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ON THE IRREDUCIBILITY OF DELIGNE-LUSZTIG VARIETIES

CÉDRIC BONNAFÉ & RAPHAËL ROUQUIER

Abstract. Let $G$ be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F : G \to G$ be an endomorphism such that $F^\delta$ is a Frobenius endomorphism for some $\delta \geq 1$. Let $P$ be a parabolic subgroup of $G$. We prove that the Deligne-Lusztig variety $\{ gP | g^{-1}F(g) \in P \cdot F(P) \}$ is irreducible if and only if $P$ is not contained in a proper $F$-stable parabolic subgroup of $G$.

Let $G$ be a connected reductive group over an algebraic closure of a finite field and let $F : G \to G$ be an endomorphism such that some power of $F$ is a Frobenius endomorphism of $G$. If $P$ is a parabolic subgroup, we set $X_P = \{ gP \in G/P | g^{-1}F(g) \in P \cdot F(P) \}$. This is the Deligne-Lusztig variety associated to $P$. The aim of this note is to prove the following result:

**Theorem A.** Let $P$ be a parabolic subgroup of $G$. Then $X_P$ is irreducible if and only if $P$ is not contained in a proper $F$-stable parabolic subgroup of $G$.

Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [DiMi2, Proposition 8.4] in the case where $P$ is a Borel subgroup: both proofs are obtained by counting rational points of $X_P$ in terms of the Hecke algebra. We present here a geometric proof (inspired by an argument of Deligne [Lu, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne-Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [Lu, Proposition 4.8].

Before starting the proof of this Theorem, we first describe an equivalent statement. Let $B$ be an $F$-stable Borel subgroup of $G$, let $T$ be an $F$-stable maximal torus of $B$, let $W$ be the Weyl group of $G$ relative to $T$ and let $S$ be the set of simple reflections of $W$ with respect to $B$. We denote again by $F$ the automorphism of $W$ induced by $F$. Given $I \subset S$, let $W_I$ denote the standard parabolic subgroup of $W$ generated by $I$ and let $P_I = BW_IB$. We denote by $\mathcal{P}_I$ the variety of parabolic subgroups of $G$ of type $I$ (i.e. conjugate to $P_I$) and by $B$ the variety of Borel subgroups of $G$ (i.e. $B = \mathcal{P}_\emptyset$). For $w \in W$, we denote by $\mathcal{O}_I(w)$ the $G$-orbit of $(P_I, wP_{F(I)})$ in $\mathcal{P}_I \times \mathcal{P}_{F(I)}$. Note that $\mathcal{O}_I(w)$ depends only on the double coset $W_IwW_{F(I)}$. We define now

$$X_I(w) = \{ P \in \mathcal{P}_I | (P, F(P)) \in \mathcal{O}_I(w) \}.$$

The group $G^F$ acts on $X_I(w)$ by conjugation. We set $\mathcal{O}(w) = \mathcal{O}_\emptyset(w)$ and $X(w) = X_\emptyset(w)$.

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Theorem A’. Let $I \subset S$ and let $w \in W$. Then $X_I(w)$ is irreducible if and only if $W_Iw$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$.

Remark 1 - Let us explain why the Theorems A and A’ are equivalent. Let $P_0$ be a parabolic subgroup of $G$. Let $I$ be its type and let $g_0 \in G$ be such that $P_0 = g_0 P_I$. Let $w \in W$ be such that $g_0^{-1} g_0 \in P_I w P_{F(I)}$. The pair $(I, W_I w W_{F(I)})$ is uniquely determined by $P_0$. Then, the map $X_{P_0} \to X_I(w)$, $g P_0 \mapsto g g_0^{-1} P_I$ is an isomorphism of varieties (indeed, it is straightforward that $g^{-1} F(g) \in P_0 \cdot F(P_0)$ if and only if $(g g_0)^{-1} F(g g_0) \in P_I w P_{F(I)}$).

Let $Q$ be a parabolic subgroup of $G$ containing $P$. Let $J$ be its type. Then $I \subset J$, $Q = g_0 P_J$ and $g_0^{-1} g_0 \in P_J w P_{F(I)}$. Now, $Q$ is $F$-stable if and only if $F(J) = J$ and $w \in W_J$. This shows the equivalence of the two Theorems.

Remark 2 - The condition “$W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$” is equivalent to “$W_I w W_{F(I)}$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$”.

The rest of this paper is devoted to the proof of Theorem A’. We fix a subset $I$ of $S$ and an element $w$ of $W$. We first recall two elementary facts. If $I \subset J$, let $\tau_{IJ} : P_I \to P_J$ be the morphism of varieties that sends $P \in P_I$ to the unique parabolic subgroup of type $J$ containing $P$. It is surjective. Moreover,

$$\tau_{IJ}(X_I(w)) \subset X_J(w)$$

and

$$\tau_{IJ}^{-1}(X_J(w)) = \bigcup_{I \in \mathcal{P}_I \subset W_I w W_{F(I)}} X_J(x).$$

First step: the “only if” part. Assume that there exists a proper $F$-stable subset $J$ of $S$ such that $W_I w \subset W_J$. Then, by [1], we have $\tau_{IJ}(X_I(w)) \subset X_J(1) = \mathcal{P}^F_J$. Since $G^F$ acts transitively on $\mathcal{P}^F_J$, we get $\tau_{IJ}(X_I(w)) = X_J(1)$. This shows that $X_I(w)$ is not irreducible.

Second step: reduction to Borel subgroups. By the previous step, we can concentrate on the “if” part. So, from now on, we assume that $W_I w$ is not contained in a proper $F$-stable parabolic subgroup of $W$. Then, by [2], we have

$$\tau_{\emptyset I}^{-1}(X_I(w)) = \bigcup_{x \in W_I w W_{F(I)}} X(x).$$

Let $v$ denote the longest element of $W_I w W_{F(I)}$. Then every element $x$ of the double coset $W_I w W_{F(I)}$ satisfies $x \leq v$ (here, $\leq$ denotes the Bruhat order on $W$): this follows for instance from the fact that $P_I w P_{F(I)}$ is irreducible and is equal to $\bigcup_{x \in W_I w W_{F(I)}} B w B$. In particular, $v$ is not contained in a proper $F$-stable parabolic subgroup of $W$.

Now, let $X' = \bigcup_{x \in W_I w W_{F(I)}} X(x)$. Then, since $X(v) = \bigcup_{x \leq v} X(x)$, we have

$$X(v) \subset X' \subset X(v).$$
So, since $\tau_{\sigma I}(X') = X_I(w)$, it is enough to show that $X(v)$ is irreducible. In other words, we may, and we will, assume that $I = \emptyset$.

**Third step: smooth compactification.** Let $(s_1, \ldots, s_n)$ be a finite sequence of elements of $S$. Let

$$\hat{X}(s_1, \ldots, s_n) = \{(B_1, \ldots, B_n) \in B^n \mid (B_n, F(B_1)) \in \mathcal{O}(s_n) \text{ and } (B_i, B_{i+1}) \in \mathcal{O}(s_i) \text{ for } 1 \leq i \leq n-1\}.$$ 

If $\ell(s_1 \cdots s_n) = n$, then $\hat{X}(s_1, \ldots, s_n)$ is a smooth compactification of $X(s_1 \cdots s_n)$ (see [Del], Lemma 9.11]: in this case,

$$(3) \quad X(s_1 \cdots s_n) \text{ is irreducible if and only if } \hat{X}(s_1, \ldots, s_n) \text{ is irreducible.}$$

Note that $(B_1, \ldots, B) \in \hat{X}(s_1, \ldots, s_n)$. We denote by $\hat{X}^\circ(s_1, \ldots, s_n)$ the connected (i.e. irreducible) component of $\hat{X}(s_1, \ldots, s_n)$ containing $(B_1, \ldots, B)$. Let $H(s_1, \ldots, s_n) \subset G^F$ be the stabilizer of $\hat{X}^\circ(s_1, \ldots, s_n)$. Let us now prove the following fact:

$$(4) \quad \text{if } 1 \leq i_1 < \cdots < i_r \leq n, \text{ then } H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n).$$

**Proof of (4).** The map $f: \hat{X}(s_{i_1}, \ldots, s_{i_r}) \longrightarrow \hat{X}(s_1, \ldots, s_n)$ defined by

$$f(B_1, \ldots, B_1) = \begin{cases} (B_1, \ldots, & B_2, \ldots, B_{i_1-1}, B_{i_1}, \ldots, B_{i_r}, F(B_1), \ldots, F(B_{i_1})) \\ & \text{times} \end{cases} \begin{cases} \text{position} & \text{position} \\ t_1 & t_{i_1-1} & t_r \\ \text{times} & \text{position} \end{cases}$$

is a $G^F$-equivariant morphism of varieties. Moreover,

$$f(B_1, \ldots, B) = (B_1, \ldots, B).$$

In particular, $f(\hat{X}^\circ(s_{i_1}, \ldots, s_{i_r}))$ is contained in $\hat{X}^\circ(s_1, \ldots, s_n)$. This proves the expected inclusion between stabilizers. ■

**Last step: twisted Coxeter element.** The quotient variety

$$G^F \setminus \{g \in G \mid g^{-1}F(g) \in BwB\}$$

is irreducible (it is isomorphic to $BwB$ through the Lang map $G^F g \mapsto g^{-1}F(g)$), hence $G^F \setminus X(w)$ is irreducible as well. So,

$$(5) \quad G^F \text{ permutes transitively the irreducible components of } X(w).$$

Let $w = s_1 \cdots s_n$ be a reduced decomposition of $W$ as a product of elements of $S$. By (3) and (5), it suffices to show that $H(s_1, \ldots, s_n) = G^F$. Since $w$ does not belong to any $F$-stable proper parabolic subgroup of $W$, there exists a sequence $1 \leq i_1 < \cdots < i_r \leq n$ such that $(s_{i_k})_1 \leq k \leq r$ is a family of representatives of $F$-orbits in $S$. By (4) we have $H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n)$. But, by [Lu], Proposition 4.8], $X(s_{i_1}, \ldots, s_{i_r})$ is irreducible so, again by (3) and (5), $H(s_{i_1}, \ldots, s_{i_r}) = G^F$. Therefore, $H(s_1, \ldots, s_n) = G^F$, as expected.
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Références


Cédric Bonnafé: Laboratoire de Mathématiques de Besançon (CNRS: UMR 6623), Université de Franche-Comté, 16 Route de Gray, 25030 Besançon Cedex, France

E-mail address: bonnafe@math.univ-fcomte.fr

URL: http://www-math.univ-fcomte.fr/ppAnnu/CBONNAFE/


E-mail address: rouquier@maths.leeds.ac.uk

URL: http://www.maths.leeds.ac.uk/~rouquier/