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ON THE IRREDUCIBILITY OF DELIGNE-LUSZTIG VARIETIES

CÉDRIC BONNAFÉ & RAPHAËL ROUQUIER

Abstract. Let $G$ be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F : G → G$ be an endomorphism such that $F^δ$ is a Frobenius endomorphism for some $δ ≥ 1$. Let $P$ be a parabolic subgroup of $G$. We prove that the Deligne-Lusztig variety $\{gP \mid g^{-1}F(g) ∈ P \cdot F(P)\}$ is irreducible if and only if $P$ is not contained in a proper $F$-stable parabolic subgroup of $G$.

Let $G$ be a connected reductive group over an algebraic closure of a finite field and let $F : G → G$ be an endomorphism such that some power of $F$ is a Frobenius endomorphism of $G$. If $P$ is a parabolic subgroup of $G$, we set $X_P = \{gP ∈ G/P \mid g^{-1}F(g) ∈ P \cdot F(P)\}$. This is the Deligne-Lusztig variety associated to $P$. The aim of this note is to prove the following result:

**Theorem A.** Let $P$ be a parabolic subgroup of $G$. Then $X_P$ is irreducible if and only if $P$ is not contained in a proper $F$-stable parabolic subgroup of $G$.

Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [DiMi2, Proposition 8.4] in the case where $P$ is a Borel subgroup: both proofs are obtained by counting rational points of $X_P$ in terms of the Hecke algebra. We present here a geometric proof (inspired by an argument of Deligne [Lu, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne-Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [$L$, Proposition 4.8].

Before starting the proof of this Theorem, we first describe an equivalent statement. Let $B$ be an $F$-stable Borel subgroup of $G$, let $T$ be an $F$-stable maximal torus of $B$, let $W$ be the Weyl group of $G$ relative to $T$ and let $S$ be the set of simple reflections of $W$ with respect to $B$. We denote again by $F$ the automorphism of $W$ induced by $F$. Given $I ⊂ S$, let $W_I$ denote the standard parabolic subgroup of $W$ generated by $I$ and let $P_I = BW_I$. We denote by $P_I$ the variety of parabolic subgroups of $G$ of type $I$ (i.e. conjugate to $P_I$) and by $B$ the variety of Borel subgroups of $G$ (i.e. $B = P_∅$). For $w ∈ W$, we denote by $O_I(w)$ the $G$-orbit of $(P_I, wP_{F(I)})$ in $P_I × P_{F(I)}$. Note that $O_I(w)$ depends only on the double coset $W_IwW_{F(I)}$. We define now $X_I(w) = \{P ∈ P_I \mid (P, F(P)) ∈ O_I(w)\}$. The group $G^F$ acts on $X_I(w)$ by conjugation. We set $O(w) = O_{∅}(w)$ and $X(w) = X_{∅}(w)$.
Theorem A'. Let $I \subset S$ and let $w \in W$. Then $X_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$.

Remark 1 - Let us explain why the Theorems A and A' are equivalent. Let $P_0$ be a parabolic subgroup of $G$. Let $I$ be its type and let $g_0 \in G$ be such that $P_0 = g_0 P_I$. Let $w \in W$ be such that $g_0^{-1} F(g_0) \in P_I w P_{F(I)}$. The pair $(I, W_I w W_{F(I)})$ is uniquely determined by $P_0$. Then, the map $X_{P_0} \rightarrow X_I(w)$, $g P_0 \mapsto g g_0 P_I$ is an isomorphism of varieties (indeed, it is straightforward that $g^{-1} F(g) \in P_0 \cdot F(P_0)$ if and only if $(g g_0)^{-1} F(g g_0) \in P_I w P_{F(I)}$).

Let $Q$ be a parabolic subgroup of $G$ containing $P$. Let $J$ be its type. Then $I \subset J$, $Q = g_0 P_J$ and $g_0^{-1} F(g_0) \in P_J w P_{F(I)}$. Now, $Q$ is $F$-stable if and only if $F(J) = J$ and $w \in W_J$. This shows the equivalence of the two Theorems.

Remark 2 - The condition “$W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$” is equivalent to “$W_I w W_{F(I)}$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$”.

The rest of this paper is devoted to the proof of Theorem A’. We fix a subset $I$ of $S$ and an element $w$ of $W$. We first recall two elementary facts. If $I \subset J$, let $\tau_{IJ} : P_I \rightarrow P_J$ be the morphism of varieties that sends $P \in P_I$ to the unique parabolic subgroup of type $J$ containing $P$. It is surjective. Moreover,

\begin{equation}
\tau_{IJ}(X_I(w)) \subset X_J(w)
\end{equation}

and

\begin{equation}
\tau_{IJ}^{-1}(X_J(w)) = \bigcup_{W_I w W_{F(I)} \subset W_J w W_{F(I)}} X_I(x).
\end{equation}

First step: the “only if” part. Assume that there exists a proper $F$-stable subset $J$ of $S$ such that $W_I w \subset W_J$. Then, by [1], we have $\tau_{IJ}(X_I(w)) \subset X_J(1) = P_J^F$. Since $G^F$ acts transitively on $P_J^F$, we get $\tau_{IJ}(X_I(w)) = X_J(1)$. This shows that $X_I(w)$ is not irreducible.

Second step: reduction to Borel subgroups. By the previous step, we can concentrate on the “if” part. So, from now on, we assume that $W_I w$ is not contained in a proper $F$-stable parabolic subgroup of $W$. Then, by [2], we have

$$\tau_{IJ}^{-1}(X_I(w)) = \bigcup_{x \in W_I w W_{F(I)}} X(x).$$

Let $v$ denote the longest element of $W_I w W_{F(I)}$. Then every element $x$ of the double coset $W_I w W_{F(I)}$ satisfies $x \leq v$ (here, $\leq$ denotes the Bruhat order on $W$); this follows for instance from the fact that $P_I w P_{F(I)}$ is irreducible and is equal to $\bigcup_{x \in W_I w W_{F(I)}} B w B$. In particular, $v$ is not contained in a proper $F$-stable parabolic subgroup of $W$.

Now, let $X' = \bigcup_{x \in W_I w W_{F(I)}} X(x)$. Then, since $X(v) = \bigcup_{x \leq v} X(x)$, we have $X(v) \subset X' \subset \overline{X(v)}$. 


Let \( (\text{Third step: smooth compactification.} \) Let \((s_1, \ldots, s_n)\) be a finite sequence of elements of \(S\). Let
\[
\hat{X}(s_1, \ldots, s_n) = \{(B_1, \ldots, B_n) \in B^n \mid (B_n, F(B_1)) \in \mathcal{O}(s_n) \quad \text{and} \quad (B_i, B_{i+1}) \in \mathcal{O}(s_i) \text{ for } 1 \leq i \leq n-1\}.
\]
If \( \ell(s_1 \cdots s_n) = n \), then \(\hat{X}(s_1, \ldots, s_n)\) is a smooth compactification of \(X(s_1 \cdots s_n)\) (see [DeLi, Lemma 9.11]): in this case,
\[
\text{(3) \quad } X(s_1 \cdots s_n) \text{ is irreducible if and only if } \hat{X}(s_1, \ldots, s_n) \text{ is irreducible.}
\]
Note that \((B_1, \ldots, B) \in \hat{X}(s_1, \ldots, s_n)\). We denote by \(\hat{X}^\circ(s_1, \ldots, s_n)\) the connected (i.e. irreducible) component of \(\hat{X}(s_1, \ldots, s_n)\) containing \((B_1, \ldots, B)\). Let \(H(s_1, \ldots, s_n) \subset G^F\) be the stabilizer of \(\hat{X}^\circ(s_1, \ldots, s_n)\). Let us now prove the following fact:
\[
\text{(4) \quad } \text{if } 1 \leq i_1 < \cdots < i_r \leq n, \text{ then } H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n).
\]
\textbf{Proof of (4)} - The map \(f : \hat{X}(s_{i_1}, \ldots, s_{i_r}) \to \hat{X}(s_1, \ldots, s_n)\) defined by
\[
f(B_1, \ldots, B_1) = (B_{i_1}, \ldots, B_{i_r}, F(B_1), \ldots, F(B_1))
\]
is a \(G^F\)-equivariant morphism of varieties. Moreover,
\[
f(B_1, \ldots, B_r) = (B_1, \ldots, B_r),
\]
in particular, \(f(\hat{X}^\circ(s_{i_1}, \ldots, s_{i_r}))\) is contained in \(\hat{X}^\circ(s_1, \ldots, s_n)\). This proves the expected inclusion between stabilizers. \(\blacksquare\)

\textbf{Last step: twisted Coxeter element.} The quotient variety
\[
G^F \backslash \{g \in G \mid g^{-1}F(g) \in BwB\}
\]
is irreducible (it is isomorphic to \(BwB\) through the Lang map \(G^F g \mapsto g^{-1}F(g)\)), hence \(G^F \backslash X(w)\) is irreducible as well. So,
\[
\text{(5) \quad } G^F \text{ permutes transitively the irreducible components of } X(w).
\]
Let \(w = s_1 \cdots s_n\) be a reduced decomposition of \(W\) as a product of elements of \(S\). By \(\text{(3)}\) and \(\text{(5)}\), it suffices to show that \(H(s_1, \ldots, s_n) = G^F\). Since \(w\) does not belong to any \(F\)-stable proper parabolic subgroup of \(W\), there exists a sequence \(1 \leq i_1 < \cdots < i_r \leq n\) such that \((s_{i_k})_1 \leq k \leq r\) is a family of representatives of \(F\)-orbits in \(S\). By \(\text{(4)}\) we have \(H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n)\). But, by [Lu, Proposition 4.8], \(X(s_{i_1}, \ldots, s_{i_r})\) is irreducible so, again by \(\text{(3)}\) and \(\text{(5)}\), \(H(s_{i_1}, \ldots, s_{i_r}) = G^F\). Therefore, \(H(s_1, \ldots, s_n) = G^F\), as expected.
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Références


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