i_2}(q)) \cdots \Theta_\zeta(\pi_{i_r}(q)),$$

indexed by the compositions  $I$  of the set  $\mathfrak{F}_n^{(N)} := \{I = (i_1, \dots, i_r) \mid n \mid \forall k \in [1, r], i_k \not\equiv 0 [N]\}$  forms a basis of  $\mathbf{Sym}(N)$ . Summarizing, we obtain the following proposition.

**Proposition 3.4.** *Let  $\zeta$  and  $\eta$  be two primitive  $N$ -th roots of unity. Then the two  $\mathbb{C}$ -algebras  $\Theta_\zeta(\mathbf{Sym})$  and  $\Theta_\eta(\mathbf{Sym})$  are isomorphic. Moreover their common dimension is*

$$\dim(\mathbf{Sym}_n(N)) = \#\mathfrak{F}_n^{(N)}.$$

■



. At this stage, it is interesting to calculate the dimension of  $\mathbf{Sym}_n(N)$ .

**Proposition 3.5.** *The Hilbert series of  $\mathbf{Sym}(N)$  is*

$$(19) \quad \sum_{n \geq 0} \dim(\mathbf{Sym}_n(N)) t^n = \frac{1 - t^N}{1 - t - t^2 - \dots - t^N}.$$

*Proof* – The right-hand side of (19) is the generating function of the generalized Fibonacci numbers  $(f_n)_{n \geq 0}$ . It is easily seen to be equal to the generating function of  $\#\mathfrak{F}_n^{(N)}$ , which is

$$(20) \quad \sum_{n \geq 0} \#\mathfrak{F}_n^{(N)} t^n = \left( 1 - \sum_{j \neq 0 \pmod N} t^j \right)^{-1}.$$

■

**Note 3.6.** Observe that  $\mathbf{Sym}_n(N) = \mathbf{Sym}_n$  for  $n \in [1, N-1]$ .

Though the family  $(\mathfrak{F}_n^{(N)})_{n \geq 0}$  was useful for the calculation of the graded dimension of  $\mathbf{Sym}(N)$ , the associated compositions are not always well-suited as labellings of the combinatorial bases that we want to construct. The following result provides an alternative labelling scheme.

**Proposition 3.7.** *Let us define*

$$\mathfrak{G}_n^{(N)} := \{ I = (i_1, \dots, i_r) \mid n \mid \forall k \in [1, r-1], i_k \in [1, N] \text{ and } i_r \in [1, N-1] \}.$$

*Then, there exists a bijection between  $\mathfrak{G}_n^{(N)}$  and  $\mathfrak{F}_n^{(N)}$ .*

*Proof* – Observe that  $\mathfrak{G}_n^{(N)}$  is just the set of all compositions of  $n$  that belong to the rational language  $[1, N]^*[1, N-1]$ . In other words,

$$\bigsqcup_{n \geq 1} \mathfrak{G}_n^{(N)} = [1, N]^*[1, N-1] = (N^*[1, N-1])^*N^*[1, N-1] = (N^*[1, N-1])^+.$$

This implies that

$$\bigsqcup_{n \geq 1} \mathfrak{G}_n^{(N)} = \{ N^{i_1} j_1 \cdots N^{i_r} j_r \mid i_1, i_2, \dots, i_r \geq 0, j_1, \dots, j_r \in [1, N-1] \}.$$

The map  $\varepsilon$  defined by

$$N^{i_1} j_1 N^{i_2} j_2 \cdots N^{i_r} j_r \xleftarrow{\varepsilon} (N \times i_1 + j_1, N \times i_2 + j_2, \dots, N \times i_r + j_r),$$

for  $i_1, i_2, \dots, i_r \geq 0$  and  $j_1, \dots, j_r \in [1, N-1]$  is then a bijection between  $\mathfrak{G}_n^{(N)}$  and  $\mathfrak{F}_n^{(N)}$ . ■

**Note 3.8.** For  $N = 2$  (remembering that  $\mathbf{Sym}(2)$  is the usual peak algebra  $\Pi$ ),  $\mathfrak{G}_n^{(N)}$  is the set of conjugates of all peak compositions of  $n$ , as introduced in Section 2.1.

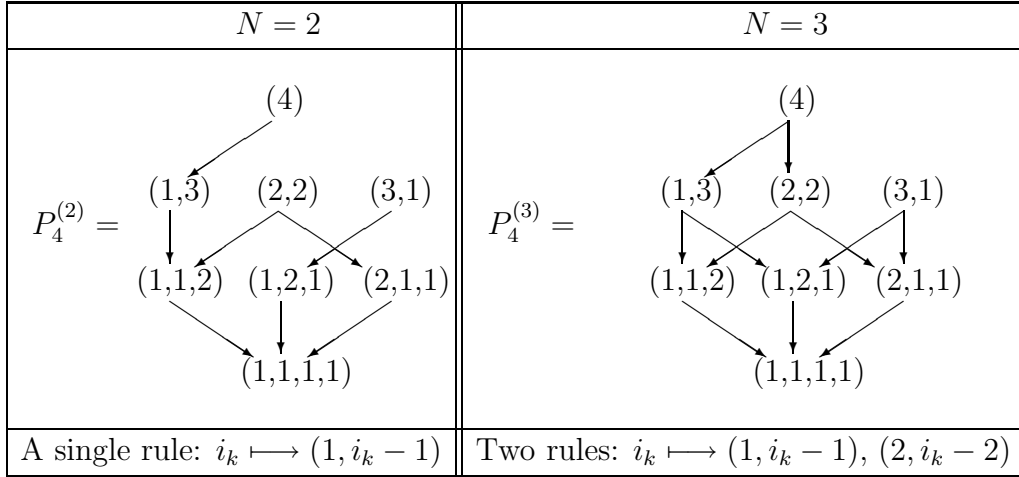
**3.2. Analogues of complete functions in higher order peak algebra.** If  $I = (i_1, i_2, \dots, i_r)$  and  $J = (j_1, j_2, \dots, j_s)$  are two compositions of  $n$ , let us write  $I \leq_{P_n^{(N)}} J$  if and only if  $J$  can be obtained from  $I$  by one of the following elementary rewriting rules:

$$\forall k \in [1, r], \begin{cases} i_k \mapsto (1, i_k - 1), \\ i_k \mapsto (2, i_k - 2), \\ \dots \\ i_k \mapsto (N - 1, i_k - N + 1). \end{cases}$$

We denote by  $\leq_{P_n^{(N)}}$  the transitive closure of  $\leq_{P_n^{(N)}}$ .

**Definition 3.9.** For  $n \geq 0$ ,  $P_n^{(N)}$  is the poset of all compositions of  $n$  endowed with the partial ordering  $\leq_{P_n^{(N)}}$ .

**Example 3.10.** On compositions of 4, we have



We can now state one of the main results of this article.

**Theorem 3.11.** For  $I$  in  $\mathfrak{G}_n^{(N)}$ , let us define the complete peak function of order  $N$   $\Sigma_I^{(N)}$  by

$$\Sigma_I^{(N)} := \sum_{J \leq_{P_n^{(N)}} I} R_J .$$

Then, the family  $(\Sigma_I^{(N)})_{I \in \mathfrak{G}_n^{(N)}}$  forms a basis of  $\mathbf{Sym}_n(N)$ .

**Example 3.12.**

$$\begin{aligned} \Sigma_{(1,2,1)}^{(2)} &= R_{(1,2,1)} + R_{(3,1)}, \\ \Sigma_{(1,2,1)}^{(3)} &= R_{(1,2,1)} + R_{(3,1)} + R_{(1,3)} + R_{(4)}, \\ \Sigma_{(1,1,2)}^{(3)} &= \Sigma_{(1,1,2)}^{(2)} = R_{(1,1,2)} + R_{(2,2)} + R_{(1,3)} + R_{(4)}. \end{aligned}$$

We shall deduce Theorem 3.11 from the following two Lemmas.

**Lemma 3.13.** *For all  $i \geq 0$  and  $j \in [1, N-1]$ ,*

$$R_{N^i j} \in \mathbf{Sym}(N),$$

with  $N^i j = (\underbrace{N, \dots, N}_i, j)$ .

**Note 3.14.** Since  $N^i j$  cannot have any predecessor in  $P_n^{(N)}$ , we have, for  $i \geq 0$  and  $j \in [1, N-1]$ ,

$$\Sigma_{N^i j}^{(N)} = R_{N^i j}.$$

**Lemma 3.15.** *For  $I, J \in \mathfrak{G}_n^{(N)}$ ,*

$$\Sigma_I^{(N)} \times \Sigma_J^{(N)} = \Sigma_{I.J}^{(N)}.$$

*Proof of Theorem 3.11* - Let  $I = (N^{i_1}, j_1, \dots, N^{i_r}, j_r)$  be a generic element of  $\mathfrak{G}_n^{(N)}$ . According to Lemma 3.15, we can write

$$\Sigma_I^{(N)} = \Sigma_{N^{i_1} j_1}^{(N)} \times \dots \times \Sigma_{N^{i_r} j_r}^{(N)}.$$

Using now Lemma 3.13 and Note 3.14, we deduce from this identity that  $\Sigma_I^{(N)} \in \mathbf{Sym}(N)$ . But the  $\Sigma_I^{(N)}$  are linearly independent by construction. Hence, they form a basis of  $\mathbf{Sym}(N)$ , since the cardinality of  $\mathfrak{G}_n^{(N)}$  is equal to the dimension of  $\mathbf{Sym}_n(N)$ , according to Propositions 3.4 and 3.7.  $\blacksquare$

It remains to establish the Lemmas.

*Proof of Lemma 3.13* - For  $j \in [1, N-1]$ , let

$$(21) \quad \varrho_j(z) = \sum_{m \geq 0} (-1)^{m+1} R_{N^m j} z^{mN+j}.$$

Substituting

$$R_{N^m j} = \sum_{\substack{\alpha_1 + \dots + \alpha_r = m \\ \alpha_1, \dots, \alpha_r \geq 0}} (-1)^{m+1-r} S_{\alpha_1 N} \dots S_{\alpha_{r-1} N} S_{\alpha_r N+j}$$

in (21) yields

$$\varrho_j(z) = \sum_{n \geq 0} \sum_{\substack{\alpha_1 + \dots + \alpha_r = n \\ \alpha_1, \dots, \alpha_r \geq 0}} (-1)^r S_{\alpha_1 N} \dots S_{\alpha_{r-1} N} S_{\alpha_r N} z^{nN+j}.$$

Rearranging the sum, we obtain

$$\varrho_j(z) = \sum_{m \geq 0} \left( \sum_{\substack{\alpha_1 + \dots + \alpha_s \geq 0 \\ s \geq 0}} (-1)^s S_{\alpha_1 N} \dots S_{\alpha_s N} z^{(\alpha_1 + \dots + \alpha_s)N} \right) S_{mN+j} z^{mN+j},$$

where the two summations are now independent. So, Equation (21) reduces to (22)

$$\varrho_j(z) = \left( \sum_{\substack{\alpha_1 + \dots + \alpha_s \geq 0 \\ s \geq 0}} (-1)^s S_{\alpha_1 N} \dots S_{\alpha_s N} z^{(\alpha_1 + \dots + \alpha_s)N} \right) \times \left( \sum_{m \geq 0} S_{mN+j} z^{mN+j} \right),$$

which can itself be rewritten as

$$(23) \quad \varrho_j(z) = \left( \sum_{n \geq 0} S_{nN} z^{nN} \right)^{-1} \times \left( \sum_{m \geq 0} S_{mN+j} z^{mN+j} \right).$$

It follows then from (23) that

$$\begin{aligned} \sum_{j=1}^{N-1} \varrho_j(z) &= \left( \sum_{n \geq 0} S_{nN} z^{nN} \right)^{-1} \times \left( \sum_{m \geq 0} S_{mN+1} z^{mN+1} + \dots + \sum_{m \geq 0} S_{mN+N-1} z^{mN+N-1} \right) \\ &= \left( \sum_{n \geq 0} S_{nN} z^{nN} \right)^{-1} \times \left( \sum_{m \neq 0 [N]} S_m z^m \right) \\ &= \left( \sum_{n \geq 0} S_{nN} z^{nN} \right)^{-1} \times \left( \sum_{m \geq 0} S_m z^m - \sum_{n \geq 0} S_{nN} z^{nN} \right) \\ &= \left( \sum_{n \geq 0} S_{nN} z^{nN} \right)^{-1} \times \left( \sum_{m \geq 0} S_m z^m \right) - 1, \end{aligned}$$

so that

$$(24) \quad 1 + \sum_{j=1}^{N-1} \varrho_j(z) = \left( \sum_{n \geq 0} S_{nN} z^{nN} \right)^{-1} \times \left( \sum_{m \geq 0} S_m z^m \right).$$

We want to prove that this series in  $z$  has its coefficients in  $\mathbf{Sym}(N)$ . Equivalently, we can prove this for its inverse

$$(25) \quad \begin{aligned} \left( 1 + \sum_{j=1}^{N-1} \varrho_j(z) \right)^{-1} &= \lambda_{-z}(A) \sum_{m \geq 0} S_{mN}(A) z^{mN} \\ &= \sum_{n \geq 0} z^n \sum_{Ni+j=n} (-1)^j \Lambda_j(A) S_{Ni}(A). \end{aligned}$$

The coefficient  $C_n$  of  $z^n$  in this expression can be rewritten as

$$(26) \quad \sum_{Ni+j=n} (-1)^j S_j(-A) S_{Ni}(qA)|_{q=\zeta} = E_0[S_n((q-1)A)]$$

where for a polynomial  $f(q)$  we set

$$(27) \quad f(q) = f_0(q^N) + qf_1(q^N) + q^2f_2(q^N) + \cdots + q^{N-1}f_{N-1}(q^N)$$

and

$$(28) \quad f_j = E_j[f].$$

Now, the inversion formula for the discrete Fourier transform on  $N$  points shows that the polynomials  $q^j f_j(q^N)$  are linear combinations with complex coefficients of  $f(q)$ ,  $f(\zeta q)$ ,  $f(\zeta^2 q), \dots, f(\zeta^{N-1} q)$ . In particular,  $E_0[S_n((q-1)A)]$  is a linear combination of

$$S_n((q-1)A), S_n(\zeta q - 1)A, \dots, S_n(\zeta^{N-1} q - 1)A,$$

and  $C_n$ , which is its specialization at  $q = \zeta$ , is therefore in the subspace spanned by

$$S_n((\zeta-1)A), S_n((\zeta^2-1)A), \dots, S_n((\zeta^N-1)A) = 0$$

which are all in  $\mathbf{Sym}(N)$ , thanks to the identity  $\zeta^k - 1 = (1 - \zeta)(-1 - \zeta - \cdots - \zeta^{k-1})$ .

■

*Proof of Lemma 3.15* - Observe first that for  $I, J \in \mathfrak{G}_n^{(N)}$ , the concatenation  $I \cdot J$  is also in  $\mathfrak{G}_n^{(N)}$ . Consider now the product

$$(29) \quad \left( \sum_{K \leq_{P_n^{(N)}} I} R_K \right) \times \left( \sum_{L \leq_{P_n^{(N)}} J} R_L \right).$$

According to the product formula for ribbons (Proposition 3.13 of [5]), the product in Formula (29) is seen to be the sum of all compositions of  $n$  which can be refined into the concatenation  $I \cdot J$ , whence the Lemma. ■

**Note 3.16.** One should observe that the converse of Lemma 3.13 is false: there exists  $R_I \in \mathbf{Sym}(N)$  such that the associated composition  $I$  is different from  $N^i j$  for all  $i \geq 0$  and  $j \in [1, N-1]$ . For example,

$$R_{(2,1,1)} = \left( \Sigma_{(2,1,1)}^{(3)} - \Sigma_{(3,1)}^{(3)} - \Sigma_{(2,2)}^{(3)} \right) \in \mathbf{Sym}(N),$$

and  $(2, 1, 1)$  is obviously not a composition of the type  $(3^i, 1)$  or  $(3^i, 2)$ .

For  $k \neq 0 [N]$ , let us now set

$$T_k = R_{N^i j} = \Sigma_{N^i j}$$

where  $(i, j)$  is the unique pair of  $\mathbb{N} \times [1, N-1]$  such that  $k = N \times i + j$ . More generally, for  $K = (k_1, \dots, k_r) \in \mathfrak{F}_n^{(N)}$ , we set

$$T_K = T_{k_1} \times \cdots \times T_{k_r}.$$

An alternative definition of  $T_K$  is

$$\Sigma_I = T_{\varepsilon(I)},$$

where  $\varepsilon$  is the bijection introduced in the proof of Proposition 3.7, which allows us to rephrase Theorem 3.11 in the form

**Corollary 3.17.** *The family  $(T_K)_{K \in \mathfrak{S}_n^{(N)}}$  forms a basis of  $\mathbf{Sym}_n^{(N)}$ .*

**3.3. Generalized peak ribbons.** We are now in a position to introduce analogues of ribbons in the higher order peak algebras.

**Definition 3.18.** *Let  $I$  be a composition of  $\mathfrak{S}_n^{(N)}$ . Then the peak ribbon of order  $N$  labelled by  $I$  is defined as*

$$(30) \quad \rho_I^{(N)} = \sum_{\substack{J \leq_{P_n^{(N)}} I, \\ J \in \mathfrak{S}_n^{(N)}}} (-1)^{l(I)-l(J)} \Sigma_J^{(N)}.$$

The following Proposition results immediately from Theorem 3.11.

**Proposition 3.19.**  *$(\rho_I^{(N)})_{I \in \mathfrak{S}_n^{(N)}}$  is a basis of  $\mathbf{Sym}(N)$ .*

It follows that each  $\Sigma_I^{(N)}$  has a decomposition on the generalized peak ribbons. This decomposition is given by the Möbius-like inverse of Formula (30), justifying the claim that the families  $(\rho_I^{(N)})_{I \in \mathfrak{S}_n^{(N)}}$  and  $(\Sigma_I^{(N)})_{I \in \mathfrak{S}_n^{(N)}}$  are in a similar relation as  $(R_I)_{I \models n}$  and  $(S_I)_{I \models n}$ .

**Proposition 3.20.** *For all  $I \in \mathfrak{S}_n^{(N)}$ ,*

$$(31) \quad \Sigma_I^{(N)} = \sum_{\substack{J \leq_{P_n^{(N)}} I \\ J \in \mathfrak{S}_n^{(N)}}} \rho_J^{(N)}$$

*Proof* – See Note 4.5. ■

**Note 3.21.** We can introduce others analogues of the usual ribbon functions, such as, for example, the following two families, which are indexed by compositions  $I \in \mathfrak{S}_n^{(N)}$ :

$$\rho_I^{(N)}(t) = \sum_{\substack{J \leq_{P_n^{(N)}} I \\ J \in \mathfrak{S}_n^{(N)}}} t^{l(I)-l(J)} \Sigma_J \quad \text{or} \quad \rho_I^{\prime(N)}(t) = \sum_{\substack{J \leq_{P_n^{(N)}} I \\ J \in \mathfrak{S}_n^{(N)}}} t^{l(I)+l(J)} \Sigma_J,$$

and which also are clearly bases of  $\mathbf{Sym}(N)$  for  $t \in \mathbb{C} - \{0\}$ . We have of course

$$\rho_I^{(N)} = \rho_I^{(N)}(-1) = \rho_I^{\prime(N)}(-1),$$

for  $I \in \mathfrak{S}_n^{(N)}$ . Note also that  $\rho_I^{(N)}(t)$  and  $\rho_I^{\prime(N)}(t)$  are connected by

$$\rho_I^{\prime(N)}(t) = t^{2l(I)} \rho_I^{(N)}\left(\frac{1}{t}\right).$$

Due to this relation, we will only make use of  $(\rho_I^{(N)}(t))_{I \in \mathfrak{S}_n^{(N)}}$  in the sequel.

**Example 3.22.** We now give explicitly some elements of the family  $(\rho_I^{(N)})_{I \in \mathfrak{G}_n^{(N)}}$ :

$$\rho_{(1,1,1)}^{(2)} = \Sigma_{(1,1,1)}^{(2)} - \Sigma_{(2,1)}^{(2)}, \quad \begin{cases} \rho_{(1,1,1)}^{(3)} = \Sigma_{(1,1,1)}^{(3)} - \Sigma_{(1,2)}^{(3)} - \Sigma_{(2,1)}^{(3)} = R_{(1,1,1)} - R_{(3)}, \\ \rho_{(1,2)}^{(3)} = \Sigma_{(1,2)}^{(3)} = R_{(1,2)} + R_{(3)}, \\ \rho_{(2,1)}^{(3)} = \Sigma_{(2,1)}^{(3)} = R_{(2,1)} + R_{(3)}. \end{cases}$$

Unfortunately the internal product of two elements of the family  $(\rho_I^{(N)}(t))_{I \in \mathfrak{G}_n^{(N)}}$  does not always decompose with non-negative coefficients on this family, contrary to the case of the usual ribbon basis of **Sym** (cf. section 5 of [5]). This shows that the family defined by Definition 3.18 is probably not a perfect analogue of the usual non commutative ribbons.

**Example 3.23.**

$$\begin{cases} \rho_{(1,1,1)}^{(3)} * \rho_{(1,1,1)}^{(3)} = (-2) \times \rho_{(1,1,1)}^{(3)}, \\ \rho_{(1,2)}^{(3)} * \rho_{(1,1,1)}^{(3)} = \rho_{(1,1,1)}^{(3)} + \rho_{(2,1)}^{(3)} - \rho_{(1,2)}^{(3)}. \end{cases}$$

#### 4. DECOMPOSITIONS ON PEAK BASES

**4.1. A projector.** Let us define a linear map  $\pi_N$  from **Sym** onto **Sym**( $N$ ) by

$$(32) \quad \pi_N(S^I) = \begin{cases} \Sigma_I & \text{if } I \in \mathfrak{G}_n^{(N)}, \\ 0 & \text{otherwise.} \end{cases}$$

**Note 4.1.** Note that  $\pi_N(R_I) = \rho_I^{(N)}$  for every  $I \in \mathfrak{G}_n^{(N)}$ .

Let  $T^{(N)}$  be the left-ideal of **Sym** generated by the  $S_j$  such that  $j \not\equiv 0 \pmod{N}$ .

**Lemma 4.2.** **Sym**( $N$ ) is a subalgebra of  $T^{(N)}$ .

*Proof* – The generators  $R_{N^i j}$  of the algebra **Sym**( $N$ ) are in  $T^{(N)}$ , since

$$R_{N^i j} = \sum_{\substack{\alpha_1 + \dots + \alpha_r = i \\ \alpha_1, \dots, \alpha_r \geq 0}} (-1)^{i+1-r} S^{(\alpha_1 N, \dots, \alpha_{r-1} N, \alpha_r N + j)}.$$

■

**Proposition 4.3.**  $\pi_N|_{\mathbf{Sym}(N)}$  is an algebra morphism.

*Proof* – We will in fact show the slightly stronger property

$$(33) \quad \pi_N(F \times G) = \pi_N(F) \times \pi_N(G),$$

for all  $F \in T^{(N)}$  and  $G \in \mathbf{Sym}$ . We establish (33) by proving that it holds for products of complete functions, that is

$$(34) \quad \pi_N(S^I \times S^J) = \pi_N(S^I) \times \pi_N(S^J),$$

for  $S^I \in T_n^{(N)}$  and all  $S^J$ . Two cases are to be considered.

Suppose first that  $I \cdot J \notin \mathfrak{G}_n^{(N)}$ . Then one has either  $I \notin \mathfrak{G}_n^{(N)}$ , or  $J \notin \mathfrak{G}_n^{(N)}$  by definition of  $\mathfrak{G}_n^{(N)}$ . So, both sides of (34) are 0 according to the definition of  $\pi_N$ .

Assume now that  $I \cdot J \in \mathfrak{G}_n^{(N)}$ . Then,  $I$  and  $J$  are also elements of  $\mathfrak{G}_n^{(N)}$ , since we have assumed that the last part of  $I$  is not a multiple of  $N$ . Then, by definition of  $\pi_N$ ,

$$(35) \quad \pi_N(S^{I \cdot J}) = \Sigma_{I \cdot J}^{(N)}.$$

But from Lemma 3.15, we know that that

$$(36) \quad \Sigma_{I \cdot J}^{(N)} = \Sigma_I^{(N)} \times \Sigma_J^{(N)}.$$

So it results from (35) and (36) that

$$\pi_N(S^{I \cdot J}) = \pi_N(S^I) \times \pi_N(S^J),$$

since  $\Sigma_I^{(N)}$  and  $\Sigma_J^{(N)}$  are respectively the images of  $S^I$  and  $S^J$  under  $\pi_N$ . Hence, (34) also holds in this case. ■

We can now give an important property of  $\pi_N$ .

**Proposition 4.4.**  $\pi_N$  is a projector from  $\mathbf{Sym}$  onto  $\mathbf{Sym}(N)$ .

*Proof* – We have to prove that

$$(37) \quad \pi_N(\Sigma_I) = \Sigma_I,$$

for  $I \in \mathfrak{G}_n^{(N)}$ . Using Proposition 4.3, which gives that  $\pi_N$  is multiplicative on  $\mathbf{Sym}(N)$ , and Lemma 3.15, we only have to prove that

$$\pi_N(R_{N^i j}) = R_{N^i j},$$

for  $i \geq 0$  and  $j \in [1, N-1]$ . But we have already seen (Note 4.1) that  $\pi_N(R_I) = \rho_I^{(N)}$  for every  $I \in \mathfrak{G}_n^{(N)}$ , and for  $I = (N^i j)$ ,  $\rho_I^{(N)} = \Sigma_I = R_I$ . ■

**Note 4.5.** Remark that for  $I \in \mathfrak{G}_n^{(N)}$ ,

$$\Sigma_I^{(N)} = \pi_N(S^I) = \sum_{J \leq I} \pi_N(R_J) = \sum_{\substack{J \leq_{P_n^{(N)}} I \\ J \in \mathfrak{G}_n^{(N)}}} \rho_J,$$

which gives an alternative proof of Proposition 3.20.



**4.2. Some interesting decompositions.** We record in this section a number of remarkable decompositions with respect to the bases of complete and ribbon peak functions of order  $N$ .

**Proposition 4.6.** *Let  $\zeta$  be a primitive  $N$ -th root of unity. Then, for a composition  $I$  of  $n$ ,*

$$(38) \quad \vartheta_\zeta(S^I) = \sum_{\substack{J = (j_1, \dots, j_s) \leq I \\ J \in \mathfrak{G}_n^{(N)}}} (-1)^{l(I)-l(J)} \prod_{k \in \mathcal{H}(I, J)} \left( \left( \frac{1}{\zeta} \right)^{j_k} - 1 \right) \Sigma_J^{(N)},$$

where

$$\mathcal{H}(I, J) := \{l \in [1, s], |\exists k \in [1, r], j_1 + \dots + j_l = i_k\}.$$

*Proof* – Formula (38) results by applying  $\pi_N$  to the decomposition of  $S^I((1-q)A)$  on products of complete functions, specialized at  $q = \zeta$ , which is given by Formula (105) of Proposition 5.30 of [7]. ■

**Proposition 4.7.** *Let  $\zeta$  be a primitive  $N$ -th root of unity. Then, for a composition  $I$  of  $n$ ,*

$$(39) \quad \vartheta_\zeta(R_I) = \sum_{\substack{J = (j_1, \dots, j_s) \leq I \\ J \in \mathfrak{G}_n^{(N)}}} (-1)^{l(I)-l(J)} q^{\alpha(I, J)} (1 - q^{j_s}) \Sigma_J^{(N)},$$

where

$$\alpha(I, J) = \sum_{k=1}^{l(J)-1} j_k \times \delta_{j_1 + \dots + j_k \notin D(I)}.$$

*Proof* – As for Proposition 4.6, Formula (39) is obtained by applying  $\pi_N$  on the decomposition of  $R_I((1-\zeta)A)$  on the basis  $S^J$ , as given in [7]. ■

**Note 4.8.** Note that Propositions 4.6 or 4.7 give another proof of Theorem 3.11, since Formulas (38) or (39) show that  $(\Sigma_J^{(N)})_{J \in \mathfrak{G}_n^{(N)}}$  is a generating family of  $\mathbf{Sym}(N)$ . Since this family is obviously linearly free by construction, it follows that it is a basis of  $\mathbf{Sym}(N)$ .

In order to state the next Proposition, we need a definition. For each pair of compositions  $I = (i_1, \dots, i_r)$  and  $J = (j_1, \dots, j_s)$  of the same weight, let us introduce the sequence of compositions  $H(I, J) = (H_1, \dots, H_r)$  of length  $l(I)$  which is uniquely determined by the conditions  $|H_k| = i_k$  for  $k \in [1, l(I)]$ , and

$$H_1 \bullet \dots \bullet H_r = J,$$

where  $\bullet$  denotes either the concatenation of compositions, or the operation  $\triangleright$ , defined by

$$H \triangleright K = (k_1, \dots, k_{r-1}, k_r + l_1, l_2, \dots, l_s).$$

**Example 4.9.** Let  $I = (3, 2, 1, 4)$  and  $J = (2, 5, 2, 1)$ . Then, we have

$$H(I, J) = ((2, 1), (2), (1), (1, 2, 1)).$$

We can now give the following definition.

**Definition 4.10.** For any two compositions  $I$  and  $J$  of the same weight, we set

$$(40) \quad h(I, J) = \begin{cases} -\infty & \text{if there exists } k \in [1, l(I)], \alpha \geq 0, \beta \geq 1, H_k \neq (1^\alpha, \beta), \\ \sum_{i=1}^{l(I)} \alpha_i & \text{if for every } k \in [1, l(I)], \alpha_k \geq 0, \beta_k \geq 1, H_k = (1^{\alpha_k}, \beta_k). \end{cases}$$

**Proposition 4.11.** Let  $\zeta$  be a primitive  $N$ -th root of unity. Then, for a composition  $I$  of  $n$ ,

$$(41) \quad \vartheta_\zeta(S^I) = (1 - \zeta)^{l(I)} \sum_{\substack{J \in \mathfrak{G}_n^{(N)} \\ h(I, J) \neq -\infty}} (-\zeta)^{h(I, J)} \rho_J^{(N)}$$

*Proof* – The decomposition of  $\vartheta_q(S^I)$  on the ribbon basis follows from Formula (67) of Proposition 5.2 of [7]. Formula (41) comes by applying  $\pi_N$  to this decomposition, specialized at  $q = \zeta$ . ■

Note that an arbitrary composition  $I$  can always be uniquely written as

$$I = H_1 \cdot H_2 \cdots H_{hl(I)},$$

where  $H_k = (1^\alpha, \beta)$ ,  $k \in [1, hl(I)]$ . We denote by  $H_I$  the composition

$$H_I = (|H_1|, |H_2|, \dots, |H_{hl(I)}|).$$

**Example 4.12.** For  $I = (1, 3, 1, 4, 2)$ , we have

$$I = H_1 \cdot H_2 \cdot H_3,$$

with  $H_1 = (1, 3)$ ,  $H_2 = (1, 4)$ ,  $H_3 = (2)$ . Hence  $hl(I) = 3$  and  $D(H_I) = \{4, 9\}$ , in this case.

**Definition 4.13.** Let  $I, J$  be two compositions of  $n$ . We set

$$(42) \quad b(I, J) = \begin{cases} |(1 + (D(I) - D(J))) \cup (D(J) - D(I))| & \text{if } D(H_J) \subset S(I), \\ -\infty & \text{otherwise,} \end{cases}$$

where  $S(I) := ((1 + D(I)) - D(I)) \cup (D(I) - (1 + D(I)))$ .

**Proposition 4.14.** Let  $\zeta$  be a primitive  $N$ -th root of unity. Then, for a composition  $I$  of  $n$ ,

$$(43) \quad \vartheta_\zeta(R_I) = \sum_{\substack{J \in \mathfrak{G}_n^{(N)} \\ b(I, J) \neq -\infty}} (1 - \zeta)^{hl(J)} (-\zeta)^{b(I, J)} \rho_J^{(N)}.$$

*Proof* – This follows by applying  $\pi_N$  to the ribbon decomposition of  $\vartheta_q(R_I)$ , specialized at  $q = \zeta$ , as given in Formula (121) of Proposition 5.41 of [7]. ■

**4.3. Higher order “noncommutative tangent numbers”.** We will finally present in this last section, a number of formulas generalizing the known relations involving the so-called *noncommutative tangent numbers*, introduced in [5]. The formulas of [5] are obtained for  $N = 1$ , and those of [2] for  $N = 2$ .

**Proposition 4.15.** *Let*

$$\mathbf{t} := \sum_{i \geq 0} (-1)^{i+1} \left( \sum_{j=1}^{N-1} \Sigma_{N^i j}^{(N)} \right),$$

*Then,*

$$(44) \quad (1 - \mathbf{t})^{-1} = \sum_{n \geq 0} (-1)^n \rho_{1^n}^{(N)},$$

*Proof* – We know that

$$(45) \quad (1 - t)^{-1} = \sum_{\substack{i_1, \dots, i_r \geq 0 \\ r \geq 0}} (-1)^{i_1 + \dots + i_r + r} \sum_{j_1, \dots, j_r \in [1, N-1]} \Sigma_{N^{i_1 j_1} \dots N^{i_r j_r}}^{(N)},$$

where  $\Sigma_{\emptyset}^{(N)} = 1$ . But for a composition  $K = (N^{i_1 j_1}, \dots, N^{i_r j_r})$ , one has

$$l(K) = i_1 + \dots + i_r + r,$$

so that (45) can be rewritten as

$$\begin{aligned} (1 - \mathbf{t})^{-1} &= 1 + \sum_{\substack{K \in \mathfrak{G}_n^{(N)} \\ n \geq 1}} (-1)^{l(K)} \Sigma_K^{(N)} \\ &= 1 + \sum_{n \geq 1} (-1)^n \left( \sum_{K \in \mathfrak{G}_n^{(N)}} (-1)^{l(K) - |K|} \Sigma_K^{(N)} \right), \\ &= 1 + \sum_{n \geq 1} (-1)^n \rho_{1^n}^{(N)}, \end{aligned}$$

which is the required expression ( $\rho_{1^0}^{(N)} = 1$ ). ■

**Note 4.16.** Equation (44) can be regarded as a kind of generalization of the trivial relation  $\sigma(1) \lambda(-1) = 1$ , where  $\sigma$  and  $\lambda$  are respectively the generating functions of

complete and elementary non commutative symmetric functions. More generally, we can set

$$\begin{cases} \sigma_N(t) = 1 + \sum_{i=1}^{N-1} t^i \Sigma_i^{(N)} + \sum_{i \geq 1} (-1)^i \left( \sum_{j=1}^{N-1} t^{iN+j} \Sigma_{N^i j}^{(N)} \right), \\ \lambda_N(t) = \sum_{n \geq 0} (-t)^n \rho_{1^n}^{(N)}, \end{cases}$$

and one can check, following the lines of the proof of Proposition 4.15, that  $\sigma_N(t) \lambda_N(-t) = 1$ , another generalization of the same relation.

**Proposition 4.17.** *Let  $\zeta$  be a primitive  $N$ -th root of unity and let  $\mathbf{t}_\zeta$  be defined by*

$$\mathbf{t}_\zeta := \sum_{i \geq 0} \sum_{j=1}^{N-1} \zeta^{j-i-1} \Sigma_{N^i j}^{(N)}.$$

Then,

$$(46) \quad (1 - \mathbf{t}_\zeta)^{-1} = \sum_{n \geq 0} (-1)^n \rho_{1^n}^{(N)}(\zeta).$$

**Note 4.18.** Note that for  $N = 2$ , we get  $\mathbf{t}_{-1} = TH$ , in the notation of [5].

*Proof* – According to Lemma 3.15,

$$(47) \quad (1 - \mathbf{t}_\zeta)^{-1} = \sum_{\substack{i_1, \dots, i_r \geq 0 \\ j_1, \dots, j_r \in [1, N-1] \\ r \geq 0}} \zeta^{j_1 + \dots + j_r - (i_1 + \dots + i_r + 1)} \Sigma_{N^{i_1} j_1 \dots N^{i_r} j_r}^{(N)}.$$

But for a composition  $K = (N^{i_1} j_1, \dots, N^{i_r} j_r)$ ,

$$|K| - l(K) = \underbrace{N \times (i_1 + \dots + i_r) + j_1 + \dots + j_r}_{|K|} - \underbrace{(r + i_1 + \dots + i_r)}_{l(K)}.$$

So, (47) becomes

$$(1 - \mathbf{t}_\zeta)^{-1} = \sum_{\substack{K \in \mathfrak{G}_n^{(N)} \\ n \geq 0}} \zeta^{|K| - l(K)} \Sigma_K^{(N)},$$

whence the proposition. ■

*Acknowledgements.* - This project has been partially supported by EC's IHRP Programme, grant HPRN-CT-2001-00272, "Algebraic Combinatorics in Europe". The authors are also grateful to Maxime Rey for his help with the redaction of this paper.

## REFERENCES

- [1] M. AGUIAR, N. BERGERON, KATHRYN NYMAN, *The peak algebra and the descent algebras of types B and D*, preprint math.CO/0302278.
- [2] BERGERON N., HIVERT F., THIBON J.Y., *The peak algebra and the Hecke-Clifford algebras at  $q = 0$* , J. Combinatorial Theory A **117** (2004),1–19.
- [3] DUCHAMP G., HIVERT F., THIBON J.Y., *Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras*, Intern. J. Alg. Comput., **12**, no. 5, (2002), 671-717.
- [4] DUCHAMP G., KLYACHKO A., KROB D., THIBON J.Y., *Noncommutative symmetric functions III: Deformations of Cauchy and convolutive algebras*, Discrete Math. Theoret. Comput. Sci., **1**, (1997), 159-216.
- [5] GELFAND I.M., KROB D., LASCoux A., LECLERC B., RETAKH V.S., THIBON J.Y., *Noncommutative symmetric functions*, Adv. in Math., **112**, (1995), 218–348.
- [6] KNUTSON D.,  *$\Lambda$ -rings and the representation theory of the symmetric group*, Lecture Notes in Mathematics, **308**, Springer, 1973.
- [7] KROB D., LECLERC B., THIBON J.Y., *Noncommutative symmetric functions II: Transformations of alphabets*, Intern. J. Alg. Comput., **7**, (2), (1997), 181–264.
- [8] KROB D., THIBON J.Y., *Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at  $q = 0$* , J. Alg. Comb., **6**, (1997), 339-397.
- [9] KROB D., THIBON J.Y., *Noncommutative symmetric functions V: A degenerate version of  $U_q(\mathfrak{gl}_N)$* , Intern. J. Alg. Comput., **9**, no. 3/4, (1999), 405-430.
- [10] AL. LASCoux, B. LECLERC, J.-Y. THIBON, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Commun. Math. Phys.**181** (1996), 205-263.
- [11] LASCoux A., SCHÜTZENBERGER M.P., *Formulaire raisonné de fonctions symétriques*, Université Paris 7, 1985.
- [12] MACMAHON P.A., *Combinatory analysis*, Cambridge University Press, 1915, 1916; Chelsea reprint, 1960.
- [13] MACDONALD I.G., *Symmetric functions and Hall polynomials*, Oxford Math. Monographs, Oxford University Press, 2nd ed., 1994.
- [14] NYMAN K.L., *The peak algebra of  $k[\mathfrak{S}_n]$* , J. of Alg. Comb., **17**, (2003), 309-322.
- [15] SOLOMON L., *A Mackey formula in the group ring of a Coxeter group*, J. Algebra, **41**, no. 2, (1976), 255-268.
- [16] SCHOCKER M., *The peak algebra revisited*, Preprint math.RA/0209376, 2002.
- [17] J. STEMBRIDGE, *Enriched P-partitions*, Trans. Amer. Math. Soc. **349** (1997), 763–788.

(D. Krob) LIX (CNRS), ECOLE POLYTECHNIQUE,, ROUTE DE SACLAY,, 91128 PALAISEAU CEDEX, FRANCE

(J.-Y. Thibon) INSTITUT GASPARD MONGE, UNIVERSITÉ DE MARNE-LA-VALLÉE, 5 BOULEVARD DESCARTES, CHAMPS-SUR-MARNE, 77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

*E-mail address*, D.Krob: dk@lix.polytechnique.fr

*E-mail address*, Jean-Yves Thibon: jyt@univ-mlv.fr