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A note on a.s. finiteness of perpetual integral functionals of diffusions

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Abstract

In this note, with the help of the boundary classification of diffusions, we derive a criterion of the convergence of perpetual integral functionals of transient real-valued diffusions.

In the particular case of transient Bessel processes, we note that this criterion agrees with the one obtained via Jeulin’s convergence lemma.

Keywords: Brownian motion, random time change, exit boundary, local time, additive functional, stochastic differential equation.

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1. Consider a diffusion $Y$ on an open interval $I = (l, r)$ determined by the SDE

$$dY_t = \sigma(Y_t) \, dW_t + b(Y_t) \, dt,$$

where $W$ is a standard Wiener process defined in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. It is assumed that $\sigma$ and $b$ are continuous and $\sigma(x) > 0$ for all $x \in I$. We assume also that $Y$ is transient and

$$\lim_{t \to \zeta} Y_t = r \quad \text{a.s.,}$$

(1)
where $\zeta$ is the life time of $Y$. Hence, if $\zeta < \infty$ then
\[ \zeta = H_r(Y) := \inf\{t : Y_t = r\}. \]
For the speed and the scale measure of $Y$ we use
\[ m^Y(dx) = 2\sigma^2(x)e^{B^Y(x)}dx \quad \text{and} \quad S^Y(dx) = e^{-B^Y(x)}dx, \]
respectively, where
\[ B^Y(x) = 2 \int_x^\infty \frac{b(z)}{\sigma^2(z)}dz. \]

Let $f$ be a positive and continuous function defined on $I$, and consider the perpetual integral functional
\[ A_\zeta(f) := \int_0^\zeta f(Y_s)ds. \]

We are interested in finding necessary and sufficient conditions for a.s. finiteness of $A_\zeta(f)$. When $Y$ is a Brownian motion with drift $\mu > 0$ such a condition is that the function $f$ is integrable at $+\infty$ (see Engelbert and Senf [4] and Salminen and Yor [9]). This condition is derived in [9] via Ray-Knight theorems and the stationarity property of the local time processes (which makes Jeulin’s lemma applicable). In this note a condition (see Theorem 2) valid for general $Y$ is deduced by exploiting the fact that $A_{H_x}(f)$ for $x < r$ can, via random time change, be seen as the first hitting time of a point for another diffusion.

2. The next proposition presents the key result connecting perpetual integral functionals to first hitting times. The result is a generalization of Proposition 2.1 in [8] discussed in Propositions 2.1 and 2.3 in [2].

**Proposition 1.** Let $Y$ and $f$ be as above, and assume that there exists a two times continuously differentiable function $g$ such that
\[ f(x) = (g'(x)\sigma(x))^2, \quad x \in I. \]  

Set for $t > 0$
\[ A_t := \int_0^t f(Y_s)ds. \]
and let $\{a_t : 0 \leq t < A_\zeta\}$ denote the inverse of $A$, that is,
\[ a_t := \min\{s : A_s > t\}, \quad t \in [0, A_\zeta). \]
Then the process $Z$ given by

$$Z_t := g(Y_{a_t}), \quad t \in [0, A_{\xi}),$$

is a diffusion satisfying the SDE

$$dZ_t = d\tilde{W}_t + G(g^{-1}(Z_t)) \, dt, \quad t \in [0, A_{\xi}),$$

where $\tilde{W}_t$ is a Brownian motion and

$$G(x) = \frac{1}{f(x)} \left( \frac{1}{2} \sigma(x)^2 g''(x) + b(x) g'(x) \right).$$

Moreover, for $l < x < y < r$

$$A_{H_{z(y)}} = \inf\{t : Z_t = g(y)\} =: H_{g(y)}(Z) \quad a.s.$$  \hspace{1cm} (6)

with $Y_0 = x$ and $Z_0 = g(x)$.

3. To fix ideas, assume that the function $g$ as introduced in Proposition 1 is increasing. We define $g(r) := \lim_{x \to r} g(x)$, and use the same convention for any increasing function defined on $(l, r)$. The state space of the diffusion $Z$ is the interval $(g(l), g(r))$ and a.s. $\lim_{t \to \xi(Z)} Z_t = g(r)$. Clearly, letting $y \to r$ in (6) it follows that

$$A_{H_{z(y)}} = \inf\{t : Z_t = g(r)\} \quad a.s.,$$

where both sides in (7) are either finite or infinite. Now we have

**Theorem 2.** For $Y$, $A$, $f$ and $g$ as above it holds that $A_{\xi}$ is a.s. finite if and only if for the diffusion $Z$ the boundary point $g(r)$ is an exit boundary, i.e.,

$$\int_{g(r)}^{g(r)} S^Z(d\alpha) \int_{\alpha}^{\infty} m^Z(d\beta) < \infty,$$  \hspace{1cm} (8)

where the scale $S^Z$ and the speed $m^Z$ of the diffusion $Z$ are given by

$$S^Z(d\alpha) = e^{-B^Z(\alpha)} d\alpha \quad \text{and} \quad m^Z(d\beta) = 2 e^{B^Z(\beta)} d\beta$$

with

$$B^Z(\beta) = 2 \int_{\beta}^{\infty} G \circ g^{-1}(z) \, dz.$$

The condition (8) is equivalent with the condition

$$\int_{r}^{r} \left( S^Y(r) - S^Y(v) \right) f(v) m^Y(dv) < \infty.$$  \hspace{1cm} (9)
Proof. As is well known from the standard diffusion theory, a diffusion hits its exit boundary with positive probability and an exit boundary cannot be unattainable (see [3] or [1]). This combined with (7) and the characterization of an exit boundary (see [1] No. II.6 p.14) proves the first claim. It remains to show that (8) and (9) are equivalent. We have

\[ B^Z(\alpha) = 2 \int g^{-1}(\alpha) G(u) g'(u) \, du \]

\[ = 2 \int g^{-1}(\alpha) \left( \frac{1}{2} g''(u) + \frac{b(u)}{\sigma^2(u)} \right) \, du \]

\[ = \log(g'(g^{-1}(\alpha)))) + B^Y(g^{-1}(\alpha)). \]

Consequently,

\[ S^Z(d\alpha) = e^{-B^Z(\alpha)} \, d\alpha = \frac{1}{g'(g^{-1}(\alpha))} \exp\left(-B^Y(g^{-1}(\alpha))\right) \, d\alpha \]

and

\[ m^Z(d\alpha) = 2 e^{B^Z(\alpha)} \, d\alpha = 2 g'(g^{-1}(\alpha)) \exp\left(B^Y(g^{-1}(\alpha))\right) \, d\alpha. \]

Substituting first \( \alpha = g(u) \) in the outer integral in (8) and after this \( \beta = g(v) \) in the inner integral yield

\[ \int_{g(r)}^{g(s)} S^Z(d\alpha) \, m^Z(d\beta) = 2 \int_r^u du \, e^{-B^Y(u)} \int_v^u dv \, (g'(v))^2 e^{B^Y(u)} \]

\[ = 2 \int_r^u dv \, (g'(v))^2 e^{B^Y(v)} \int_v^u dv \, e^{-B^Y(u)} \]

by Fubini’s theorem. Using the expressions given in (2) for the speed and the scale of \( Y \) and the relation (4) between \( f \) and \( g \) complete the proof. \( \square \)

4. It is easy to derive a condition that the mean of \( A_\zeta(f) \) is finite. Indeed,

\[ E_x(A_\zeta(f)) = \int_0^\infty E_x(f(Y_s)) \, ds \]

\[ = \int_t^r G^Y_0(x,y) f(y) \, m^Y(dy) < \infty, \]

where \( G^Y_0 \) is the Green kernel of \( Y \) w. r. t. \( m^Y \). Under the assumption (1) we may take for \( x \geq y \)

\[ G^Y_0(x,y) = S^Y(r) - S^Y(x). \]
Consequently, the condition (9) may be viewed as a part of the condition (10).

5. Since the exit condition (8) plays a crucial rôle in our approach we discuss here shortly two proofs of this condition, thus making the paper as self-contained as possible.

Let $Y$ be an arbitrary regular diffusion living on the interval $I$ with the end points $l$ and $r$. The scale function of $Y$ is denoted by $S$ and the speed measure by $m$. It is also assumed that the killing measure of $Y$ is identically zero. Recall the definition due to Feller

$$ r \text{ is exit } \iff \int^r S(da) \int^a m(d\beta) < \infty. \quad (11) $$

Note that by Fubini’s theorem

$$ \int^r S(da) \int^a m(d\beta) = \int^r m(d\beta)(S(r) - S(\beta)), $$

and, hence, $S(r) < \infty$ if $r$ is exit. Moreover, if $r$ is exit then $H_r < \infty$ with positive probability.

5.1. We give now some details of the proof of (11) following closely Kallenberg [7] (see also Breiman [3]). For $l < a < b < r$ let $H_{ab} := \inf\{t : Y_t = a \text{ or } b\}$. Then for $a < x < b$

$$ E_x(H_{ab}) = \int^b_x \hat{G}_0^Y (x, z) m(dz), \quad (12) $$

where $\hat{G}_0^Y$ is the (symmetric) Green kernel of $Y$ killed when it exits $(a, b)$, i.e.,

$$ \hat{G}_0^Y (x, z) = \frac{(S(b) - S(x))(S(y) - S(a))}{S(b) - S(a)} \quad x \geq y. $$

If $r$ is exit there exists $h > 0$ such that $P_x(H_r < h) > 0$ for any fixed $x \in (a, r)$. Using this property it can be deduced (see [7] p. 377) that for any $a \in (l, r)$

$$ E_x(H_{ar}) < \infty, $$

which, from (12), is seen to be equivalent with (11).

5.2. Another proof of (11) can be found in Itô and McKean [5] p. 130. To present also this briefly recall first the formula

$$ E_x(\exp(-\lambda H_b)) = \frac{\psi_\lambda(x)}{\psi_\lambda(b)}, \quad (13) $$
where \( \lambda > 0 \) and \( \psi_{\lambda} \) is an increasing solution of the generalized differential equation
\[
\frac{d}{dm} \frac{d}{dS} u = \lambda u.
\] (14)

Letting \( b \to r \) in (13) it is seen that
\[
r \text{ is exit } \iff \lim_{b \to r} \psi_{\lambda}(b) < \infty.
\]

Let \( \psi_{\lambda}^+ \) denote the (right) derivative of \( \psi_{\lambda} \) with respect to \( S \). Since \( \psi_{\lambda} \) is increasing it holds that \( \psi_{\lambda}^+ > 0 \). The fact that \( \psi_{\lambda} \) solves (14) yields for \( z < r \)
\[
\psi_{\lambda}^+(r) - \psi_{\lambda}^+(z) = \lambda \int_z^r \psi_{\lambda}(a) \, m(da).
\]

In particular, \( \psi_{\lambda}^+ \) is increasing and \( \psi_{\lambda}^+(r) > 0 \). Hence, assuming now that \( \psi_{\lambda}(r) < \infty \) we obtain \( S(r) < \infty \), and, further,
\[
\lambda \psi_{\lambda}(z) \int_z^r S(d\alpha) \int_z^\alpha m(d\beta) \leq \lambda \int_z^r S(d\alpha) \int_z^\alpha \psi_{\lambda}(\alpha) m(d\beta)
= \int_z^r S(d\alpha) (\psi_{\lambda}^+(\alpha) - \psi_{\lambda}^+(z))
= \psi_{\lambda}(r) - \psi_{\lambda}(z) - \psi_{\lambda}^+(z) (S(r) - S(z)) < \infty,
\]

which yields the condition on the right hand side of (11). Assume next that the condition on the right hand side of (11) holds, and consider for \( z < \beta \)
\[
0 \leq (\psi_{\lambda}(\beta))^{-1} (\psi_{\lambda}^+(\beta) - \psi_{\lambda}^+(z)) = (\psi_{\lambda}(\beta))^{-1} \int_z^\beta \psi_{\lambda}(\alpha) m(d\alpha).
\]
Integrating over \( \beta \) gives
\[
\log(\psi_{\lambda}(r)) - \log(\psi_{\lambda}(z)) - \psi_{\lambda}^+(z) \int_z^r (\psi_{\lambda}(\beta))^{-1} S(d\beta)
= \int_z^r S(d\beta) (\psi_{\lambda}(\beta))^{-1} \int_z^\beta \psi_{\lambda}(\alpha) m(d\alpha)
\leq \int_z^r S(d\beta) \int_z^\beta m(d\alpha) < \infty,
\]

which implies that \( \psi_{\lambda}(r) < \infty \), thus completing the proof.
6. As an application of Theorem 2, we consider a Bessel process with
dimension parameter $\delta > 2$. Let $R$ denote this process. It is well known that
$\lim_{t \to \infty} R_t = +\infty$ and that $R$ solves the SDE
\[ dR_t = dW_t + \frac{\delta - 1}{2R_t^2} dt, \]
where $W$ is a standard Brownian motion. Here the function $B^R$ (cf. (3))
takes the form
\[ B^R(v) = (\delta - 1) \log v, \]
and, consequently,
\[
\int_{v}^{\infty} dv \left( g'(v) \right)^2 e^{B^R(v)} \int_{v}^{\infty} du e^{-B^R(u)}
= \int_{v}^{\infty} dv \left( g'(v) \right)^2 v^{\delta - 1} \int_{v}^{\infty} du u^{-\delta + 1}
= \int_{v}^{\infty} dv \left( g'(v) \right)^2 v^{\delta - 1} \frac{1}{\delta - 2} v^{-\delta + 2}
\]
leading to
\[
\int_{0}^{\infty} f(R_s) \, ds < \infty \iff \int_{0}^{\infty} u \, f(u) \, du < +\infty.
\]

Another way to derive this condition is via local times and Jeulin’s lemma
[6]. Indeed, by the occupation time formula and Ray-Knight theorem for the
total local times of $R$ (see, e.g. [10] Theorem 4.1 p. 52) we have
\[
\int_{0}^{\infty} f(R_s) \, ds \overset{(d)}{=} \int_{0}^{\infty} f(a) \, \frac{\rho_{a^\gamma}}{\gamma a^\gamma} \, da
= \frac{1}{\gamma} \int_{0}^{\infty} a f(a) \frac{\rho_{a^\gamma}}{a^\gamma} \, da
\]
where $\delta = 2 + \gamma$ and $\rho$ is a squared 2-dimensional Bessel process. Using
the scaling property, it is seen that the distribution of the random variable
$\rho_{a^\gamma}/a^\gamma$ does not depend on $a$. Hence, we obtain by Jeulin’s lemma that if the
function $a \mapsto a f(a)$, $a > 0$, is locally integrable on $[0, \infty)$ then
\[
\int_{0}^{\infty} f(R_s) \, ds < \infty \iff \int_{0}^{\infty} a f(a) \, da < \infty. \tag{15}
\]
The same argument allows us to recover the result in [9], that is,
\[
\int_0^\infty g(W_s^{(\mu)}) \, ds < \infty \iff \int_0^\infty g(x) \, dx < \infty.
\]  
(16)

where \( g \) is any non-negative locally integrable function and \( W^{(\mu)} \) denotes a Brownian motion with drift \( \mu > 0 \). To see this, write \( g(x) = f(e^x) \) and use Lamperti’s representation
\[
\exp(W_s^{(\mu)}) = R_u^{(\mu)}, \quad s \geq 0,
\]
where
\[
A_s^{(\mu)} = \int_0^s du \exp(2W_u^{(\mu)})
\]
and \( R^{(\mu)} \) is a Bessel process with dimension \( d = 2(1 + \mu) \) starting from 1, we obtain (cf. [8] Remark 3.3.(3))
\[
\int_0^\infty f(\exp(W_s^{(\mu)})) \, ds = \int_0^\infty (R_u^{(\mu)})^{-2} f(R_u^{(\mu)}) \, du \quad \text{a.s.,}
\]
and, in order to get (16) it now only remains to use the equivalence (15).

We wish to underline the fact that in Theorem 2 it is assumed that the function \( f \) is continuous whereas the approach via Jeulin’s lemma, which we developed above, demands only local integrability.

References


