Conjectures on the enumeration of tableaux of bounded height

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Some Conjectures on the Enumeration of Tableaux of Bounded Height (1)

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Abstract

We express general conjectures for explicit forms of P-recurrences for the number of Young standard tableaux of height bounded by h. These recurrences are compatible with known results and Regev's asymptotic evaluations.

Résumé

Le but de cette note est essentiellement de présenter des conjectures ainsi que des indications sur les raisons qui nous ont portées à les énoncer. Nos conjectures concernent les formes explicites d'équations de récurrence que semble satisfaire la suite $t_h(n)$ des nombres de tableaux de Young standard de hauteur bornée par h, ainsi que la suite $t_h^{(2)}(n)$ des nombres de paires de tels tableaux ayant même forme (voir (2)). La forme des récurrences (1a et 2a) et certains aspects explicites des coefficients (1b, 1c, 2b, 2c) font l'objet de ces conjectures dans le cas général, et nous donnons encore plus de détails (1c, 1e et 2c) pour le cas h impair. Un résultat de Zeilberger [4] assure que de telles récurrences existent, mais sa démonstration ne semble pas permettre de déduire une forme explicite aussi précise que (4) ou (6). On montre aussi que le comportement asymptotique des solutions des récurrences obtenues par le biais de nos conjectures est compatible avec les résultats de Regev [3] sur le comportement asymptotique des nombres $t_h(n)$.

1. Introduction

Let us first fix some notations. A partition λ of a positive integer n is a sequence of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ such that $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ to express this fact. The number k of parts involved in λ is denoted $l(\lambda)$ and called the *height* of λ. The height of the empty partition (of 0) is set to be 0. The (Ferrer's) diagram of a partition is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$.

A Young standard tableau T is an injective labeling of a Ferrer's diagram by the elements of $\{1, 2, \ldots, n\}$, such that $T(i,j) < T(i+1,j)$, for $1 \leq i < k$, and $T(i,j) < T(i,j+1)$, for $1 \leq j < \lambda_i$. We further say that λ is the *shape* of the tableau T. For a given $\lambda \vdash n$, the number f_λ of tableaux of shape λ is obtained by the *hook* formula

$$f_\lambda = \frac{n!}{\prod_c h_c} \,,$$

where $c = (i,j)$ runs over the set of points in the diagram of λ, and

$$h_c = \lambda_i + \#\{j \mid \lambda_j \geq i\} - i - j + 1 \,.$$

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Other classical results in this context are

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!\,,$$

and

$$\sum_{\lambda \vdash n} f_\lambda = \text{ coeff of } \frac{x^n}{n!} \text{ in } e^{x+x^2/2}.$$

We refer for instance the reader to [**2**] for more details.

We are interested in the enumeration of tableaux of height bounded by some integer h, this is to say that we want to compute the numbers

$$t_h(n) = \sum_{\lambda \vdash n,\, l(\lambda) \leq h} f_\lambda\,, \tag{1}$$

as well as in the enumeration of the numbers

$$t_h^{(2)}(n) = \sum_{\lambda \vdash n,\, l(\lambda) \leq h} f_\lambda^2\,. \tag{2}$$

For the cases $h = 2,3,4,5$ (see Regev [**3**] and Gouyou-Beauchamps [**1**]), nice explicit expressions have been given for the $t_h(n)$'s or their generating function. We also should mention at this point that Zeilberger has shown in [**4**] that the $t_h(n)$'s are P-recursive, this is to say that they satisfy a recurrence of the form

$$\sum_{k=0}^{m} p_k(n)\, t_h(n-k) = 0\,, \tag{3}$$

for some polynomials $p_k(n)$ and some integer m. The same kind of result holds also for the $t_h(n)^{(2)}$'s. However Zeilberger's argument gives no clear indication on the bounds for m or the respective degrees of the $p_k(n)$'s. We propose, in this note, explicit values for the degree of the polynomials appearing in (3) as well as for the value of m.

2. Conjectures for $t_h(n)$

Using the first values of the numbers $t_h(n)$ for small h's, and an undetermined coefficient method, we looked for simple P-recurrences for these numbers. The surprising outcome of these experiments was that these recurrences were of relatively low degree. A careful study of the first of these recurrences led us to the conjectures that are given below. We then predicted the form of the recurrences for larger h's using these conjectures and further computations showed that these conjectured recurrences agreed with those obtained by the previous undetermined coefficient method.

Let us first introduce the new sequence $T_h(n)$ defined by

$$T_h(n) = \frac{t_h(n)}{n!} = \sum_{\lambda \vdash n,\, l(\lambda) \leq h} \frac{1}{\prod_c h_c}\,.$$

We use in fact these numbers $T_h(n)$ in order to state our conjectures, rather than the sequence $t_h(n)$ itself. As we will see below, the nature of the recurrences conjectured for $T_h(n)$ will allow easily to conjecture equivalent P-recurrences for $t_h(n)$.

(1a) *The numbers $T_h(n)$ satisfy a recurrence of the form*

$$\sum_{k=0}^{\lfloor h/2 \rfloor + 1} P_k(n)\, T_h(n-k) = 0\,, \tag{4}$$

with polynomials $P_k(n)$ each of degree $\leq \lfloor h/2 \rfloor - k + 1$.

(1b) *The coefficient of $T_h(n)$ in (4) is*

$$P_0(n) = n \prod_{k=1}^{\lfloor h/2 \rfloor} (n + k(h-k))\,.$$

(1c) *For odd $h = 2m+1$, the coefficient of $T_h(n-1)$ in (4) is*

$$P_1(n) = P_0(n-1) - P_0(n)$$

and the leading coefficient of $P_k(n)$ is the coefficient of z^k in the polynomial

$$\prod_{j=0}^{m} (1 - (-1)^{(m-j)}(2j+1)z)\,.$$

Also the degree of $P_k(n)$ is exactly $m - k + 1$.

It is easy to see that if the above conjectures are true, then the sequence $t_h(n)$ satisfy to the following P-recurrence :

$$\sum_{k=0}^{\lfloor h/2 \rfloor + 1} p_k(n)\, T_h(n-k) = 0\,, \tag{5}$$

where we have

$$p_0(n) = \prod_{k=1}^{\lfloor h/2 \rfloor} (n + k(h-k)) \quad \text{and} \quad p_k(n) = P_k(n) \prod_{i=1}^{k-1} (n-k)$$

for every $1 \leq k \leq \lfloor h/2 \rfloor + 1$. We now have the following conjecture concerning the $P_k(n)$'s :

(1d) *The polynomials $P_k(n)$ are such that the recurrence (5) is true with the unique initial condition $t_h(0) = 1$.*

Using these conjectures and an indeterminate coefficients method, we obtain the following recurrences for $h = 1, 3, 5, 7$ (the case $h = 1$ is trivial).

$$n\,T_1(n) = T_1(n-1)$$

$$n\,(n+2)\,T_3(n) = (2\,n+1)\,T_3(n-1) + 3\,T_3(n-2)$$

$$n\,(n+4)\,(n+6)\,T_5(n) = (3\,n^2 + 17\,n + 15)\,T_5(n-1) + (13\,n+9)\,T_5(n-2) - 15\,T_5(n-3)$$

$$n\,(n+6)\,(n+10)\,(n+12)\,T_7(n) = (4\,n^3 + 78\,n^2 + 424\,n + 495)\,T_7(n-1) + (34\,n^2 + 280\,n + 305)\,T_7(n-2) - (76\,n + 290)\,T_7(n-3) - 105\,T_7(n-4)\,.$$

And for $h = 2, 4, 6$,

$$n\,(n+1)\,T_2(n) = 2\,T_2(n-1) - 4\,T_2(n-2)$$

$$n\,(n+3)\,(n+4)\,T_4(n) = 4\,(3+2\,n)\,T_4(n-1) + 16\,n\,T_4(n-2)$$

$$n\,(n+5)\,(n+8)\,(n+9)\,T_6(n) = 4\,(84 + 46\,n + 5\,n^2)\,t_6(n-1) + 4\,(10\,n^2 + 58\,n + 33)\,T_6(n-2) - 144\,T_6(n-3) - 144\,T_6(n-4)\,.$$

Recurrences for bigger h's are easy to obtain in the same manner. But, the computation time gets to be quite large for $h \simeq 20$. We have checked that these recurrences are consistent with explicit computation (using (1) and (5)) of the $t_h(n)$ as far as reasonable computation time allowed ($n \simeq 40$). Moreover, for very large values of n ($n \simeq 2000$), the values of $t_h(n)$, obtained through (4) and (5), are strikingly consistent with the asymptotic expression given by Regev in [**3**] which is

$$cte\,\frac{h^n}{n^{h(h-1)/4}}\,.$$

In fact, a simple translation of (5) in term of a differential equation for the generating function $y(x)$ of the numbers $t_h(n)$, gives for instance the following for $h = 7$,

$$\begin{aligned}
&(1-7\,x)\,(1+5\,x)\,(1-3\,x)\,(1+x)\,x^3\,\frac{d^3}{dx^3}\,y(x) \\
&\quad + \left(-552\,x^2 + 974\,x^3 - 102\,x + 31 + 945\,x^4\right)\,x^2\,\frac{d^2}{dx^2}\,y(x) \\
&\quad + \left(1890\,x^4 - 1901\,x^2 + 2528\,x^3 - 686\,x + 281\right)\,x\,\frac{d}{dx}\,y(x) \\
&\quad + \left(720 - 1001\,x - 1001\,x^2 + 630\,x^4 + 1036\,x^3\right)\,y(x) = 720
\end{aligned}$$

From this differential equation we easily find the (regular) singularity of smallest module of $y(x)$ since it is a root of the dominating polynomial. Using $y(x) \sim (1/7 - x)^r$, we solve for r and find $r = 19/2$. Thus the the coefficient of z^n in $y(x)$ is asymptotically equivalent to

$$cte\,\frac{7^n}{n^{21/2}},$$

since the asymptotic behavior of the coefficients of $(1-hx)^r$ is $cte\,h^n/n^{r+1}$.

For odd h ($h = 2m+1$), we have also obtained the following candidate for the generating function $\sum_m c_m z^m$ of coefficients c_m of n^{m-2} in the polynomial $P_2(n)$ of (4) (of degree $m-1$)

(1e) *One has the generating function*

$$\sum_m c_m\, z^m = \frac{9\,x^3 + 217\,x^4 + 91\,x^5 + 3\,x^6}{(1-x)^7}$$

Recall that conjecture (1c) implicitly gives the coefficient of n^{m-1} in these polynomials. It appears that similar generating functions can be found for all coefficients of the $P_k(n)$'s. The $P_2(n)$ for $h = 3, 5, 7, 9$ are the polynomials given below :

$$3$$

$$13\,n + 9$$

$$34\,n^2 + 280\,n + 305$$

$$70\,n^3 + 1862\,n^2 + 13433\,n + 18991\,.$$

3. Conjectures for $t_h^{(2)}(n)$

We introduce as in the previous section the sequence $T_h^{(2)}(n)$ defined by

$$T_h^{(2)}(n) = \frac{t_h^{(2)}(n)}{n!^2}$$

and we will state our conjectures with this new sequence rather than with $t_h^{(2)}(n)$.

(2a) *The numbers $T_h^{(2)}(n)$ satisfy a recurrence of the form*

$$\sum_{k=0}^{\lfloor h/2\rfloor+1} P_k^{(2)}(n)\,T_h^{(2)}(n-k) = 0, \qquad (6)$$

with polynomials $P_k^{(2)}(n)$ each of degree $\le h-2k$.

(2b) *The coefficient of* $T_h^{(2)}(n)$ *in (6) is*

$$P_0^{(2)}(n) = n^2 \prod_{k=1}^{\lfloor h/2 \rfloor} (n + k(h-k))^2 .$$

(2c) *When* $h = 2m+1$ *is odd, the leading coefficient of* $P_k^{(2)}(n)$ *is the coefficient of* z^k *in the polynomial*

$$\prod_{j=0}^{m} (1 - (2j+1)^2 z) .$$

All remarks that we have made about the $T_h(n)$'s also apply to the $T_h^{(2)}(n)$'s with the necessary modifications. In particular, it follows easily that if the above conjectures are true, then $t_h^{(2)}(n)$ satisfy to a P-recurrence of the form :

$$\sum_{k=0}^{\lfloor h/2 \rfloor + 1} p_k^{(2)}(n)\, t_h^{(2)}(n-k) = 0, \tag{7}$$

where the polynomials $p_k^{(2)}(n)$ are easily deduced from the $P_k^{(2)}(n)$ using the same method than in the previous section. We can now state our last conjecture :

(1d) *The polynomials* $P_k^{(2)}(n)$ *of (6) are such that the recurrence (7) is true with the unique initial condition* $t_h^{(2)}(0) = 1$.

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