Approximation results for the weighted P4 partition problem

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Abstract

We present several new standard and differential approximation results for the P_4-partition problem using the Hassin and Rubinstein' algorithm (Information Processing Letters, 63: 63-67, 1997). Those results concern both minimization and maximization versions of the problem. However, the main point of this paper lies on the establishment it does of the robustness of this algorithm, in the sense that this latter provides good quality solutions, whatever version of the problem is addressed, whatever approximation framework is considered in order to evaluate the approximate solutions.

Key words: graph partitioning, P_4-packing, approximation algorithms, performance ratio, standard approximation, differential approximation.

1 Introduction

Consider an instance I of an **NP**-hard optimization problem Π and a polynomial-time algorithm `A` that computes feasible solutions for Π. Denote respectively by $\mathrm{apx}_\Pi(I)$ the value of a solution computed by `A` on I, by $\mathrm{opt}_\Pi(I)$ the value of an optimum solution and by $\mathrm{wor}_\Pi(I)$ the value of a worst solution (that corresponds to the optimum value when reversing the optimization goal). The quality of `A` is expressed by the means of approximation ra-

⋆ A preleminary version of this paper appeared in the proceedings the 15th International Symposium on Fundamentals of Computation Theory, FCT 2005, LNCS 3623, (2005) 377-385 tios that somehow compare the approximate value to the optimum one. So far, two measures stand out from the literature: the *standard* ratio [2] (the most widely used) and the *differential* ratio [3,4,7,10]. The standard ratio is defined by $\rho_\Pi(I,\mathtt{A}) = \mathrm{apx}_\Pi(I)/\mathrm{opt}_\Pi(I)$ if Π is a maximization problem, by $\rho_\Pi(I,\mathtt{A}) = \mathrm{opt}_\Pi(I)/\mathrm{apx}_\Pi(I)$ otherwise, whereas the differential ratio is defined by $\delta_\Pi(I,\mathtt{A})(\mathrm{wor}_\Pi(I) - \mathrm{apx}_\Pi(I))/(\mathrm{wor}_\Pi(I) - \mathrm{opt}_\Pi(I))$. Instead of dividing the approximate value by the optimum one, this latter measure divides the distance from a worst solution to the approximate value by the instance diameter. Within the worst case analysis framework and given a universal constant $\varepsilon \leq 1$ (*resp.*, $\varepsilon \geq 1$), an algorithm A is said to be an ε-standard approximation for a maximization (*resp.* a minimization) problem Π if $\rho_{\mathtt{A}_\Pi}(I) \geq \varepsilon$ $\forall I$ (*resp.*, $\rho_{\mathtt{A}_\Pi}(I) \leq \varepsilon$ $\forall I$). With respect to differential approximation, A is said to be ε-differential approximate for Π if $\delta_{\mathtt{A}_\Pi}(I) \geq \varepsilon$, $\forall I$, for a universal constant $\varepsilon \leq 1$. Equivalently, because any solution value is a convex combination of the two values $\mathrm{wor}_\Pi(I)$ and $\mathrm{opt}_\Pi(I)$, an approximate solution value $apx_\Pi(I)$ will be an ε-differential approximation if for any instance I, $apx_\Pi(I) \geq \varepsilon \times \mathrm{opt}_\Pi(I) + (1-\varepsilon) \times \mathrm{wor}_\Pi(I)$ (for the maximization case; reverse the sense of the inequality when minimizing). Within the worst case analysis framework and considering both standard and differential ratios, we focus on a special problem, the weighted P_4-partition problem. Furthermore, we study the performance of a single algorithm on various versions of this problem. Doing so, we put to the fore the effectiveness of this algorithm by proving that it provides approximation ratios for both standard and differential measures, for both maximization and minimization versions of the problem.

In the weighted P_k-partition problem ($\mathrm{P}_k\mathrm{P}$ in short), we are given a complete graph K_{kn} together with a distance function $d : E \to \mathbb{N}$ on its edges. A P_k is an induced path of length $k-1$ (or, equivalently, an induced path on k vertices) and the cost of such a path is the sum of its edge weight. Given an instance $I = (K_{kn}, d)$, the aim is to compute a partition $T^* = \{P_1^*, \ldots, P_n^*\}$ of $V(K_{kn})$ into n vertex-disjoint P_k (what we call a P_k-partition) that is of optimum weight (that is, of maximum weight if the goal is to maximize ($\textsc{Max}\mathrm{P}_k\mathrm{P}$), of minimum weight otherwise ($\textsc{Min}\mathrm{P}_k\mathrm{P}$), where the value of a solution T^* is given by $d(T^*) = \sum_{i=1}^{q} d(P_i^*)$. When considering the minimization version, we will more often assume that the distance function satisfies the triangular inequality, *i.e.*, $d(x,y) \leq d(x,z) + d(z,y), \forall x,y,z$; $\textsc{MinMetric}\mathrm{P}_k\mathrm{P}$ will refer to this restriction. Finally, we also deal with a special case of metric instances where the distance function is worth either 1 or 2; the corresponding problems will be denoted by $\textsc{Max}\mathrm{P}_k\mathrm{P}_{1,2}$ and $\textsc{Min}\mathrm{P}_k\mathrm{P}_{1,2}$. Note that for $k = 2$, a P_2-partition is a perfect matching and hence, $\textsc{Min}\mathrm{P}_2\mathrm{P}$ and $\textsc{Max}\mathrm{P}_2\mathrm{P}$ both are polynomial. On the other hand, all these problems turn to be **NP**-hard for $k \geq 3$, [9,16]. Nevertheless, $\textsc{Max}\mathrm{P}_k\mathrm{P}$ is standard-approximable for any k, [11]. In particular, $\textsc{Max}\mathrm{P}_3\mathrm{P}$ and $\textsc{Max}\mathrm{P}_4\mathrm{P}$ are respectively $35/67-\varepsilon$, [12] and $3/4$, [11] approximable. On the other hand, $\textsc{Min}\mathrm{P}_k\mathrm{P}$ may not be approximated within $2^{p(n)}$ for any polynomial p, for any k; this is due to the fact that the P_k-partition problem, which consists in deciding whether a graph does or not admit a partition of its vertex set into P_k, is **NP**-complete, [9,15,16]. Furthermore, even when restricting to metric instances and more specifically for $k = 4$, no approximation rate has (to our knowledge) been established for MinMetricP_kP so far. Finally, note that this latter problem (and P_kP in general) is very close to the vehicle routing problem when restricting the route of each vehicle to at most k intermediate stops, [1,8].

In the second section, we study the relationship between TSP and P_kP under differential ratio; namely, we show how a differential approximation for TSP enables a differential approximation for P_kP. In the third section, that contains the main result of this paper, we propose a complete analysis, from both a standard and a differential point of view, of an algorithm proposed by Hassin and Rubinstein [11]. We prove that, with respect to the standard ratio, this algorithm provides new approximation rates for MetricP_4P, namely: the approximate solution respectively achieves a 3/2-, a 7/6- and a 9/10-standard approximation for MinMetricP_4P, Min$\mathrm{P}_4\mathrm{P}_{1,2}$ and Max$\mathrm{P}_4\mathrm{P}_{1,2}$. Under differential ratio, the approximate solution is a 1/2-approximation for general P_4P, a 2/3-approximation for $\mathrm{P}_4\mathrm{P}_{a,b}$. The gap between differential and standard ratios that might be reached for a maximization problem may be explained by the fact that, within the differential framework, the approximate value has to be located within the interval $[\mathrm{wor}(I), \mathrm{opt}(I)]$, instead of $[0, \mathrm{opt}(I)]$ when considering the standard measure. That is the aim of differential approximation: thanks to the reference it does to $\mathrm{wor}(I)$, this measure is both more precise (relevant with respect to the notion of guaranteed performance) and more robust (since minimizing and maximizing turn to be equivalent and more generally, differential ratio is stable under affine transformation of the objective function). In addition to the new approximation bounds that they provide, the obtained results enable to establish the robustness of the algorithm that is addressed here, since this latter provides good quality solutions, whatever version of the problem we deal with, whatever approximation framework within which we estimate the approximate solutions.

2 From Traveling salesman problem to P_kP

A common technic in order to obtain an approximate solution for MaxP_kP from a Hamiltonian cycle is called the *deleting and turning around* method, see [11,12,8]. Starting from a tour, this method builds k solutions of MaxP_kP and picks the best among them, where the ith solution is obtained by deleting 1 edge upon k from the input cycle, starting from its ith edge. The quality of the output T' obviously depends on the quality of the initial tour; in this way it is proven in [11,12], that any ε-standard approximation for MaxTSP provides a $\frac{k-1}{k}\varepsilon$-standard approximation for MaxP_kP. From a differential point of

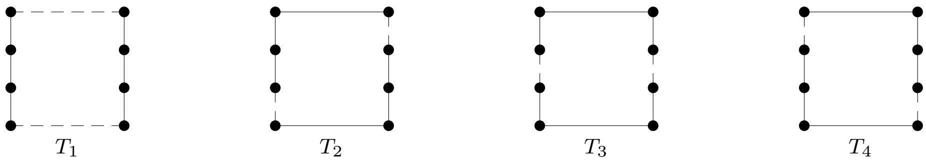

Fig. 1. An example of the 4 solutions $T_1, \ldots, T_4$.

Fig. 2. A worst and an optimum solutions when $n = 1$.

view, things are less optimistic: even for $k = 4$, there exists an instance family $(I_n)_{n \geq 1}$ that verifies $\text{apx}(I_n) = \frac{1}{2}\text{opt}_{\text{MaxP}_4\text{P}}(I_n) + \frac{1}{2}\text{wor}_{\text{MaxP}_4\text{P}}(I_n)$. This instance family is defined as $I_n = (K_{8n}, d)$ for $n \geq 1$, where the vertex set $V(K_{8n})$ may be partitioned into two sets $L = \{\ell_1, \ldots, \ell_{4n}\}$ and $R = \{r_1, \ldots, r_{4n}\}$ in such a way that the associated distance function d is worth 0 on $L \times L$, 2 on $R \times R$ and 1 on $L \times R$. Thus, for any $n \geq 1$, the following property holds:

Property 1 $\text{apx}(I_n) = 6n$*,* $\text{opt}_{\text{MaxP}_4\text{P}}(I_n) = 8n$*,* $\text{wor}_{\text{MaxP}_4\text{P}}(I_n) = 4n$*.*

If the initial tour is described as $\Gamma = \{e_1, \ldots, e_n, e_1\}$, then the *deleting and turning around* method produces 4 solutions $T_1, \ldots, T_4$ where $T_i = \cup_{j=0}^{n-1}\{\{e_{j+i}, e_{j+i+1}, e_{j+i+2}\}\}$ for $i = 1, \ldots, 4$ (indexes are considered mod n). Figure 1 provides an illustration of this process (the dashed lines correspond to the edges from $\Gamma \setminus T_i$).

First, observe that any tour Γ on I_n is optimum, of total weight $8n$. Indeed, any tour contains as many edges with their two endpoints in L as edges with their two endpoints in R and thus, $d(\Gamma) = |\Gamma \cap L \times R| + 2|\Gamma \cap R \times R| = |\Gamma| = 8n$. Hence, starting from the optimum cycle $\Gamma^* = [r_1, \ldots, r_{4n}, l_1, \ldots, l_{4n}, r_1]$, the four solutions $T_1, \ldots, T_4$ outputted by the algorithm (see Figure 1) will all be worth $d(T_i) = 6n$, while an optimum solution T^* and a worst solution T_* are of total weight respectively $8n$ and $4n$ (see Figure 2). Indeed, because any P_4-partition T is a $2n$ edge cut down tour, we get, on the one hand, $\text{opt}_{\text{MaxTSP}}(I_n) \geq d(T)$ and, on the other hand, $d(T) \geq 8n - 4n = 4n$, which concludes this argument.

Nevertheless, the deleting and turning around method leads to the following weaker differential approximation relation:

Lemma 2 *¿From an ε-differential approximation of* MaxTSP*, one can polynomially compute a $\frac{\varepsilon}{k}$-differential approximation of* MaxP_kP*. In particular,* *we deduce from [10,13] that* MAXP_kP *is* $\frac{2}{3k}$*-differential approximable.*

Let us show that the following inequality holds for any instance $I = (K_{kn}, d)$ of MAXP_kP:

$$\text{opt}_{\text{MAXTSP}}(I) \geq \frac{1}{k-1}\text{opt}_{\text{MAXP}_k\text{P}}(I) + \text{wor}_{\text{MAXP}_k\text{P}}(I) \tag{1}$$

Let T^* be an optimum solution of MAXP_kP, then arbitrarily add some edges to T^* in order to obtain a tour Γ. From this latter, we can deduce $k-1$ solutions T_i for $i = 1, \ldots, k-1$, by applying the deleting and turning around method in such a way that any of the solutions T_i contains $(\Gamma \setminus T^*)$. Thus, we get $(k-1)\text{wor}_{\text{MAXP}_k\text{P}}(I) \leq \sum_{i=1}^{k-1} d(T_i) = (k-1)d(\Gamma) - \text{opt}_{\text{MAXP}_k\text{P}}(I)$. Hence, consider that $d(\Gamma) \leq \text{opt}_{\text{MAXTSP}}(I)$ and the result follows. By applying again the deleting and turning around method, but this time from a worst tour, we may obtain k approximate solutions of MAXP_kP, which allows us to deduce:

$$\text{wor}_{\text{MAXTSP}}(I) \geq \frac{k}{k-1}\text{wor}_{\text{MAXP}_k\text{P}}(I) \tag{2}$$

Finally, let Γ' be an ε-differential approximation of MAXTSP, we deduce from Γ' k approximate solutions of MAXP_kP. If T' is set to the best one, we get $d(T') \geq \frac{k}{k-1}d(\Gamma')$ and thus:

$$\text{apx}(I) \geq \frac{k}{k-1}d(\Gamma') \geq \frac{k}{k-1}(\varepsilon\text{opt}_{\text{MAXTSP}}(I) + (1-\varepsilon)\text{wor}_{\text{MAXTSP}}(I)) \tag{3}$$

Using inequalities (1), (2) and (3), we get $\text{apx}(I) \geq \frac{\varepsilon}{k}\text{opt}_{\text{MAXP}_k\text{P}}(I) + (1 - \frac{\varepsilon}{k})\text{wor}_{\text{MAXP}_k\text{P}}(I)$ and the proof is complete.

To conclude with the relationship between P_kP and TSP with respect to their approximability, observe that the minimization case also is trickier. Notably, if we consider $\text{MINMETRICP}_4\text{P}$, then the instance family $I'_n = (K_{8n}, d')$ built as the same as I_n with a distinct distance function defined as $d'(\ell_i, \ell_j) = d'(r_i, r_j) = 1$ and $d'(\ell_i, r_j) = n^2 + 1$ for any i, j, then we have: $\text{opt}_{\text{TSP}}(I'_n) = 2n^2 + 8n$ and $\text{opt}_{\text{P}_4\text{P}}(I'_n) = 6n$.

3 Approximating P_4P by the means of optimum matchings

Here starts the analysis, from both a standard and a differential point of view, of an algorithm proposed by Hassin and Rubinstein in [11], where the authors show that the approximate solution is a 3/4-standard approximation for MaxP_4P. First, dealing with the standard ratio, we prove that this algorithm provides a 3/2-approximation for $\text{MinMetricP}_4\text{P}$ and respectively a 7/6 and a 9/10-approximation for $\text{MinP}_4\text{P}_{1,2}$ and $\text{MaxP}_4\text{P}_{1,2}$. As a corollary of a more general result, we also obtain an alternative proof of the result of [11]. We then prove that, with respect to the differential measure, the approximate solution achieves a 1/2-approximation in general graphs, for both maximization and minimization versions of the problem. Finally, this latter ratio is raised up to 2/3 when restricting to bi-valuated graphs.

3.1 Description of the algorithm

The algorithm proposed in [11] runs in two stages: first, it computes an optimum weight perfect matching $M_{T'}$ on (K_{4n}, d); then, it builds on the edges of $M_{T'}$ a second optimum weight perfect matching $R_{T'}$ in order to complete the solution (note that "optimum weight" signifies "*maximum weight*" if the goal is to maximize, "*minimum weight*" if the goal is to minimize). Precisely, we define the instance (K_{2n}, d') (to any edge $e_v \in M_{T'}$ corresponds a vertex v in K_{2n}), where the distance function d' is defined as follows: for any edge $[v_1, v_2]$, $d'(v_1, v_2)$ is set to the weight of the heaviest edge that links e_{v_1} and e_{v_2}, that is, if v_1 represents $e_{v_1} = [x_1, y_1]$ and v_2 represents $e_{v_2} = [x_2, y_2]$, then $d'(v_1, v_2) = \max\{d(x_1, x_2), d(x_1, y_2), d(y_1, x_2), d(y_1, y_2)\}$ (when dealing with the minimum version of the problem, set the weight to the lightest). We thus build on (K_{2n}, d') an optimum weight matching $R_{T'}$, which is then transposed to the initial graph (K_{4n}, d) by selecting the edge that realizes the same cost. Since the computation of an optimum weight perfect matching is polynomial, the whole algorithm runs in polynomial time, whether the goal is to minimize or to maximize.

3.2 General P_4P *within the standard framework*

For any solution T, we denote respectively by M_T and R_T the set of the final edges and the set of the middle edges of its chains. Furthermore, we will consider for any chain $P_T = \{x, y, z, t\}$ of the solution the edge $[t, x]$ that completes P_T into a cycle. If $\overline{R_T}$ denotes the set of these edges, we observe that $R_T \cup \overline{R_T}$ forms a perfect matching. Finally, for any edge $e \in T$, we will denote by $P_T(e)$ the P_4 from the solution that contains e and by $C_T(e)$ the 4-length cycle that contains $P_T(e)$.

Lemma 3 *For any instance $I = (K_{4n}, d)$, if T is a feasible solution and T^* is an optimum solution, then there exist 4 pairwise disjoint edge sets A, B, C and D that verify:*

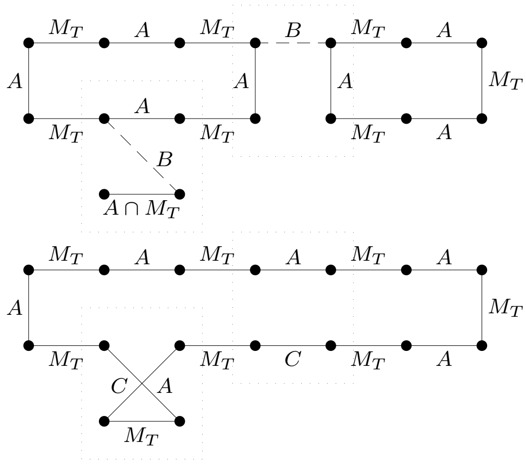

Fig. 3. The construction of sets A and C.

(i) $A \cup B = T^*$ *and* $C \cup D = \overline{R_{T^*}}$.
(ii) $A \cup C$ *and* $B \cup D$ *both are perfect matchings on* I.
(iii) $A \cup C \cup M_T$ *is a perfect 2-matching on* I *of which cycles are of length a multiple of 4.*

Let $T^* = M_{T^*} \cup R_{T^*}$ be an optimum solution, we apply the following process:

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1 Set $A = M_{T^*}$, $B = R_{T^*}$, $C = \emptyset$, $D = \overline{R_{T^*}}$;
  Set $G' = (V, A \cup C \cup M_T)$ (consider the simple graph);
2 While there exists an edge $e \in R_{T^*}$ that links two connected components of G', do:
  2.1 move $C_{T^*}(e) \cap M_{T^*}$ from A to B;
    move $C_{T^*}(e) \cap R_{T^*}$ from B to A;
    move $C_{T^*}(e) \cap \overline{R}_{T^*}$ from D to C;
  2.2 $G' \leftarrow (V, A \cup C \cup M_T)$;
3 output A, B, C and D;

At the initialization stage, the connected components of the partial graph induced by $(A \cup C \cup M_T)$ are either cycles that alternate edges from $(A \cup C)$ and M_T, or isolated edges from $M_{T^*} \cap M_T$. During step 2, at each iteration, the process merges together two connected components of G' into a single cycle; an illustration of the process is proposed in Figure 3. Note that all along the process, the sets A, B, C and D define a partition of $T^* \cup \overline{R_{T^*}}$ and thus, remain pairwise disjoint.

For (i): Immediate from definition of the process (edges from T^* are moved from A to B, from B to A, but never out of $A \cup B$; the same holds for $\overline{R_{T^*}}$

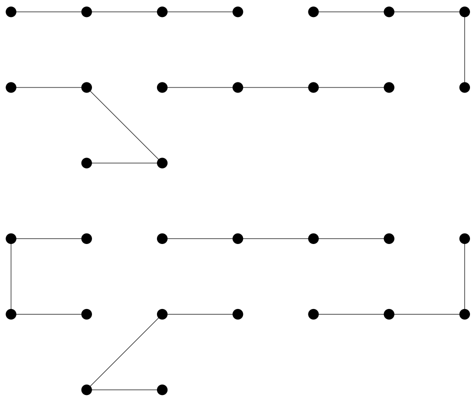

Fig. 4. Two possible P_4 partitions deduced from $A \cup C \cup M_{T'}$.

and the two sets C and D).

For (ii): At the initialization stage, $A \cup C$ and $B \cup D$ respectively coincides with M_{T^*} and $R_{T^*} \cup \overline{R}_{T^*}$, that both are perfect matchings. More precisely, for any chain P_{T^*} from the optimum solution, if C_{T^*} denotes the associated 4-length cycle, then $A \cup C$ and $B \cup D$ respectively contains the perfect matching $C_{T^*} \cap M_{T^*}$ and $C_{T^*} \cap (R_{T^*} \cup \overline{R}_{T^*})$ on $V(P_{T^*})$. Now, at each iteration, one just swaps the perfect matchings that are used in $A \cup C$ or $B \cup D$ in order to cover the vertices of a given chain P_{T^*} and thus, both $A \cup C$ and $B \cup D$ remain perfect matchings.

For (iii): At the end of the process, $(A \cup C) \cap M_T = \emptyset$ and thus, because $A \cup C$ and M_T both are perfect matchings, then $A \cup C \cup M_T$ is a perfect 2-matching. Now, consider a cycle Γ of $G' = (V, A \cup C \cup M_T)$; by definition of step 2, any edge e from R_{T^*} that is incident to Γ has its two endpoints in $V(\Gamma)$, which means that Γ contains whether the two edges of $C_{T^*}(e) \cap M_{T^*}$, or the two edges of $C_{T^*}(e) \cap (R_{T^*} \cup \overline{R}_{T^*})$. In other words, if any vertex u from any path $P_{T^*} \in T^*$ belongs to $V(\Gamma)$, then the whole vertex set $V(P_{T^*})$ actually is a subset of $V(\Gamma)$ and therefore, we deduce that $|V(\Gamma)| = 4k$.

Theorem 4 *The solution T' provided by the algorithm achieves a $\frac{3}{2}$-standard approximation for* MinMetricP$_4$P *and this ratio is tight.*

Let T^* be an optimum solution on $I = (K_{4n}, d)$, we consider 4 pairwise disjoint sets A, B, C and D in accordance with the application of Lemma 3 to the solution T'. According to property (iii), we can split $A \cup C$ into two sets A_1 and A_2 in such a way that $A_i \cup M_{T'}$ $(i = 1, 2)$ is a P_4-partition (see Figure 4 for an illustration). Hence, A_i constitutes an alternative solution for $R_{T'}$ and because this latter is optimum, we obtain:

$$2d(R_{T'}) \leq d(A) + d(C) \tag{4}$$

Moreover, item (ii) of Lemma 3 states that $B \cup D$ is a perfect matching; since $M_{T'}$ is optimum, it thus verifies:

$$d(M_{T'}) \leq d(B) + d(D) \tag{5}$$

Hence, it suffices to sum inequalities (4) and (5) (and also to consider item (i) of Lemma 3) in order to obtain:

$$d(M_{T'}) + 2d(R_{T'}) \leq d(T^*) + d(\overline{R}_{T^*}) \tag{6}$$

Now, because I satisfies the triangular inequality, we observe that $d(\overline{R}_{T^*}) \leq d(T^*)$ and thus deduce from inequality 6:

$$d(M_{T'}) + 2d(R_{T'}) \leq 2\mathrm{opt}_{\textsc{MinMetricP}_4\textsc{P}}(I) \tag{7}$$

(Note that this latter inequality is only true when minimizing.) Which enables to conclude, if we consider that $d(M_{T'}) \leq d(M_{T^*}) \leq d(T^*)$. Finally, the tightness is provided by the instance family $I_n = (K_{8n}, d)$ that has been described in Property 1.

Concerning the maximization case and using Lemma 3, one can also obtain an alternative proof of the result given in [11].

Theorem 5 *The solution T' provided by the algorithm achieves a $\frac{3}{4}$-standard approximation for* $\textsc{MaxP}_4\textsc{P}$.

The inequality (6) becomes

$$d(M_{T'}) + 2d(R_{T'}) \geq \mathrm{opt}_{\textsc{MaxP}_4\textsc{P}}(I) + d(\overline{R_{T^*}}) \tag{8}$$

Considering this time that $2 \times d(M_{T'}) \geq \mathrm{opt}_{\textsc{MaxP}_4\textsc{P}}(I) + d(\overline{R_{T^*}})$, we deduce $\mathrm{apx}_{\textsc{MaxP}_4\textsc{P}}(I) \geq \frac{3}{4}\left(\mathrm{opt}_{\textsc{MaxP}_4\textsc{P}}(I) + d(\overline{R_{T^*}})\right)$.

3.3 General $\mathrm{P}_4\mathrm{P}$ *within the differential framework*

When dealing with the differential ratio, $\textsc{MinP}_4\textsc{P}$, $\textsc{MinMetricP}_4\textsc{P}$, and $\textsc{MaxP}_4\textsc{P}$ are equivalent to approximate, since $\mathrm{P}_k\mathrm{P}$ problems belong to the class *FGNPO*, [14]. Note that such an equivalence is more generally true for any couple of problems that only differ by an affine transformation of their objective function.

Theorem 6 *The solution T' provided by the algorithm achieves a $\frac{1}{2}$-differential approximation for* $\mathrm{P_4P}$ *and this ratio is tight.*

We consider the maximization version. First, observe that $\overline{R_{T^*}}$ is a n-cardinality matching. Hence, for any perfect matching M of I such that $M \cup \overline{R_{T^*}}$ do form a P_4-partition, we have:

$$d(M) + d(\overline{R_{T^*}}) \geq \mathrm{wor}_{\mathrm{MaxP_4P}}(I) \tag{9}$$

Adding inequalities (8) and (9), we thus conclude:

$$2\mathrm{apx}_{\mathrm{MaxP_4P}}(I) \geq d(M_{T'}) + 2d(R_{T'}) + d(M) \geq \mathrm{wor}_{\mathrm{MaxP_4P}}(I) + \mathrm{opt}_{\mathrm{MaxP_4P}}(I)$$

In order to establish the tightness of this ratio, we refer to Property 1.

3.4 Bi-valuated metric $\mathrm{P_4P}$ *with weights 1 & 2 within the standard framework*

As it has been recently done for MinTSP in [5,6], we now focus on instances where any edge is worth either 1 or 2; indeed, such an analysis enables a keener comprehension of a given algorithm. Furthermore, because the $\mathrm{P_4}$-partition problem is **NP**-complete, the problems $\mathrm{MaxP_4P_{1,2}}$ and $\mathrm{MinP_4P_{1,2}}$ still are **NP**-hard.

Let us first introduce some more notations. For a given instance $I = (K_{4n}, d)$ of $\mathrm{P_4P_{1,2}}$ with $d(e) \in \{1, 2\}$, we denote by $M_{T',i}$ (*resp.*, $R_{T',i}$) the set of edges from $M_{T'}$ that are of weight i. If we aim at maximizing, then p (*resp.*, q) indicates the cardinality of $M_{T',2}$ (*resp.*, of $R_{T',2}$); otherwise, it indicates the quantity $|M_{T',1}|$ (*resp.*, $|R_{T',1}|$). In any case, p and q respectively count the number of "*good weight edges*" in the sets $M_{T'}$ and $R_{T'}$. With respect to the optimum solution, we define the sets $M_{T^*,i}$, $R_{T^*,i}$ for $i = 1, 2$ and the cardinalities p^*, q^* as the same.

Lemma 7 *For any instance $I = (K_{4n}, d)$, if T is a feasible solution and T^* is an optimum solution, then there an edge set A that verifies:*

(*i*) $A \subseteq M_{T^*,2} \cup R_{T^*,2}$ (resp., $A \subseteq M_{T^*,1} \cup R_{T^*,1}$) *and $|A| = q^*$ if the goal is to maximize* (resp., *to minimize*);
(*ii*) $G' = (V, M_{T'} \cup A)$ *is a simple graph made of pairwise disjoint chains.*

We only prove the maximization case. Wlog., we may assume that the following property always holds for T^*:

Property 8 *For any 3-length chain $P \in T^*$, $|P \cap M_{T^*,2}| \geq |P \cap R_{T^*,2}|$.*

Otherwise, T^* would contain a chain $P = \{[x,y],[y,z],[z,t]\}$ that verifies $d(x,y) = d(z,t) = 1$ and $d(y,z) = 2$; thus, by swapping P for $P' = \{[y,z],[z,t],[t,x]\}$ within T^*, one could generate an alternative optimum solution.

We now consider G' the multi-graph induced by $M_{T'} \cup R_{T^*,2}$ (the edges from $M_{T'} \cap R_{T^*,2}$ appear twice). This graph consists of elementary cycles and chains: its cycles alternate edges from $M_{T'}$ and $R_{T^*,2}$ (note that the 2-length cycles correspond to the edges from $R_{T^*,2} \cap M_{T'}$); its chains (that may be of length 1) also alternate edges from $M_{T'}$ and $R_{T^*,2}$, with the particularity that their terminal edges all belong to $M_{T'}$.

Let Γ be a cycle on G' and e be an edge from $\Gamma \cap R_{T^*,2}$. If $P_{T^*}(e) = \{x,y,z,t\}$ denotes the path from the optimum solution that contains e, then $e = [y,z]$. The initial vertex x of the chain $P_{T^*}(e)$ necessarily is the endpoint of some chain from G': otherwise, the edge $[x,y]$ from $P_{T^*}(e) \cap M_{T^*}$ would be incident to 2 distinct edges from R_{T^*}, which would contradicts the fact that T^* is a P_4 partition. The same obviously holds for t. W.l.o.g., we may assume from Property 8 $[x,y] \in M_{T^*,2}$. In the light of these remarks and in order to build an edge set A that fulfills the requirements (i) and (ii), we then proceed as follows:

---

1 Set $A = R_{T^*,2}$;
  Set $G' = (V, A \cup M_T)$ (consider the multi-graph);
2 While there exists a cycle Γ in G', do:
  2.1 pick e from $\Gamma \cap R_{T^*,2}$;
      pick f from $P_{T^*}(e) \cap M_{T^*,2}$;
      $A \leftarrow A \setminus \{e\} \cup \{f\}$;
  2.2 $G' \leftarrow (V, A \cup M_T)$;
3 output A;

By construction, the set A outputted by the algorithm is of cardinality q^* and contains exclusively edges of weight 2. Furthermore, thanks to the stopping criterion of the step 2, and because each iteration of this step merges a cycle and a chain into a chain, $G' = (V, A \cup M_T)$ is a simple graph of which connected components are elementary chains (an illustration of this step is provided by Figure 5). Finally, the validity of this process (namely, the existence of edge f at step 2.1) directly comes from the above discussion.

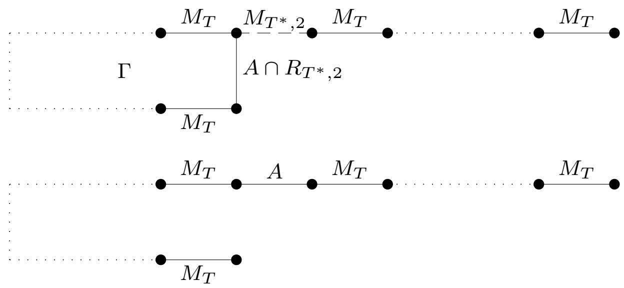

Fig. 5. The construction of set A.

Theorem 9 *The solution T' provided by the algorithm achieves a $\frac{9}{10}$-standard approximation for* $\textsc{MaxP}_4\textsc{P}_{1,2}$ *and a $\frac{7}{6}$-standard approximation for* $\textsc{MinP}_4\textsc{P}_{1,2}$. *These ratios are tight.*

Let consider A the edge subset of the optimum solution that may be deduced from the application of Lemma 7 to the approximate solution. We arbitrarily complete A by the means of an edge set B in such a way that $A \cup B \cup M_{T'}$ constitutes a perfect 2-matching. As we did while proving Theorem 4, we split the edge set $A \cup B$ into two sets A_1 and A_2 in order to obtain two P_4-partitions $M_{T'} \cup A_1$ and $M_{T'} \cup A_2$ of $V(K_{4n})$. As both $A_1 \cup B_1$ and $A_2 \cup B_2$ complete $M_{T'}$ into a P_4-partition and because $R_{T'}$ is optimum, we deduce that A_i does not contain more "*good weight edges*" than $R_{T'}$ does, that is: $q \geq |\{e \in A_i : d(e) = 2\}|$ if the goal is to maximize, $q \geq |\{e \in A_i : d(e) = 1\}|$ otherwise. Since $A \subseteq A_1 \cup A_2$ and $|A| = q^*$, we immediately deduce:

$$q \geq q^*/2 \tag{10}$$

On the other hand, the optimality of $M_{T'}$ leads to the following relation:

$$p \geq \max\{p^*, q^*\} \tag{11}$$

Moreover, the quantities p^* and q^* structurally verify:

$$n \geq \max\{p^*/2, q^*\} \tag{12}$$

Finally, whether the goal is to maximize or to minimize, we can express the value of any solution T as:

$$d(T) = \begin{cases} 3n + (p + q) & \text{when maximizing,} \\ 6n - (p + q) & \text{when minimizing.} \end{cases} \tag{13}$$

This expected results may now be obtained by the means of a little algebra

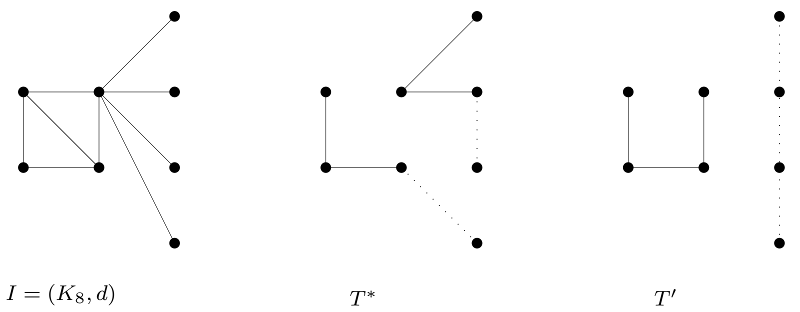

Fig. 6. Instance $I = (K_8, d)$ that establishes the tightness for MAXP$_4$P$_{1,2}$.

on relations (10), (11), (12) and (13):

$$\begin{aligned}
10\text{apx}_{\text{MaxP}_4\text{P}_{1,2}}(I) &= 10(3n + p + q) \\
&= 9(3n) + 3n + 9p + p + 10q \\
&\geq 9(3n) + 3q^* + 9p^* + q^* + 5q^* \\
&= 9(3n + p^* + q^*) = 9\text{opt}_{\text{MaxP}_4\text{P}_{1,2}}(I) \\
6\text{apx}_{\text{MinP}_4\text{P}_{1,2}}(I) &= 6(6n - p - q) \\
&= 6(6n) - 6p - 6q \\
&\leq 6(6n) - 6p^* - 3q^* + (2n - p^*) + (4n - 4q^*) \\
&\leq 7(6n - p^* - q^*) = 7\text{opt}_{\text{MinP}_4\text{P}_{1,2}}(I)
\end{aligned}$$

The tightness for MAXP$_4$P$_{1,2}$ is established thanks to the instance $I = (K_8, d)$ depicted in Figure 6, where the edges of distance 2 are drawn in continuous line, whereas the edges of distance 1 on T^* and T' are drawn in dotted line (other edges are not drawn). One can easily see $\text{opt}_{\text{MaxP}_4\text{P}_{1,2}}(I) = 10$ and $\text{apx}_{\text{MaxP}_4\text{P}_{1,2}}(I) = 9$. Concerning the minimization case, the ratio is tight on the instance $J = (K_8, d)$ that verifies: $\text{opt}(J) = d(T^*) = 6$ and $\text{apx}(J) = d(T') = 7$. $J = (K_8, d)$ is depicted in Figure 7 (the 1-weight edges are drawn in continuous line and the 2-weight edges on T^* and T' are drawn in dotted line).

3.5 *Bi-valuated metric* P$_4$P *with weights a and b within the differential framework*

As we have already mentioned, the differential measure is stable under affine transformation; now, any instance from MAXP$_4$P$_{a,b}$ may be mapped into an instance of MAXP$_4$P$_{1,2}$ or MINP$_4$P$_{a,b}$ by the way of such a transformation. Thus, proving MAXP$_4$P$_{1,2}$ is ε-differential approximable actually establishes

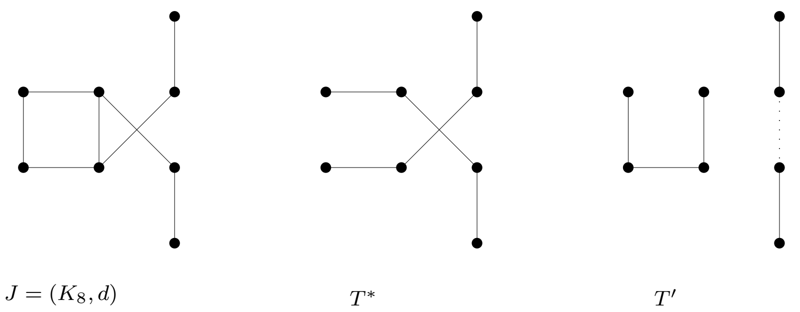

Fig. 7. Instance $I = (K_8, d)$ that establishes the tightness for MinP$_4$P$_{1,2}$.

that MinP$_4$P$_{a,b}$ and MaxP$_4$P$_{a,b}$ are ε-differential approximable for any couple of real values $a < b$.

Theorem 10 *The solution T' provided by the algorithm achieves a $\frac{2}{3}$-differential approximation for* P$_4$P$_{a,b}$ *and this ratio is tight.*

Let $I = (K_{4n}, d)$ be an instance of MaxP$_4$P$_{1,2}$. We use the notations that were introduced while proving Theorem 9, namely: $p = |M_{T',2}|$, $p^* = |M_{T^*,2}|$, $q = |R_{T',2}|$ and $q^* = |R_{T^*,2}|$. Furthermore, for $i = 1, 2$, $\mathcal{P}^i_{T'}$ will refer to the set of chains from T' of which central edge is of weight i. Note that the chains from $\mathcal{P}^1_{T'}$ may be of total weight 3, 4 or 5, whereas the chains from $\mathcal{P}^2_{T'}$ may be of total weight 5 or 6 (at least one extremal edge must be of weight 2, or $M_{T'}$ is not optimum). We will more specifically denote by $\mathcal{P}^2_{T',5}$ and $\mathcal{P}^2_{T',6}$ the chains from $\mathcal{P}^2_{T'}$ that are of total weight respectively 5 and 6. Finally, for $i = 1, 2$, $M^i_{T'}$ will refer to the set of edges $e \in M_{T'}$ such that $P_{T'}(e) \in \mathcal{P}^i_{T'}$ (that is, e is element of a chain from T' of which central edge has weight i). Thanks to relations (10) and (11), we first express some upper bounds for $\mathrm{opt}_{\mathrm{MaxP_4P_{1,2}}}(I)$:

$$\mathrm{opt}_{\mathrm{MaxP_4P_{1,2}}}(I) \leq \min\{3n + p + 2q, 3n + 2p\} \tag{14}$$

It order to obtain a differential approximation, one also has to produce an efficient bound for $\mathrm{wor}_{\mathrm{MaxP_4P_{1,2}}}(I)$. To do so, we will deduce from the optimality of $M_{T'}$ and $R_{T'}$ some edges of weight 1 that will enable us to approximate the worst solution. We first consider the vertices from $V(\mathcal{P}^1_{T'})$: they are "*easy*" to cover by the means of 3-length chains of total weight 3, since we may immediately deduce from the optimality of $R_{T'}$ the following property (an illustration is provided by Figure 8, where dotted lines indicate edges of weight 1 and dashed lines indicate unspecified weight edges):

Property 11 $[x, y] \neq [x', y'] \in M^1_{T'} \;\Rightarrow\; \forall e \in \{x, y\} \times \{x', y'\}\,,\; d(e) = 1$

We now consider the vertices from $V(\mathcal{P}^2_{T',5})$. Let $P_{T'} = \{x, y, z, t\}$ with $[x, y] \in M_{T',2}$ be a chain from $\mathcal{P}^2_{T',5}$, we deduce from the optimality of $M_{T'}$ that $d(t, x) = 1$; hence, the 3-length chain $P'_{T'} = \{y, z, t, x\}$ covers the vertices

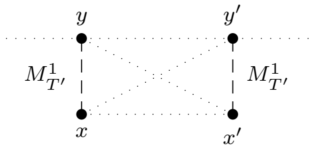

Fig. 8. 1-weight edges on $V(M^1_{T'})$.

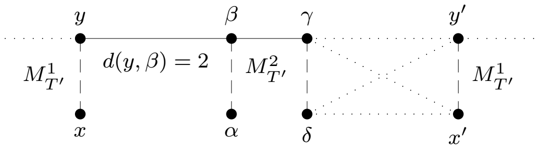

Fig. 9. 1-weight edges that may be deduced from the optimality of $R_{T'}$.

$\{x, y, z, t\}$ with a total cost 4. Let us assume that $\mathcal{P}^2_{T',6} = \emptyset$, then we are able to build a P$_4$ partition of $V(K_{4n})$ using exclusively edges of weight 1, but $|\mathcal{P}^2_{T',5}|$ edges of weight 2 in order to cover $V(\mathcal{P}^2_{T',5})$. Hence, a worst solution will cost at most $3n+q$, while the approximate solution is of total weight $3n+p+q$. Thus, using relation 14, we would be able to conclude. Of course, there is no reason for $\mathcal{P}^2_{T',6} = \emptyset$; nevertheless, this discussion has brought to the fore the following fact: the difficult point of the proof lies on the partitioning of $V(\mathcal{P}^2_{T',6})$ into "*light*" 3-length chains, what we are attempting to do by now.

We first stand two more properties that are immediate from the optimality of $M_{T'}$ and $R_{T'}$, respectively.

Property 12 $[x,y] \in M_{T',1}$ *and* $[x',y'] \in M_{T',2} \;\Rightarrow\; \min\{d(x,x'), d(y,y')\} = \min\{d(x,y'), d(y,x')\} = 1$

Property 13 *If* $[x,y] \neq [x',y'] \in M^1_{T'}$ *and* $P_{T'} = \{\alpha,\beta,\gamma,\delta\} \in \mathcal{P}^2_{T'}$*, then* $\max\{d(e)|e \in \{\alpha,\beta\} \times \{x,y\}\} = 2 \;\Rightarrow\; \max\{d(e)|e \in \{\gamma,\delta\} \times \{x',y'\}\} = 1$*. (See Figure 9 for an illustration, where continuous and dotted lines respectively indicate 2- and 1-weight edges, whereas dashed lines indicate unspecified weight edges).*

¿From Properties 12 and 13, we now are able to propose a "*light*" P$_4$ partition of $\mathcal{P}^2_{T',6}$.

Property 14 *Given a chain* $P_{T'} \in \mathcal{P}^2_{T',6}$ *and two edges* $[x,y] \neq [x'y'] \in M^1_T$*, then there exists a* P_4 *partition* $\mathcal{P} = \{P_1, P_2\}$ *of* $(V(P_{T'}) \cup \{x,y,x',y'\})$ *that is of total weight at most* 8*. Furthermore, if* $[x,y]$ *and* $[x',y']$ *both belong to* $M_{T',1}$*, then we can decrease this weight down to (at most)* 7 *(see Figure 10 for an illustration).*

Consider $P_{T'} = \{\alpha,\beta,\gamma,\delta\} \in \mathcal{P}^2_{T',6}$ and $[x,y] \neq [x'y'] \in M^1_T$. We set $P_1 = \{\alpha, x, x', \delta\}$ and $P_2 = \{\beta, y, y', \gamma\}$. If every edge from $\{\alpha,\beta,\gamma,\delta\} \times \{x,x',y,y'\}$ is of weight 1, then $P_1 \cup P_2$ has a total weight 6. Conversely, if there exists a 2-weight edge (assume that $[\beta, y]$ is such an edge), then $P_1 \cup P_2$ is of total

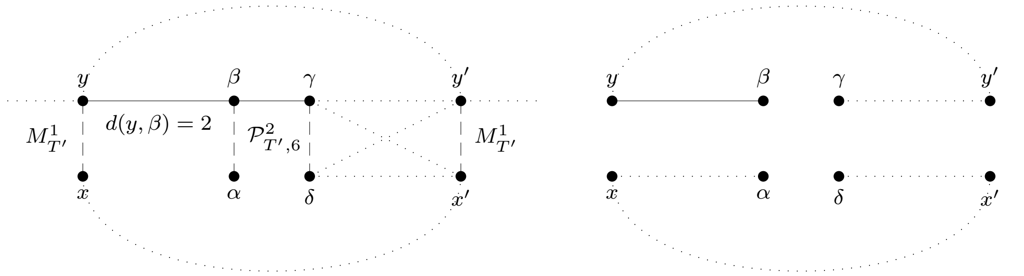

Fig. 10. A P_4 partition of $(P_{T'}, e_1, e_2) \in \mathcal{P}^2_{T',6} \times (M^1_{T'})^2$ of total weight at most 7.

weight at most 8: indeed, we get $d(x, x') = d(y, y') = 1$ from Property 11 and $d(\delta, x') = d(\gamma, y') = 1$ from Property 13. Furthermore, if $d(x, y) = 1$, then $d(\alpha, x) = 1$ from Property 12 and thus, $d(P_1) + d(P_2) = 7$. We now are able to compute an approximate worst solution that will provide an efficient upper bound for $\mathrm{wor}_{\textsc{MaxP}_4\textsc{P}_{1,2}}(I)$.

---

0 Set $T = T'$, $T_* = \emptyset$;
1 While $\exists \{P, e_1, e_2\} \subseteq T$ *s.t.* $(P, e_1, e_2) \in \mathcal{P}^2_{T',6} \times M^1_{T',1} \times M^1_{T',1}$
  1.1 compute $\mathcal{P} = \{P_1, P_2\}$ on $V(P) \cup V(e_1) \cup V(e_2)$ with $d(\mathcal{P}) \leq 7$;
  1.2 $T \leftarrow T \setminus \{P, e_1, e_2\}$, $T_* \leftarrow T_* \cup \{P_1, P_2\}$;
2 While $\exists \{P, e_1, e_2\} \subseteq T$ *s.t.* $(P, e_1, e_2) \in \mathcal{P}^2_{T',6} \times M^1_{T'} \times M^1_{T'}$
  2.1 compute $\mathcal{P} = \{P_1, P_2\}$ on $V(P) \cup V(e_1) \cup V(e_2)$ with $d(\mathcal{P}) \leq 8$;
  2.2 $T \leftarrow T \setminus \{P, e_1, e_2\}$, $T_* \leftarrow T_* \cup \{P_1, P_2\}$;
3 While $\exists P \subseteq T$ *s.t.* $P \in \mathcal{P}^2_{T',6}$
  3.1 $T \leftarrow T \setminus P$, $T_* \leftarrow T_* \cup \{P\}$;
4 While $\exists P \subseteq T$ *s.t.* $P \in \mathcal{P}^2_{T',5}$
  4.1 compute $\mathcal{P} = \{P_1\}$ on $V(P)$ with $d(\mathcal{P}) \leq 4$;
  4.2 $T \leftarrow T \setminus P$, $T_* \leftarrow T_* \cup \{P_1\}$;
5 While $\exists \{e_1, e_2\} \subseteq T$ *s.t.* $(e_1, e_2) \in M^1_{T'} \times M^1_{T'}$
  5.1 compute $\mathcal{P} = \{P_1\}$ on $V(e_1) \cup V(e_2)$ with $d(\mathcal{P}) = 3$;
  5.2 $T \leftarrow T \setminus e_1, e_2$, $T_* \leftarrow T_* \cup \{P_1\}$;
6 Output T_*;

In order to estimate the value of the approximate worst solution T_*, one has to count the number $p_* + q_*$ of 2-weight edges it contains. Let p^1_i refer to $|M^1_{T'} \cap M_{T',i}|$ for $i = 1, 2$ (the cardinality p^1_1 enables the expression of the number of iterations during step 1). Steps 1, 2 and 3 respectively put into T_* at most one, two and three 2-weight edges per iteration. Any chain from $\mathcal{P}^2_{T',6}$ is treated by one of the three steps 1 to 3. If $2|\mathcal{P}^2_{T',6}| \geq p^1_1$, only $|\mathcal{P}^2_{T',6}| - \lfloor p^1_1/2 \rfloor$ chains

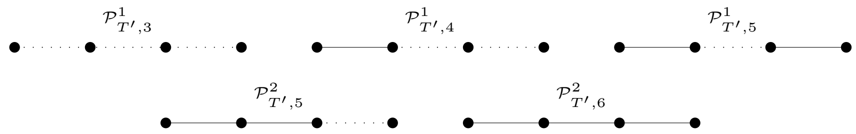

Fig. 11. A partition of T'.

from $\mathcal{P}^2_{T',6}$ are treated by one of the steps 2 and 3. Finally, if $|\mathcal{P}^2_{T',6}| \geq |\mathcal{P}^1_{T'}|$, only $|\mathcal{P}^2_{T',6}| - |\mathcal{P}^1_{T'}|$ chains from $\mathcal{P}^2_{T',6}$ are treated during step 3. Furthermore, step 4 puts at most $|\mathcal{P}^2_{T',5}|$ 2-weight edges into T_* (at most one per iteration), while steps 0 and 5 do not incorporate any 2-weight edges within T_*. Thus, considering $q = |\mathcal{P}^2_{T',5}| + |\mathcal{P}^2_{T',6}|$ and $|\mathcal{P}^1_{T'}| = n - q$, we obtain the following inequality (where expression $X^{(+)}$ is equivalent to $\max\{X, 0\}$):

$$p_* + q_* \leq q + (|\mathcal{P}^2_{T',6}| - \lfloor p^1_1/2 \rfloor)^{(+)} + (|\mathcal{P}^2_{T',6}| - n + q)^{(+)} \tag{15}$$

Let us introduce some more notations. Likewise $\mathcal{P}^2_{T'} = \mathcal{P}^2_{T',5} \cup \mathcal{P}^2_{T',6}$, we define a partition of $\mathcal{P}^1_{T'}$ into three subsets $\mathcal{P}^1_{T',3}$, $\mathcal{P}^1_{T',4}$ and $\mathcal{P}^1_{T',5}$ according to the chain total weight. Note that, since the subsets $\mathcal{P}^1_{T',j}$ define a partition of T', we have $n = |\mathcal{P}^1_{T',3}| + |\mathcal{P}^1_{T',4}| + |\mathcal{P}^1_{T',5}| + |\mathcal{P}^2_{T',5}| + |\mathcal{P}^2_{T',6}|$ (see Figure 11 for an illustration of this partition; the edges of distance 2 are drawn in continuous lines whereas the edges of distance 1 are drawn in dotted lines).

We now will establish the three following relations in order to compare the worst solution value to both the approximate solution and the optimum solution values:

$$p \geq q^* + (|\mathcal{P}^2_{T',6}| - \lfloor p^1_1/2 \rfloor)^{(+)} \tag{16}$$

$$2q \geq q^* + (|\mathcal{P}^2_{T',6}| + q - n)^{(+)} \tag{17}$$

$$q \geq p_* + q_* - (|\mathcal{P}^2_{T',6}| - \lfloor p^1_1/2 \rfloor)^{(+)} - (|\mathcal{P}^2_{T',6}| + q - n)^{(+)} \tag{18}$$

Actually, by summing inequalities 16 to 18, together with $2p \geq 2p^*$, we may obtain the expected result:

$$\begin{aligned} 3\mathrm{apx}_{\mathrm{MaxP_4P}}(I) &= 3(3n + p + q) \\ &\geq 2(3n + p^* + q^*) + (3n + p_* + q_*) \\ &= 2\mathrm{opt}_{\mathrm{MaxP_4P_{1,2}}}(I) + \mathrm{wor}_{\mathrm{MaxP_4P_{1,2}}}(I) \end{aligned}$$

Proof of inequality 16: Obvious if $2|\mathcal{P}^2_{T',6}| \leq p^1_1$, since $p \geq q^*$ (from inequality 11). Otherwise, one can write p as the sum $p = n + |\mathcal{P}^2_{T',6}| + |\mathcal{P}^1_{T',5}| - |\mathcal{P}^1_{T',3}|$. Now, $|\mathcal{P}^1_{T',5}| - |\mathcal{P}^1_{T',3}|$ is precisely the half of the difference between the number of 2-weight and of 1-weight edges in $M^1_{T'}$: since $p^1_2 = |\mathcal{P}^1_{T',4}| + 2|\mathcal{P}^1_{T',5}|$ and $p^1_1 = |\mathcal{P}^1_{T',4}| + 2|\mathcal{P}^1_{T',3}|$, then $p^1_2 - p^1_1 = 2(|\mathcal{P}^1_{T',5}| - |\mathcal{P}^1_{T',3}|)$. ¿From this latter equality, we deduce that p_1^1 and p_2^1 have the same parity; hence, we have $1/2(p_2^1 - p_1^1) = \lfloor p_2^1/2 \rfloor - \lfloor p_1^1/2 \rfloor$ and thus, $p = n + |\mathcal{P}^2_{T',6}| + \lfloor p_2^1/2 \rfloor - \lfloor p_1^1/2 \rfloor$. Just observe that n and q^* verify $n \geq q^*$ in order to conclude.

Proof of inequality 17: Obvious if $|\mathcal{P}^2_{T',6}| \leq n - q$, since $2q \geq q^*$ (from inequality 10). Otherwise, consider that q, n and $|\mathcal{P}^2_{T',6}|$ verify: $q \geq |\mathcal{P}^2_{T',6}|$ and $n \geq q^*$.

Proof of inequality 18: Immediate from Property 14.

The tightness is provided by the instance $I = (K_8, d)$ that is pictured on Figure 6; since this latter contains a vertex v such that any edge from v is of weight 2, the result follows.

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