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Federico Della Croce 1  Bruno Escoffier 2  Vangelis Th. Paschos 2  

1 D.A.I., Politecnico di Torino, Italy, federico.dellacroce@polito.it  
2 LAMSADE, CNRS UMR 7024 and Université Paris-Dauphine, France  
{escoffier,paschos}@lamsade.dauphine.fr  

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Abstract

We consider MIN SET COVERING when the subsets are constrained to have maximum cardinality three. We propose an exact algorithm whose worst case complexity is bounded above by \(O^{*}(1.3957^n)\). This is an improvement, based on a refined analysis, of a former result (\(O^{*}(1.4492^n)\)) by F. Della Croce and V. Th. Paschos, Computing optimal solutions for the MIN 3-SET COVERING problem, Proc. ISAAC’05, LNCS 3827, pp. 685–692.

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In MIN SET COVERING, we are given a universe \(U\) of elements and a collection \(S\) of (non-empty) subsets of \(U\). The aim is to determine a minimum cardinality sub-collection \(S' \subseteq S\) which covers \(U\), i.e., \(\cup_{S \in S'} S = U\) (we assume that \(S\) covers \(U\)). The frequency \(f_i\) of \(u_i \in U\) is the number of subsets \(S_j \in S\) in which \(u_i\) is contained. The cardinality \(d_j\) of \(S_j \in S\) is the number of elements \(u_i \in U\) that \(S_j\) contains. We say that \(S_j\) hits \(S_k\) if both \(S_j\) and \(S_k\) contain an element \(u_i\) and that \(S_j\) double-hits \(S_k\) if both \(S_j\) and \(S_k\) contain at least two elements \(u_i, u_l\).

Indeed, if item 1 is not verified, then the set containing \(u_i\) belongs to any feasible cover of \(U\). On the other hand, if item 2 is not verified, then \(S_i\) can be replaced by \(S_j\) in any solution containing \(S_i\) and the resulting cover will not be worse than the one containing \(S_i\). Finally, if item 3 is not verified, then element \(u_j\) can be ignored as any sub-collection \(S'\) covering \(u_i\) will necessarily cover also \(u_j\). So, for any instance of MIN SET COVERING, a preprocessing of data, obviously performed in polynomial time, leads to instances where all items 1, 2 and 3 are verified.

Let \(T(\cdot)\) be a super-polynomial and \(p(\cdot)\) be a polynomial, both on integers. In what follows, using notations in [9], for an integer \(n\), we express running-time bounds of the form \(p(n)T(n)\) as \(O^{*}(T(n))\), the asterisk meaning that we ignore polynomial factors. We denote by \(T(n)\) the

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worst case time required to exactly solve the MIN SET COVERING problem with \( n \) subsets. We recall (see, for instance, [5]) that, if it is possible to bound above \( T(n) \) by a recurrence expression of the type \( T(n) \leq \sum T(n-r_i) + O(p(n)) \), we have \( T(n) = O^*(\alpha(r_1, r_2, \ldots)^n) \) where \( \alpha(r_1, r_2, \ldots) \) is the largest zero of the function \( f(x) = 1 - \sum x^{-r_i} \).

There exist to our knowledge few results on worst-case complexity of exact algorithms for MIN SET COVERING or for cardinality-constrained versions of it. Let us note that an exhaustive algorithm computes any solution for MIN SET COVERING in \( O(2^n) \). For MIN SET COVERING the most recent non-trivial result is the one of [6] (that has improved the result of [8]) deriving a bound (requiring exponential space) of \( O^*(1.2301^{(m+n)}) \). We consider here, the most notorious cardinality-constrained version of MIN SET COVERING, the MIN 3-SET COVERING, namely, MIN SET COVERING where \( d_j \leq 3 \) for all \( S_j \in S \) (notice that the bound of [6], for the case where \( f_i = 2 \), \( u_i \in U \), and \( d_j = 3 \), for any \( S_j \in S \) corresponds to \( O^*(1.2301^{(5n/2)}) \approx O^*(1.6782^n) \)). It is well known that MIN 3-SET COVERING is \( \text{NP}-\text{hard} \), while MIN 2-SET COVERING (where any set has cardinality at most 2) is polynomially solvable by matching techniques ([2, 7]).

Our purpose is to devise an exact (optimal) algorithm with provably improved worst-case complexity for MIN 3-SET COVERING. We propose a search tree-based algorithm with running time \( O^*(1.3957^n) \). This result, largely inspired by the one of [4], further improves it by reducing the complexity of the tree-based algorithm from \( O^*(1.4492^n) \) down to \( O^*(1.3957^n) \). This outcome is due to a different complexity analysis of the algorithm by the introduction of a kind of weights on the fixed sets. This technique seems to be quite close to the one very recently introduced in [6].

The following straightforward lemma holds, inducing some useful domination conditions for the solutions of MIN SET COVERING.

**Lemma 1.** There exists at least one optimal solution of MIN SET COVERING where:

1. for any subset \( S_j \) with \( d_j = 2 \) containing elements \( u_i, u_p \), if \( S_j \) is included in \( S' \), then all subsets \( S_k \) hitting \( S_j \) are excluded from \( S' \);

2. for any subset \( S_j \) with \( d_j = 3 \) containing elements \( u_i, u_p, u_q \), where \( S_j \) double-hits another subset \( S_k \) with \( d_k = 3 \) on \( u_i \) and \( u_p \), if \( S_j \) is included in \( S' \) then \( S_k \) must be excluded from \( S' \) and viceversa;

3. for any subset \( S_j \) with \( d_j = 3 \) containing elements \( u_i, u_p, u_q \), if \( S_j \) is included in \( S' \), then either all subsets \( S_k \) hitting \( S_j \) on element \( u_i \) are excluded from \( S' \), or all subsets \( S_k \) hitting \( S_j \) on elements \( u_p \) and \( u_q \) are excluded from \( S' \).

**Proof.** We only prove item 1, items 2 and 3 being proved by the same kind of analysis. Assume, without loss of generality, that \( S_j \) hits \( S_k \) on \( u_i \) and \( S_l \) on \( u_p \). Suppose by contradiction that the optimal solution \( S' \) includes \( S_j \) and \( S_k \). Then, it cannot include no more \( S_l \), or else, it would not be optimal as a better cover would be obtained by excluding \( S_j \) from \( S' \). On the other hand, suppose that \( S' \) includes \( S_j, S_k \) but does not include \( S_l \). Then, an equivalent optimal solution can be derived by swapping \( S_j \) with \( S_l \).

In what follows, we consider the following counting. When we fix the status of a set of size 3, then our benefit is 1. When we do not fix a set of size 3 but cover one element of this set (hence this set will have size 2 is the remaining instance), we consider that our benefit is \( \alpha \leq 1 \). Obviously, when a set of size 2 is fixed, we can only consider that (in the worst case) our benefit is \( 1 - \alpha \). Hence, in some cases, the benefit is increasing with \( \alpha \) while, in other cases, it is decreasing. An optimal value for \( \alpha \) following our analysis, is \( \alpha = 0.297 \).

The rest of the paper is devoted to the proof of the following result.

**Theorem 1.** MIN 3-SET COVERING can be optimally solved within time \( O^*(1.396^n) \).
The algorithm either reduces the min 3-set covering instance according to assumptions 1, 2 and 3 on the form of the instance (by detecting a subset \( S_j \) to be immediately included in (excluded from) \( S' \) or an element \( u_i \) to be ignored (correspondingly reducing the size of several subsets), or applies a branching on subset \( S_j \), where the following exhaustive relevant branching cases may occur.

1. \( d_j = 2 \): then no double-hitting occurs to \( S_j \) or else, due to the preprocessing step of the algorithm, \( S_j \) can be excluded from \( S' \) without branching. The following subcases occur.

   (a) \( S_j \) contains elements \( u_i, u_k \) with \( f_i = f_k = 2 \) where \( S_j \) hits \( S_l \) on \( u_i \) and \( S_m \) on \( u_k \).
   
   Due to Lemma 1, if \( S_j \) is included in \( S' \), then both \( S_l \) and \( S_m \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \) and, correspondingly, both \( S_l \) and \( S_m \) must be included in \( S' \) to cover elements \( u_i, u_k \). For the analysis, consider the two following cases.

   i. \( d_l = 3 \), or \( d_m = 3 \), say \( d_l = 3 \). Then, in both cases (including or excluding \( S_j \)) we fix \( 3 - 2\alpha \) (1 for \( S_l \), (at least) \( 1 - \alpha \) for \( S_j \) and \( S_m \)).

   ii. \( d_l = d_m = 2 \), \( S_l \) contains \( u_i \) and \( u_l \) and \( S_m \) contains \( u_k \) and \( u_m \), (with \( u_l \neq u_m \), otherwise no need to branch). By including \( S_j \) we fix \( 3(1 - \alpha) \). Otherwise, \( u_l \) is contained in \( S_p \) and \( u_m \) in \( S_q \). If \( S_p \neq S_q \), then we fix at least \( 3(1 - \alpha) + 2\alpha = 3 - \alpha \). Indeed, we fix \( 1 - \alpha \) for any of the sets \( S_j, S_l, S_m \); by covering \( u_m \), we fix \( \alpha \) (resp., \( 1 - \alpha \geq \alpha \)) if \( d_p = 3 \) (resp., if \( d_p = 2 \), since we can exclude \( S_p \)), and the same holds for covering \( u_k \). Note that this is still valid if \( S_p = S_q \), since in this case we can exclude this set, which gives at least \( 1 - \alpha \geq 2\alpha \).

   In case 1(a)i, we have \( T(n) \leq 2T(n - 3 + 2\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.334^n) \). In case 1(a)ii, we have \( T(n) \leq T(n - 3 + 3\alpha) + T(n - 3 + \alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.336^n) \).

   (b) \( S_j \) contains elements \( u_i, u_k \) with \( f_i = 2 \) and \( f_k \geq 3 \), where \( S_j \) hits \( S_l \) on \( u_i \) and \( S_m, S_p \) on \( u_k \). Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \) and, correspondingly, \( S_l \) must be included in \( S' \) to cover element \( u_i \). For the analysis, consider the two following cases.

   i. \( d_l \geq 3 \), i.e., \( S_l \) contains \( u_i, u_l \); then, \( f_l \geq 3 \) (or else we are in case 1a). Then, by including \( S_j \), we fix \( 4(1 - \alpha) \) (\( (1 - \alpha) \) for any of the sets \( S_j, S_l, S_m, S_p \)); by excluding \( S_j \), we fix \( 2(1 - \alpha) + 2\alpha = 2 \) \( (1 - \alpha) \) for any of the sets \( S_j, S_l \), and (at least) \( \alpha \) for each set containing \( u_l \).

   ii. If \( d_l \geq 3 \), i.e., \( S_l \) contains at least \( u_i, u_l, u_m \), then by including \( S_j \), we fix \( 3(1 - \alpha) + 1 \) (since now fixing \( S_l \) gives benefit 1); by excluding \( S_j \), we fix \( (1 - \alpha) + 1 + 2\alpha = 2 + \alpha \) (\( \alpha \) from covering \( u_l \), and \( \alpha \) from covering \( u_m \), with the same reasoning as in case 1(a)ii).

   The worst case is 1(b)i where we get \( T(n) \leq T(n - 2) + T(n - 4 + 4\alpha) + O(p(n)) \), resulting in a time-complexity of \( O^*(1.338^n) \).

   (c) \( S_j \) contains elements \( u_i, u_k \) with \( f_i = 3 \) and \( f_k \geq 3 \) where \( S_j \) hits \( S_l \), \( S_m \) on \( u_i \) and (at least) \( S_p, S_q \) on \( u_k \). Note that we can suppose that \( S_j \) hits at least one set of size 3. Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p, S_q \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). For the analysis, consider the three following cases.

   i. If \( d_l = d_m = d_p = d_q = 3 \), then we fix either \( 5 - \alpha \), or \( 1 - \alpha \).
ii. If \( d_l = 2 \) or \( d_m = 2 \), say \( d_l = 2 \), then we fix either \( 5 - 4\alpha \), or \( 1 - \alpha \). But in the case where we exclude \( S_j \) from \( S' \), then \( S_l \) has size 2 and contains \( u_i \), whose frequency is now 2. Hence, we are either in case 1a or in case 1b. In the worst case, the branching gives (with case 1(b)) \( 5 - 4\alpha \), \( 5(1 - \alpha) \) and \( 3 - \alpha \).

iii. Finally, if \( d_l = d_m = 3 \), then we can suppose that \( f_k \geq 4 \) (otherwise we are either in case 1(c)i or in case 1(c)ii). In this case, by including \( S_j \) we fix \( 2 + 4(1 - \alpha) \) and by excluding \( S_j \) we fix \( 1 - \alpha \).

In case 1(c)i, we get \( T(n) \leq T(n - 1 + \alpha) + T(n - 5 + \alpha) + O(p(n)) \), i.e., a time-complexity of \( O^*(1.3953^n) \). In case 1(c)ii, we get \( T(n) \leq T(n - 3 + \alpha) + T(n - 5 + 5\alpha) + T(n - 5 + 4\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3942^n) \). In case 1(c)iii, we get \( T(n) \leq T(n - 6 + 4\alpha) + T(n - 1 + \alpha) + O(p(n)) \), i.e., a time-complexity of \( O^*(1.389^n) \).

(d) \( S_j \) contains elements \( u_i, u_k \) with \( f_i \geq 4 \) and \( f_k \geq 4 \) where \( S_j \) hits \( S, S_m, S_p \) on \( u_i \) and \( S_q, S_r, S_s \) on \( u_k \). Note that we can suppose that \( S_j \) hits at least one set of size 3. Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_l, S_m, S_p, S_q, S_r, S_s \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). Then, we fix either \( 7 - 6\alpha \) or \( 1 - \alpha \) getting \( T(n) \leq T(n - 1 + \alpha) + T(n - 7 + 6\alpha) + O(p(n)) \), resulting so in a time-complexity of \( O^*(1.366^n) \).

2. \( d_j = 3 \) (that is, there does not exist \( S_k \in S \) such that \( d_k = 2 \)) and there is at least one element \( u_i \) with \( f_i = 2 \). Then, \( S_j \) contains \( u_i, u_j, u_k \), and \( S_k \) contains \( u_i, u_j, u_m \) (notice that no double crossing can occur between \( S_j \) and \( S_k \) due to the preprocessing step of the algorithm). Then, either we include \( S_j \), and we fix \( 1 + 3\alpha \) new sets, or we exclude \( S_j \), and we have to include \( S_k \) fixing so \( 2 + 2\alpha \) new sets. In this case, we get \( T(n) \leq T(n - 1 - 3\alpha) + T(n - 2 - 2\alpha) + O(p(n)) \). This results in a time-complexity of \( O^*(1.366^n) \).

3. \( d_j = 3 \), all elements have a frequency at least 3, with \( S_j \) double-hitting one or more subsets. The following exhaustive subcases may occur.

(a) \( S_j \) double-hits at least three subsets \( S_k, S_l, S_m \). Due to Lemma 1, if \( S_j \) is included in \( S' \), then \( S_k, S_l, S_m \) must be excluded from \( S' \); alternatively, \( S_j \) is excluded from \( S' \). This can be seen as a binary branching where either one subset \( (S_j) \) is fixed, or four subsets \( (S_j, S_k, S_l, S_m) \) are fixed and hence, \( T(n) \leq T(n - 1) + T(n - 4) + O(p(n)) \). This results in a time-complexity of \( O^*(1.3803^n) \).

(b) \( S_j \) double-hits two subsets \( S_k, S_l \). Note that the double-hit elements must be contained by another set. Note also that (at least) one element, say \( u_i \), is in \( S_j, S_k \) and \( S_l \). Consider the two following cases.

i. If \( f_i \geq 4 \), then either we include \( S_j \) and then, by Lemma 1, we can exclude \( S_k \) and \( S_l \), or we exclude \( S_j \). Then, either we fix \( 3 + 3\alpha \) (3 for \( S_j, S_k, S_l \), and \( 3\alpha \) since \( u_i, u_j \) and \( u_k \) belong to at least one other set) or 1.

ii. If \( f_i = 3 \), then we must include at least one set among \( S_j, S_k, S_l \), but we can suppose that we do not include two such sets. In other words, we have a branching on the three following choices:

- taking \( S_j \) (and not \( S_k, S_l \)),
- taking \( S_k \) (and not \( S_j, S_l \)),
- taking \( S_l \) (and not \( S_j, S_k \)).

In any case, we fix \( 3 + 2\alpha \) (\( 2\alpha \) since each element has a frequency at least 3)
In the first case, $T(n) \leq T(n-1) + T(n-3 - 3\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.388^n)$. In the second case, $T(n) \leq 3T(n-3-2\alpha) + O(p(n))$, and this results in a time-complexity of $O^*(1.358^n)$.

(c) $S_j$ contains elements $u_i, u_k, u_l$ and double-hits one subset $S_k$ on elements $u_i, u_k$. The following exhaustive subcases must be considered.

i. $f_i = 3, f_k \geq 3, f_l \geq 3$, with $u_i$ contained by $S_j, S_k, S_m, u_k$ contained at least by $S_j, S_k, S_p$ and $u_l$ contained at least by $S_j, S_q, S_r$. A composite branching can be devised.
   - Suppose that $S_j$ is included in $S'$ and then $S_k$ is excluded from $S'$. In this case, we fix $2 + 4\alpha$ ($\alpha$ from reduction of the sizes of $S_m, S_p, S_q, S_r$).
   - Suppose that $S_j$ is included from $S'$ and $S_k$ is included in $S'$. In this case, we fix $2 + 4\alpha$ (since no other double hit occurs on $S_k$).
   - Suppose finally that $S_j$ and $S_k$ are excluded from $S'$. In this case, we have to exclude $S_m$ in $S'$. Since $d_m = 3$, all elements have frequency at least 3, and at most one double crossing occurs on $S_m$; we can see that $S_m$ hits at least three new sets. Hence, we fix $3 + 3\alpha$.

ii. $f_i \geq 4, f_k \geq 4, f_l \geq 3$, with $u_i$ contained at least by $S_j, S_k, S_m, S_p, u_k$ contained at least by $S_j, S_k, S_q, S_r$ and $u_l$ contained at least by $S_j, S_u, S_w$. Either we include $S_j$ in $S'$, and then we can exclude $S_k$ from $S'$ and fix $2 + 6\alpha$, or we exclude $S_j$ and fix 1.

In case 3(c)i, we get $T(n) \leq 2T(n-2-4\alpha) + T(n-3 - 3\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.381^n)$. In case 3(c)ii, we get $T(n) \leq T(n-1) + T(n-2 - 6\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.3957^n)$.

4. $d_j = 3$ and no double-hitting occurs to $S_j$ (nor to any other subset) that contains elements $u_i, u_k, u_l$. The following subcases occur.

(a) $f_i = 3, f_k \geq 3, f_l \geq 3$ with $u_i$ contained by $S_j, S_k, S_l, u_k$ contained by $S_j, S_m, S_p$ and $u_l$ contained at least by $S_j, S_q, S_r$. A composite branching can be devised:
   - if $S_j$ is included in $S'$, then we fix $1 + 6\alpha$ new sets;
   - if $S_j$ is excluded from $S'$ and $S_k$ is included in $S'$, then there exist at least five other subsets hitting $S_k$ and hence we fix $2 + 5\alpha$;
   - finally, if $S_j, S_k$ are excluded from $S'$, then we have to include $S_l$ in $S'$ (in order to cover $u_l$); there exist at least four other subsets hitting $S_l$ and hence we fix $3 + 4\alpha$.

Thus, $T(n) \leq T(n-1 - 6\alpha) + T(n-2 - 5\alpha) + T(n-3 - 4\alpha) + O(p(n))$, resulting in a time-complexity of $O^*(1.378^n)$.

(b) $f_i \geq 4, f_k \geq 4, f_l \geq 4$, $u_i$ is contained by $S_j, S_k, S_l, S_m, u_k$ is contained by $S_j, S_p, S_q, S_r$ and $u_l$ is contained at least by $S_j, S_t, S_u, S_w$. A composite branching on $S_j$ can be devised:
   - if $S_j$ is excluded from $S'$, then we fix 1;
   - if $S_j$ is included in $S'$, then $S_k, S_l, S_m$ are excluded from $S'$; in this case we fix $4 + 6\alpha$;
   - finally, if $S_j$ is included in $S'$, then $S_p, S_q, S_r, S_t, S_u, S_w$ are excluded from $S'$; in this case we fix $7 + 3\alpha$.

Hence, $T(n) \leq T(n-1) + T(n-4 - 6\alpha) + T(n-7 - 3\alpha) + O(p(n))$. This results in a time-complexity of $O^*(1.355^n)$.
Putting things together, the global worst case complexity is $O^*(1.3957^n)$ and the proof of the theorem is complete.

As a last word, let us note that a straightforward (improvable) analysis along the lines of Theorem 1, leads to an $O^*(1.1679^n)$ time bound for minimum vertex covering in graphs of maximum size 3. Such a bound is the best-known dealing with search tree-based algorithms and is only dominated by the bounds in [1, 3], ($O^*(1.1252^n)$ and $O^*(1.152^n)$, respectively) that are not based upon such algorithms. Note also, dealing with minimum dominating set in graphs of maximum size 3, analysis along the same lines reaches $O^*(1.344^n)$, which is always the best-known search-tree complexity.

References