Polynomial Ideals for Sandpiles and their Grobner Bases

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Abstract. A polynomial ideal encoding topplings in the abelian sandpile model on a graph is introduced. A Gröbner basis of this ideal is interpreted combinatorially in terms of well-connected subgraphs. This gives rise to algorithms to determine the identity and the operation in the group of recurrent configurations.

Introduction

The abelian sandpile model has been extensively considered since Bak, Tang and Wiesenfeld [1] introduced it in the context of self-organized critical phenomena in statistical physics.

This model can be described as a game on a graph, each configuration being a mapping of the vertices of the graph into the set of nonnegative integers. The value of the mapping at a vertex may be considered as the number of grains of sand on a sandpile placed at the vertex. The evolution is given by a toppling rule: "each vertex containing at least as many grains as it has neighbours distributes one grain to each of them." D. Dhar has considered the set of recurrent configurations, those that can be reached from any configuration by adding grains of sand and performing topplings.

Topplings can be represented by linear operators on the space of configurations. Dhar [8], using this linear algebraic setting, showed that the set of toppling operators has the structure of a finite abelian group. He also showed that the order of this group is equal to the number of spanning trees of the graph. Other approaches to this model can be found in [2, 3, 4, 5, 12]. Creutz [7] showed that this structure of abelian group carries over to the set of recurrent configurations themselves.

In this article, we associate a *toppling ideal* to a graph, encoding configurations with monomials and topplings with binomials. We show that Gröbner bases for these ideals can be interpreted (and computed) combinatorially. Moreover, we give a one-to-one mapping between recurrent configurations and monomials in the quotient of the polynomial algebra by the toppling ideal. This correspondence yields a combinatorial algorithm to compute the operation and the identity in the group of recurrent configurations.

In Section 1 we recall notation and useful results on the sandpile model and recurrent configurations. In Section 2 we define toppling polynomials and the toppling ideal and we give the dictionary for the translation between the linear algebra model for sandpiles and the model using polynomials. We show in Section 3 that the set of toppling polynomials constitutes a Gröbner basis for the ideal they generate, which we refine into a minimal basis. In the last section, we give the bijection

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between recurrent configurations and irreducible monomials, show how to compute the operation and the identity of the group of recurrent configurations and conclude with a few examples.

1. Abelian Sandpile Model

In this section the main definitions and results on the sandpile model are recalled. The main tool is linear algebra.

1.1. **Description.** Let $G = (V, E)$ be a non-oriented and connected *multi-graph* with $V = \{1, \dots, n\}$ its set of vertices and E a symmetric $n \times n$ matrix whose entry $e_{i,j}$ is the number of edges with endpoints i, j. It is assumed that for any i, $e_{i,i} = 0$ so that the multi-graph has no loops. Frequently, G is a graph, and hence $e_{i,j}$ is either 0 or 1. The *degree* of vertex i in G is $d_i := \sum_{j=1}^{n} e_{i,j}$. A multi-graph is *rooted* if one of its vertices is distinguished, it is called the *sink* and is numbered n.

A *configuration* $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ of G is a vector of non-negative integers. In the context of the sandpile model, the vertices of the graph are cells, and the number u_i may be interpreted as the height of a pile of grains of sand lying in cell i.

In the rest of this article, the number of grains in the sink is not taken into account, thus two configurations u and v which differ only in position n are considered as equal; we write $u = v$ if $u_i = v_i$ for all $i < n$. This translates the fact that the sink collects all grains of sand getting out of the system.

A *toppling* of the vertex $i < n$ in configuration u consists in decreasing the number of grains in this vertex by its degree while the number of those of each of its neighbours j increases by $e_{i,j}$. This is equivalent to the addition to u of the vector Δ_i such that $(\Delta_i)_i = -d_i$ and $(\Delta_i)_j = e_{i,j}$. The notation $u \longrightarrow v$ means that v is obtained from u by *toppling* a vertex, so that there exists an $i < n$ such that $v = u + \Delta_i$. The transitive closure of the toppling operation $\longrightarrow$ is denoted $\stackrel{*}{\longrightarrow}$: $u \stackrel{*}{\longrightarrow} v$ if v is obtained from u by a sequence of topplings.

A key observation is that the connectedness of the graph implies that for any configuration u there exists a sequence of topplings which leads to a *stable* configuration $\hat{u}$; in such a configuration the number of grains in each cell is strictly less than its degree, hence no toppling is possible. This stable configuration does not depend on the order in which topplings are performed [9]: if both Δ_i and Δ_j can be added to a configuration while leaving all its coordinates non-negative, then adding Δ_i first can only increase the number of grains in vertex j, so that this vertex can still topple. After both topplings have taken place, the configuration has been modified by $\Delta_i + \Delta_j$, which does not depend on the order.

1.2. **Recurrent Configurations.** A configuration is *recurrent* in an evolving system if it keeps reappearing during the evolution of the system. In the case of the sandpile model, the system is considered to evolve by adding a grain of sand in a random cell and then applying toppling rules until a stable configuration is reached. This translates into the following central notion.

Definition 1. *A configuration u is* recurrent *if it is stable and if there exists a vector $v \neq 0$ with non-negative coordinates such that $u + v \stackrel{*}{\longrightarrow} u$.*

In order to characterize recurrent configurations Dhar used the vector $\beta = \Delta_n = -(\Delta_1 + \cdots + \Delta_{n-1})$ corresponding to the toppling of the sink, where $\beta_j = e_{j,n}$

for $j \neq n$, and $\beta_n = -d_n$. The simplest example of a recurrent configuration is $\delta = (d_1 - 1, \dots, d_{n-1} - 1, 0)$. Indeed, $\delta + \beta$ is not stable and can topple in each vertex connected to vertex n. Performing these topplings brings grains of sand to vertices at distance 2 from the sink, and so on. The connectedness of the graph then leads to topple all vertices, which leads to the configuration

$$\delta + \beta + \Delta_1 + \cdots + \Delta_{n-1} = \delta,$$

thereby showing that δ is recurrent.

Lemma 1. *There exists $N > 0$ and a configuration ϵ such that $N\beta \xrightarrow{*} \epsilon$ and $\epsilon_i \geq d_i$ for $i = 1, \dots, n-1$.*

Proof. We prove that for each vertex i there exists an integer k_i and a configuration c_i such that $k_i\beta \xrightarrow{*} c_i$ and $(c_i)_i \neq 0$. Adding the $d_i c_i$'s then gives the result.

The proof is an induction on the distance between a vertex i and the sink n. The basis of the induction is provided by β for vertices at distance 1 from the sink. Now, let i be a vertex of G. Since G is connected, there exists a neighbour j of i which is closer to the sink than i. By the induction hypothesis on j, $d_j k_j \beta \xrightarrow{*} d_j c_j$ with $(d_j c_j)_j \geq d_j$. Then toppling vertex j leads to a configuration c_i with $(c_i)_i \neq 0$. Using ϵ leads to the following property of recurrent configurations.

Proposition 1. *For any configuration u, there exists a unique recurrent configuration $\tilde{u}$ such that $u - \tilde{u} \in \Delta = \oplus_{i=1}^{n} \mathbb{Z}\Delta_i$.*

Proof. The proof is given in detail in [5]. We reproduce it here for completeness.

Using ϵ, we have that for any configuration u, $\widehat{u+\epsilon}$ is recurrent, thanks to

$$u + \epsilon + \beta = u + (\epsilon - \delta) + \delta + \beta.$$

Indeed, toppling $\delta+\beta$ leads to δ. When this is added to $u+(\epsilon-\delta)$, the configuration topples to $\widehat{u+\epsilon}$.

To show uniqueness, consider two recurrent configurations u and v such that $u - v \in \Delta$. Then we have

$$u - \sum \beta_i \Delta_i = v - \sum \alpha_i \Delta_i$$

for some positive integers α_i and β_i. Adding a multiple of ϵ on both sides if necessary constructs a configuration where the topplings $\beta_i \Delta_i$ and $\alpha_i \Delta_i$ can be performed, so that this configuration can topple either to u or v. By confluence they have to be equal. This proposition leads to another characterization of recurrent configurations.

Corrolary 1. *The set of recurrent configurations is isomorphic to the set of equivalence classes defined by the symmetric closure $\equiv$ of $\xrightarrow{*}$.*

Combinatorially, this symmetric closure corresponds to allowing topplings and reverse topplings (when all the neighbours of a vertex are non-empty, they can give it a grain of sand each).

Theorem 1. [7] *Given two recurrent configurations u, v, define $\oplus$ by $u \oplus v = \widehat{u+v}$. The set of recurrent configurations is a group for $\oplus$. This group is isomorphic to $\mathbb{Z}^n/\Delta$.*

This group is a central object in the study of sandpiles. The rest of this article illustrates how properties of this group can be computed combinatorially. A combinatorial proof of this result is given in [5].

Dhar also obtained the following characterisation of recurrent configurations.

Proposition 2. *The configuration u is recurrent if and only if $u+\beta \xrightarrow{*} u$. Moreover, in the sequence of topplings leading to u, each vertex topples exactly once.*

2. Toppling Ideal

In this section, we introduce a polynomial ideal associated with a graph, which translates the group of recurrent configurations to a commutative algebra setting.

2.1. **Dictionary.** Configurations and topplings are easily translated from the linear algebra setting into polynomial operations by associating to a configuration $u=(u_1,u_2,\ldots,u_n)\in\mathbb{N}^n$ a monomial $x_u = x_1^{u_1}x_2^{u_2}\cdots x_n^{u_n}\in\mathbb{Q}[x_1,\ldots,x_n]$. To a toppling Δ_i is associated the binomial $T(x_i)=x_i^{d_i}-\prod_j x_j^{e_{i,j}}$.

The addition of two configurations translates into the multiplication of the corresponding monomials and toppling vertex i in u translates into the division of x_u by $x_i^{d_i}$ followed by the multiplication by $\prod_j^n x_j^{e_{i,j}}$.

Given a vector α in $\mathbb{Z}^n$, we write $\alpha=\alpha^+-\alpha^-$, where α^+ and α^- are in $\mathbb{N}^n$ and for each i, either $\alpha_i=\alpha_i^+$ or $\alpha_i=-\alpha_i^-$. The central part of our dictionary is the following equivalence.

Lemma 2. [11] *Let $\alpha,\beta,\ldots$ be in $\mathbb{Z}^n$ and $\sim$ be the symmetric transitive closure of the relations:*

$$u+\alpha^- = v+\alpha^+,\ u+\beta^- = v+\beta^+,\ldots$$

in $\mathbb{N}^n$. Then $u\sim v$ if and only if the binomial $\prod x_j^{u_j}-\prod x_j^{v_j}$ belongs to the ideal generated by the polynomials:

$$\prod {x_j}^{\alpha_j^+}-\prod {x_j}^{\alpha_j^-},\prod {x_j}^{\beta_j^+}-\prod {x_j}^{\beta_j^-},\ldots$$

in $\mathbb{Q}[x_1,\ldots,x_n]$.

Definition 2. *The* toppling ideal *$\mathcal{I}_G$ is generated by x_n-1 and the toppling polynomials $T(x_i)$, for $i\in\{1,\ldots,n\}$.*

The toppling ideal is generated by binomials. Such ideals are called *binomial ideals* and were studied in detail by Eisenbud and Sturmfels [10]. The binomials considered here are called “pure binomials” and the corresponding ideals are akin to toric ideals. In particular, their reduced Gröbner bases consist of pure binomials.

Proposition 3. *Two configurations u and v are equivalent by $\equiv$ if and only if $x_u-x_v\in\mathcal{I}_G$ or equivalently $u-v\in\Delta$.*

Proof. This is a consequence of Lemma 2 using Corollary 1, which showed that introducing Δ_n gives the required symmetric transitive closure.

Note that the number of recurrent configurations is equal to the number of equivalence classes for $\equiv$. In terms of polynomial rings, this is also the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q}[x_1,\ldots,x_n]/\mathcal{I}_G$ [6, Chap. 5].

2.2. **Set Topplings.** A toppling polynomial can also be associated to a subset X of the set V of vertices as follows.

For a vertex i of V, define

$$d_i(X)=\sum_{j\in X} e_{i,j},$$

the number of edges with endpoints i and a vertex of X.

The *set toppling* of the set X in configuration u consists in adding the vector Δ_X to u, where

$$(\Delta_X)_i = \begin{cases} -d_i(\overline{X}), & \text{for } i \in X, \\ d_i(X), & \text{for } i \in \overline{X}, \end{cases}$$

where $\overline{X}$ denotes $V \setminus X$.

Accordingly, the *toppling polynomial* of the subset X of V is defined by

$$T(X) = \prod_{i \in X} x_i^{d_i(\overline{X})} - \prod_{i \in \overline{X}} x_i^{d_i(X)}.$$

The binomial $T(x_i)$ defined above corresponds to the special case $X = \{x_i\}$.

3. Gröbner Bases for the Toppling Ideal

Gröbner bases are a classical computational tool for dealing with polynomial ideals. Given an ordering on monomials which is compatible with the product (a so-called *admissible* ordering) and a set of generators of an ideal $\mathcal{I}$, one can compute a Gröbner basis for $\mathcal{I}$ and from there test ideal membership and more generally compute normal forms in the quotient of the algebra by $\mathcal{I}$. The rest of this article makes use of the notation and basic results from [6, Chapter 2].

In particular, the *graded reverse lexicographic order* (grevlex) denoted $\underset{\text{tdeg}}{<}$, is defined as follows. If $A = \prod_{i=1}^n x_i^{\alpha_i}$ and $B = \prod_{i=1}^n x_i^{\beta_i}$ are two monomials in the variables x_i, $i = 1, \ldots, n$, then $A \underset{\text{tdeg}}{<} B$ if

$$|\alpha| = \sum_{i=1}^n \alpha_i < |\beta| = \sum_{i=1}^n \beta_i$$

or $|\alpha| = |\beta|$ and in $(\alpha_1, \ldots, \alpha_n) - (\beta_1, \ldots, \beta_n)$ the right-most nonzero entry is positive.

¿From there a *toppling order* is defined as follows: let σ be a permutation of $\{1, \ldots, n\}$ such that $\sigma(n) = n$ and if the distance from vertex i to the sink is larger than the distance from vertex j to the sink, then $\sigma(i) > \sigma(j)$. The toppling order is the graded reverse lexicographic order on $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$. To simplify notations in the rest of this article, we assume that the vertices of G have been renumbered so that the graded reverse lexicographic order on $x_1, \ldots, x_n$ is a toppling order.

With such an order, the leading monomial of $T(X)$ for $X \subset \{1, \ldots, n-1\}$ is the product indexed by elements of X. Indeed, $T(X)$ is homogeneous and its leading term is therefore determined by the vertex which is closer to n. Since the graph is connected, one of the vertices of $\overline{X}$ which is connected to X is either n or closer to n than all the vertices of X.

When a Gröbner basis is known for $\mathcal{I}_G$, a unique reduced form $\rho(P)$ is associated to a polynomial P of $\mathbb{Q}[x_1, x_2, \ldots, x_n]$, such that $P - \rho(P) \in \mathcal{I}_G$. Hence, in order to test whether two configurations u and v are equivalent, it is sufficient to check whether the reduced forms $\rho(x_u)$ and $\rho(x_v)$ are equal.

It is easy to prove that the Gröbner basis of a binomial ideal consists of binomials, and that the reduced form of a binomial is also a binomial.

3.1. **A Basis of Toppling Polynomials.** Our main result is the following.

Theorem 2. *A Gröbner basis of the ideal $\mathcal{I}_G$ with respect to a toppling order is given by*

$$\mathcal{T} := \{T(X), X \subset \{1, \ldots, n-1\}\} \cup \{x_n - 1\}.$$

Proof. The proof proceeds in two steps. First the elements of $\mathcal{T}$ are proved to generate $\mathcal{I}_G$. Then, for any pair of polynomials p, q in $\mathcal{T}$, the S-polynomial $S(p, q)$ is shown to reduce to 0 by $\mathcal{T}$. Both these results are obtained using combinatorial interpretations of these binomials in terms of topplings.

First, since $\mathcal{T}$ contains the generators of $\mathcal{I}_G$, the ideal generated by $\mathcal{T}$ contains $\mathcal{I}_G$. The converse inclusion is given by the following lemma.

Lemma 3. *A set toppling can be achieved by a sequence of topplings. Consequently $\mathcal{T} \subset \mathcal{I}_G$.*

Proof. For $X \subset G$, this is equivalent to

$$\Delta_X = \sum_{j \in X} \Delta_j.$$

Extracting the ith coordinate of both terms recovers the definition of $\Delta(X)$ given above:

$$\begin{cases} -d_i(\overline{X}) = -d_i + \sum_{j \in X} e_{i,j}, & \text{if } i \in X, \\ d_i(X) = \sum_{j \in X} e_{i,j}, & \text{if } i \in \overline{X}. \end{cases}$$

We now turn to the last part of the proof of the theorem, which relies on a confluence property.

Lemma 4. *Let X and Y be two subsets of G. Given any configuration u, toppling X and then $Y \setminus X$ leads to the same configuration as that obtained by toppling Y and then $X \setminus Y$. Moreover, if $u \geq 0$, $u + \Delta(X) \geq 0$ and $u + \Delta(Y) \geq 0$, then $u + \Delta(X) + \Delta(Y \setminus X) \geq 0$.*

Proof. For the first part of the lemma, it is sufficient to prove that

$$\Delta(X) + \Delta(Y \setminus X) = \Delta(Y) + \Delta(X \setminus Y).$$

This is obtained by considering the ith coordinate of these vectors in the four cases $i \in X \cap Y$, $i \in \overline{X \cup Y}$, $i \in X \setminus Y$, $i \in Y \setminus X$. The proof is then concluded by the following identities:

$$\begin{aligned} d_i(\overline{Y}) + d_i(Y \setminus X) = \quad & d_i(\overline{X \cap Y}) = d_i(\overline{X}) + d_i(X \setminus Y), && i \in X \cap Y, \\ d_i(X) + d_i(Y \setminus X) = \quad & d_i(X \cup Y) = d_i(Y) + d_i(X \setminus Y), && i \in \overline{X \cup Y}, \\ d_i(Y) - d_i(Y \setminus X) = \quad & d_i(X \cup Y) = d_i(\overline{X}) - d_i(\overline{X \setminus Y}), && i \in X \setminus Y. \end{aligned}$$

The case when $i \in Y \setminus X$ is obtained by symmetry.

Conservation of positivity is also obtained by considering the ith coordinate of the final vector. The only non-obvious case is when $i \in Y \setminus X$. Then the corresponding coordinate is

$$u_i + d_i(X) - d_i(\overline{Y \setminus X}) = u_i - d_i(\overline{Y}) \geq 0.$$

This lemma is applied to the proof of Theorem 2 as follows. Given X, Y two subsets of $G \setminus \{n\}$, let their corresponding toppling polynomials be written

$$T(X) = B(X) - W(X), \qquad T(Y) = B(Y) - W(Y),$$

with $B(X)$ and $B(Y)$ their leading monomials.

The S-polynomial $S = S(T(X), T(Y))$ is obtained by multiplying both $T(X)$ and $T(Y)$ by the smallest monomials $m(X)$ and $m(Y)$ such that $m(X)B(X) = m(Y)B(Y)$, and then subtracting these polynomials. Thus

$$S = m(X)W(X) - m(Y)W(Y).$$

Combinatorially, the monomial $m(X)B(X)$ corresponds to a configuration u where both X and Y can topple. (Here the fact that $n \notin X \cup Y$ is used to determine the leading term of $T(X)$ and $T(Y)$ for the toppling order.) In the S-polynomial, $m(X)W(X)$ thus corresponds to the configuration obtained from u after toppling X, while $m(Y)W(Y)$ corresponds to the result of toppling Y. Without loss of generality, assume that the leading monomial of S is $m(X)W(X)$. Now reduce S by $T(Y \setminus X)$. This replaces $m(X)W(X)$ by a monomial μ corresponding to the result of toppling X and then $Y \setminus X$ from u. The other monomial is $m(Y)W(Y)$ which is now the leading monomial since it contains variables with indices in Y. Then perform a reduction by $T(X \setminus Y)$. By the lemma, this replaces $m(Y)W(Y)$ by μ and leads to 0, as was to be proved.

To conclude the proof of Theorem 2, it remains to be shown that the S-polynomials $S(T(X), x_n - 1)$ also reduce to 0, but that follows from Buchberger's first criterion (when the leading terms are relatively prime, the S-polynomial reduces to 0).

3.2. **Minimal Gröbner Basis.**

The Gröbner basis introduced in Theorem 2 contains 2^{n-1} elements, where n is the number of vertices of the graph. In this section we exhibit a *minimal* Gröbner basis for the toppling ideal with respect to the same reverse lexicographic order.

Recall that a Gröbner basis is *minimal* when its elements have leading coefficient 1 and no leading monomial divides another leading monomial in the basis.

Definition 3. *A subset X of vertices of the graph $G = (V, E)$ is well-connected if the subgraphs of G induced by X and $\overline{X}$ are both connected.*

Theorem 3. *The set S_c of toppling polynomials corresponding to the sets $X \subseteq \{1, \ldots, n-1\}$ which are well-connected is a minimal Gröbner basis for the toppling order.*

Proof. The proof consists in pruning $\mathcal{T}$ by successively removing polynomials whose leading monomial is divisible by the leading monomial of another element of $\mathcal{T}$. Let X be a set of vertices which is not well-connected, we show that $T(X)$ is removed during this process.

First, if X is not connected, then for any connected component C of X the leading monomial of $T(C)$ divides that of $T(X)$ since $d_i(\overline{C}) = d_i(\overline{X})$ for $i \in C$.

If $\overline{X}$ is not connected, one of its connected components, say C, does not contain n. Then the leading monomial of $T(X \cup C)$ divides that of $T(X)$: for $i \in X$, $\overline{X \cup C} \subset \overline{X}$ implies that $d_i(\overline{X \cup C}) \leq d_i(\overline{X})$, while for $i \in C$, $d_i(\overline{X \cup C}) = 0$.

We now prove that no toppling polynomial corresponding to a well-connected set can be removed. Let X and Y be elements of S_c and assume the leading monomial of $T(Y)$ divides that of $T(X)$. We now show that either $Y \subset X$ and the subgraph induced by X is not connected or $Y \not\subset X$ and the subgraph induced by $\overline{X}$ is not connected. Both cases lead to a contradiction.

If $Y \subset X$, for any $i \in Y$, $d_i(\overline{Y}) \leq d_i(\overline{X})$, so that any neighbour of i in $\overline{Y}$ is also in $\overline{X}$, hence there is no edge from Y to $X \setminus Y$ and the subgraph induced by X is not connected.

Otherwise, if $Y \not\subset X$, for any $i \in Y \setminus X$, $d_i(\overline{Y}) = 0$, hence there is no edge from $Y \setminus X$ to $\overline{X} \setminus Y$, which is not empty since it contains the sink n. Therefore the subgraph induced by $\overline{X}$ is not connected.

Note that in the worst case, the minimal Gröbner basis still contains 2^{n-1} elements for the complete graph, but it can be much smaller, as shown by the examples below.

4. Recurrent Configurations and Irreducible Monomials

As mentioned before, the quotient $\mathbb{Q}[x_1, \ldots, x_n]/\mathcal{I}_G$ is a $\mathbb{Q}$-vector space whose dimension is the order of the group of recurrent configurations. From a Gröbner basis for $\mathcal{I}_G$, a basis of this vector space is given by the set of monomials that do not reduce to 0 by the basis. We call these *reduced* monomials. In this section we exhibit a simple bijection between reduced monomials for the toppling order and recurrent configurations.

4.1. **Bijection.** Let $\delta = (d_1 - 1, \ldots, d_n - 1)$ and Φ be the mapping from the set of stable configurations onto itself given by $\Phi(u) = \delta - u$. We also denote $\Phi(M) := \Phi(a_1, \ldots, a_n)$ for a monomial $M = x_1^{a_1} \cdots x_n^{a_n}$.

Theorem 4. *The mapping Φ defines a bijection between the set of reduced monomials with respect to the toppling order and the set of recurrent configurations.*

Proof. Since the number of reduced monomials is the order of the group, it is sufficient to prove that for each reduced monomial M, $\Phi(M)$ is recurrent.

We use Proposition 2 to characterize recurrent configurations. Suppose that $u = \Phi(M)$ is not recurrent and consider $v = u + \beta$. Stabilizing the configuration v yields the configuration $\hat{v}$. In the sequence of topplings, the subset X of vertices that do not topple is not empty. Let j be a vertex in X, since it does not topple, $\hat{v}_j < d_j - 1$. However, in the sequence $v \xrightarrow{*} \hat{v}$, j has received $d_j(\overline{X})$ grains since only vertices not in X topple. Hence $v_j < d_j - 1 - d_j(\overline{X})$. On the other hand, if $B(X)$ is the leading monomial of $T(X)$,

$$w = \Phi(B(X)) = (d_1 - 1 - d_1(\overline{X}), \ldots, d_n - 1 - d_n(\overline{X})).$$

Thus, w contains more grains on each cell than u. This implies that $\Phi^{-1}(w)$ divides M which is impossible since M is reduced.

4.2. **Group Operation.** For a configuration u, let $\rho(u)$ denote the reduced configuration obtained from the monomial associated to u by performing reductions in the Gröbner basis of $\mathcal{I}_G$ associated with the toppling order. We now exploit Φ and ρ to make the link between the operation of the group of recurrent configurations and reduction by the Gröbner basis.

Proposition 4. *If u is a configuration then the recurrent configuration equivalent to u is $\Phi(\rho(\Phi(\rho(u))))$. The identity in the group of recurrent configurations is $\Phi(\rho(\delta))$.*

Proof. For any configuration u, $\rho(u)$ is a configuration which is equivalent to u. By Theorem 4, $\Phi(\rho(v))$ is a recurrent configuration for any configuration v. This

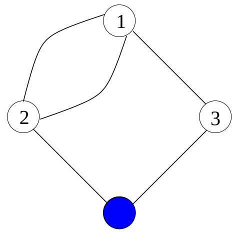

Figure 1. Multigraph with 4 vertices

configuration is equivalent to $\Phi(v)$. Since Φ is an involution, taking $v = \Phi(\rho(u))$ concludes the first part of the proof.

The identity of the group is the recurrent configuration equivalent to $0 = \Phi(\delta)$, whence the second part.

Corrolary 2. *For two recurrent configurations u and v,*

$$u \oplus v = \Phi(\rho(\Phi(u) + \Phi(v))).$$

4.3. **Application: Computation of the Identity.** Proposition 4 yields the following algorithm to compute the identity on a graph G with sink s: beginning with configuration $\delta = (d_1 - 1, \ldots, d_n - 1)$, perform the set topplings for all well-connected subgraphs of $G \setminus \{s\}$ (this is equivalent to reducing by the Gröbner basis for the toppling order). When no further set toppling can be performed, for each cell i replace its number of grains n_i with $d_i - n_i$. The resulting configuration is the identity.

The set of well-connected subgraphs can be identified for special classes of graphs. For instance, the (p, q)-*grid*, whose vertex set C consists of cells (i, j) $(1 \leq i \leq p$ and $1 \leq j \leq q)$ and a sink s, and where each cell (i, j) in the *boundary* of the grid (i.e., $i \in \{1, p\}$ or $j \in \{1, q\}$) is adjacent to the sink. A path is a sequence of adjacent cells. A *polyomino* Π is a subset of $C \setminus \{s\}$ such that two elements of Π are connected by a path consisting of cells of Π and two elements $\overline{\Pi}$ are connected by a path in $\overline{\Pi}$. In this configuration, polyominoes correspond to the well-connected subgraphs of the grid.

When a configuration contains two reducible polyominoes Π_1 and Π_2, it may happen that the toppling of Π_1 leads to a configuration in which Π_2 is no further reducible. However the algorithm yields the same reduced configuration when toppling Π_1 or Π_2 first. This is a consequence of the fact that these topplings correspond to reductions by the Gröbner basis.

Note however, that our algorithm is not very efficient for this type of graph since the determination of a reducible polyomino is not an elementary operation. The following examples are given to illustrate the different aspects of the correspondence between sandpiles and Gröbner bases of toppling ideals.

4.4. **Examples.**

4.4.1. *Multigraph with 4 vertices.* This example corresponds to the graph displayed on Fig. 1. The structure of the graph is reflected by the toppling polynomials for the vertices:

$$x_1^3 - x_2^2 x_3, x_2^3 - x_1^2 x_4, x_3^2 - x_1 x_4, x_4^2 - x_2 x_3, x_4 - 1.$$

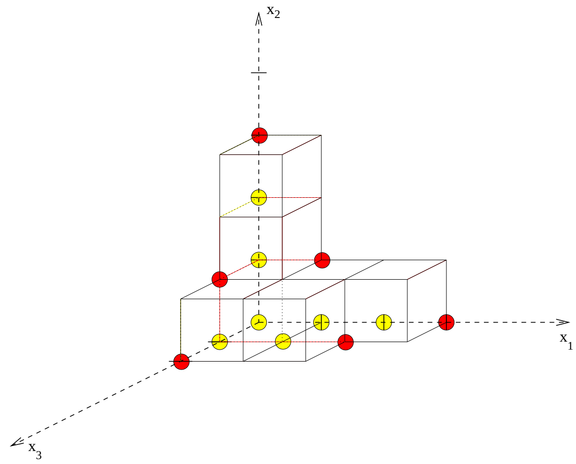

FIGURE 2. Staircase of the Gröbner basis

The minimal Gröbner basis for the graded reverse lexicographic order on monomials is:

$$x_3^2 - x_1, x_2^3 - x_1^2, x_1^3 - x_2, x_2x_3 - 1, x_2x_1 - x_3, x_3x_1^2 - x_2^2, x_4 - 1.$$

Apart from the last, these polynomials correspond respectively to well-connected subgraphs with vertices:

$$\{3\}, \{2\}, \{1\}, \{1, 2, 3\}, \{1, 2\}, \{1, 3\}.$$

Given a Gröbner basis $G = \{p_1, \dots, p_k\} \subset \mathbb{K}[x_1, \dots, x_n]$ for any field $\mathbb{K}$, it is usual to represent the leading monomials of the p_i on an integer lattice in n dimensions. Each polynomial p is associated to a point $c(p)$ whose coordinates are the exponents of its leading monomial. The leading terms of the p_i generate the ideal of leading terms of polynomials in the ideal. These leading terms are thus exactly represented by $\cup c(p_i) + \mathbb{N}^n$. This removes from $\mathbb{N}^n$ a staircase shape whose lattice points correspond to the quotient (see Fig. 2). Their number is exactly the order of the group of recurrent configurations. Note that in our example, those seven monomials are $\{1, x_1, x_1^2, x_2, x_2^2, x_3, x_2x_3\}$, none of which correspond to a recurrent configuration. However, applying Φ yields the recurrent configurations as explained above.

4.4.2. *The* 2×2 *grid.* Our second example is the 2×2 grid consisting of 4 cells, each connected twice to the sink. The sandpile group of this grid, computed for instance in [8], is the product of two cyclic groups of orders 24 and 8.

The toppling polynomials of vertices, including the sink, are

$$x_1^4 - x_2x_3x_5^2, x_2^2 - x_1x_4x_5^2, x_3^4 - x_1x_4x_5^2, x_4^4 - x_2x_3x_5^2, x_5^8 - x_1^2x_2^2x_3^2x_4^2.$$

The computation of the Gröbner basis of the ideal generated by these polynomials and $x^5 - 1$ with the graded lexicographic order yields:

$$x_1^4 - x_2x_3, x_2^4 - x_1x_4, x_3^4 - x_1x_4, x_4^4 - x_2x_3, x_5 - 1, x_1^2x_2^2x_3^2x_4^2 - 1,$$
$$x_1^3x_2^3 - x_3x_4, x_1^3x_3^3 - x_2x_4, x_2^3x_4^3 - x_3x_1, x_3^3x_4^3 - x_2x_1,$$
$$x_1^3x_3^2x_4^3 - x_2^2, x_1^3x_2^2x_4^3 - x_3^2, x_1^2x_2^3x_3^3 - x_4^2, x_2^3x_3^3x_4^2 - x_1^2.$$

3	3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3	3

4	4	4	4	4	4	4	4	4	4	4	4	4	4
0	1	1	1	1	1	1	1	1	1	1	1	1	0
4	4	4	4	4	4	4	4	4	4	4	4	4	4

1	3	3	3	3	3	3	3	3	3	3	3	3	1
3	0	1	1	1	1	1	1	1	1	1	1	0	3
1	3	3	3	3	3	3	3	3	3	3	3	3	1

2	0	2	2	2	2	2	2	2	2	2	2	0	2
3	3	0	1	1	1	1	1	1	1	1	0	3	3
2	0	2	2	2	2	2	2	2	2	2	2	0	2

3	2	0	1	1	1	1	1	1	1	1	0	2	3
0	1	0	1	1	1	1	1	1	1	1	0	1	0
3	2	0	1	1	1	1	1	1	1	1	0	2	3

0	1	3	2	2	2	2	2	2	2	2	3	1	0
3	2	3	2	2	2	2	2	2	2	2	3	2	3
0	1	3	2	2	2	2	2	2	2	2	3	1	0

FIGURE 3. Computation of Identity on the $3 \times (n+6)$ Grid

The irreducible configurations correspond to monomials which are not divisible by one of the following monomials:

$$x_1^4, x_2^4, x_3^4, x_4^4, x_5, x_1^2x_2^2x_3^2x_4^2, x_1^3x_2^3, x_3^3x_4^3, x_1^3x_3^2x_4^3, x_1^3x_2^2x_4^3, x_1^2x_2^3x_3^3, x_2^3x_3^3x_4^2.$$

It is easy to compute that the number of these irreducible monomials is 192, as expected.

Note that the dimension of the quotient is an invariant of the ideal and thus does not depend on the order for which the Gröbner basis has been computed. Computing the Gröbner basis with the pure lexicographic order gives

$$x_1 - x_4^{23}x_3^4, x_2 - x_3^7x_4^{12}, x_3^8 - x_4^{16}, x_4^{24} - 1, x_5 - 1.$$

¿From this follows that x_4 is of order 24 and that any element can be expressed as a product $x_3^ix_4^j$ where $0 \le i \le 7$ and $0 \le j \le 23$, which gives that the order of the group is 192. Also, since x_1 and x_2 can be expressed in terms of x_3 and x_4, it is seen that the group has two generators.

4.4.3. *The $3 \times n$ grid.* Our last example is a $3 \times n$ grid for which we compute the identity using the algorithm described in the previous section. Each element of the border of the grid is connected once to the sink, except the corners, which are connected twice to it. We first consider the grid corresponding to δ, with 3 grains of sand in each cell. Then we compute the reduced form of this configuration by successively finding a well-connected subset X of cells, such that each cell x in the boundary of X contains at least as many grains as x has neighbours in $\overline{X}$. At each step we perform the toppling of the whole set of cells which are in X. The process ends when there is no such set X. Note that since the $T(X)$ constitute a Gröbner basis the order in which the topplings are performed and the choice of the subsets X which are toppled have no influence on the final result. The successive steps of the algorithm and its result are displayed in Fig. 3. The number of grains is indicated in each cell and the coloured area corresponds to the polyomino being used for the toppling. The last grid gives the identity, obtained by complementing with δ. The process depicted in this Figure applies to any $3 \times (n + 6)$ grid. Experiments also lead us to the following.

Conjecture 1. *The identity in the $k \times (n + 2k)$ grid contains a $k \times n$ rectangle of 2's in the middle.*

However, we do have an (inelegant) proof for a $k \times (n + 4k)$ grid.

4.4.4. *The square grid.* We briefly comment on an experiment on the 100×100 grid. The minimal Gröbner basis is clearly out of reach because of its cardinality. However, the polyomino approach is still possible, provided the polyominoes are chosen in an appropriate way. We use the algorithm from [9]: if u is the configuration for which we want to find a reducing polyomino, we add Δ_n to u and topple, the set of vertices that do not topple is an appropriate polyomino. We display on Fig. 4 the number of polyomino topplings in which each cell has been involved during the computation of the identity using this technique. This is related to the intrinsic complexity of computing the identity.

References

[1] Per Bak, Chao Tang, and Kurt Wiesenfeld, *Self-organized criticality*, Physical Review. A. General Physics. Third Series **38** (1988), no. 1, 364–374.

[2] N. L. Biggs, *Chip-firing and the critical group of a graph*, Journal of Algebraic Combinatorics **9** (1999), no. 1, 25–45.

[3] Norman Biggs, *Chip firing on distance-regular graphs*, Tech. Report LSE-CDAM-96-11, CDAM Research Report Series, June 1996.

[4] Anders Björner, László Lovász, and Peter W. Shor, *Chip-firing games on graphs*, European Journal of Combinatorics **12** (1991), no. 4, 283–291.

[5] R. Cori and D. Rossin, *On the sandpile group of a graph*, European Journal of Combinatorics **21** (2000), no. 4, 447–459.

[6] David Cox, John Little, and Donal O'Shea, *Ideals, varieties, and algorithms*, second ed., Springer-Verlag, New York, 1997, An introduction to computational algebraic geometry and commutative algebra.

[7] Michael Creutz, *Abelian sandpile*, Computers in Physics **5** (1991), 198–203.

[8] D. Dhar, P. Ruelle, S. Sen, and D.-N. Verma, *Algebraic aspects of abelian sandpile models*, Journal of Physics. A. Mathematical and General **28** (1995), no. 4, 805–831.

[9] Deepak Dhar, *Self-organized critical state of sandpile automaton models*, Physical Review Letters **64** (1990), no. 14, 1613–1616.

[10] David Eisenbud and Bernd Sturmfels, *Binomial ideals*, Duke Mathematical Journal **84** (1996), no. 1, 1–45.

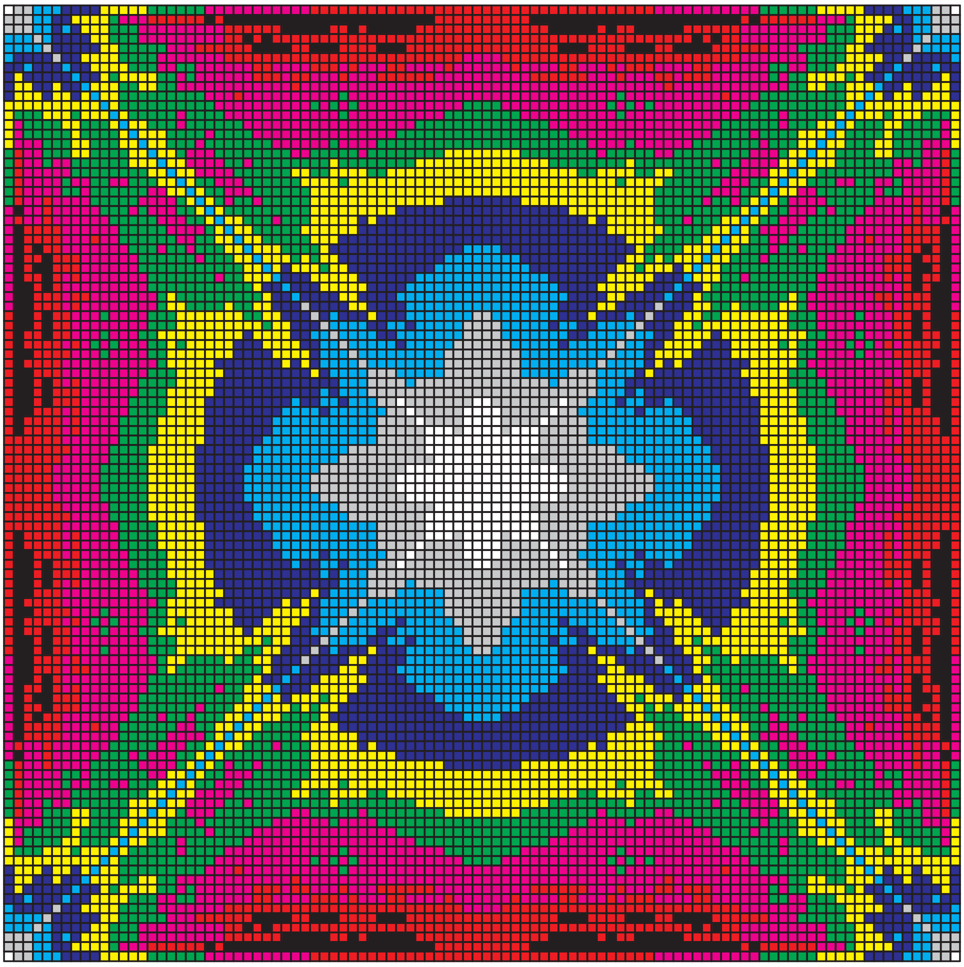

FIGURE 4. Number of topplings to compute the identity on the 100×100 grid. From white (less than 10) to black (more than 80).

[11] Ernst W. Mayr and Albert R. Meyer, *The complexity of the word problems for commutative semigroups and polynomial ideals*, Advances in Mathematics **46** (1982), no. 3, 305–329.
[12] Cristopher Moore and Martin Nilsson, *The computational complexity of sandpiles*, Journal of Statistical Physics **96** (1999), no. 1-2, 205–224.

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