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Jacky Cresson, Stephen Wiggins

► **To cite this version:**

Jacky Cresson, Stephen Wiggins. A lambda-lemma for normally hyperbolic invariant manifolds. 2005.  
hal-00012889

**HAL Id: hal-00012889**

**<https://hal.science/hal-00012889>**

Preprint submitted on 28 Oct 2005

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# A $\lambda$ -LEMMA FOR NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

JACKY CRESSON AND STEPHEN WIGGINS

ABSTRACT. Let  $N$  be a smooth manifold and  $f : N \rightarrow N$  be a  $C^\ell$ ,  $\ell \geq 2$  diffeomorphism. Let  $M$  be a normally hyperbolic invariant manifold, not necessarily compact. We prove an analogue of the  $\lambda$ -lemma in this case.

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## 1. INTRODUCTION

In a recent paper, Richard Moeckel [20] developed a method for proving the existence of drifting orbits on Cantor sets of annuli. His result is related to the study of Arnold diffusion in Hamiltonian systems [1], and provides a way to overcome the so called *gaps problems* for transition chains in Arnold's original mechanism [17]. We refer to Lochak [17] for a review of this problem.

The principal assumption of his work is the existence of a symbolic dynamics for a compact normally hyperbolic invariant annulus. His assumptions can be formulated as follows ([20],p.163):

Let  $\Sigma$  denote the Cantor set of all bi-infinite sequences of 0's and 1's, and  $\sigma : \Sigma \rightarrow \Sigma$  be the shift map defined for  $s = (s_i)_{i \in \mathbb{Z}}$  as  $\sigma(s)_i = s_{i+1}$ .

Let  $N$  be a smooth manifold and  $F : N \rightarrow N$  be a  $C^\ell$  diffeomorphism. Let  $M \subset N$  be a  $C^\ell$  normally hyperbolic invariant manifold for  $F$  such that the stable and unstable manifolds  $W^{s,u}(M)$  intersect transversally in  $N$ . Then there exists in a neighbourhood of  $M$  an invariant hyperbolic set  $\Lambda$  with the following properties:

- i)  $\Lambda \sim \Sigma \times M$ .
- ii)  $\Lambda$  is a  $C^{0,\ell}$  Cantor set of manifolds, i.e. each leaf  $M_s = \{s\} \times M$ ,  $s \in \Sigma$  is  $C^\ell$  and depends continuously on  $s$  in the  $C^\ell$  topology.
- iii) some iterate  $F^n|_\Sigma$  is conjugate to a skew product over the shift

$$\phi : \Sigma \times M \rightarrow \Sigma \times M, \phi(s, p) = (\sigma(s), g_s(p)),$$

where  $g_s : M_s \rightarrow M_{\sigma(s)}$ .

Moeckel [20] refers to previous work of Shilnikov [24], Meyer and Sell [19] and Wiggins [27].

Wiggins [27] proves an analogue of the Smale-Birkhoff theorem near a transversal homoclinic normally hyperbolic invariant torus. However, this result can not be used to justify Moeckel's assumptions. Indeed, Wiggins's result is based on:

- a particular normal form near the normally hyperbolic invariant torus obtained by Shilnikov [24].
- an annulus is not a compact boundaryless manifold, contrary to the torus.

Moreover, in most applications the compact annulus is obtained by truncating a normally hyperbolic invariant cylinder, which is not compact. However, non-compactness can be easily handled as it only provides technical difficulties. This is not the case when the normally hyperbolic invariant manifold has a boundary, which leads to technical as well as dynamical problems.

Other problems of importance deal with general compact boundaryless invariant normally hyperbolic manifolds, as for example normally hyperbolic invariant spheres. It has recently been shown that normally hyperbolic invariant spheres are an important phase space structure in Hamiltonian systems with three or more degrees-of-freedom. Specific applications where they play a central role are cosmology [22], reaction dynamics [31], [25], and celestial mechanics [26].

The proof of the Smale-Birkhoff theorem for normally hyperbolic invariant tori by Wiggins [27] is based on a generalized  $\lambda$ -lemma. This  $\lambda$ -lemma has been generalized by E. Fontich and P. Martin [12] under more general assumptions and  $C^2$  regularity for the map.

There are three settings where a new type of  $\lambda$ -lemma would be useful. What characterizes the difference in each case is the geometrical structure of the normally hyperbolic invariant manifold  $M$ .

- $M$  is non-compact, and can be characterized by a global coordinate chart. This situation arises when we consider normally hyperbolic invariant cylinders in Hamiltonian systems. Non-compactness is dealt with by assuming uniform bounds on first and second derivatives of certain functions (cf. [13]).
- $M$  is compact with a boundary. This situation arises when we truncate normally hyperbolic invariant cylinders to form normally hyperbolic invariant annuli. The technical difficulty is controlling the dynamics at the boundary.
- $M$  is compact, but it cannot be described globally by a single coordinate chart. This situation arises when we consider normally hyperbolic invariant spheres.

In this paper, we prove a  $\lambda$ -lemma for normally hyperbolic invariant manifolds, which are not necessarily compact. This result allows us to prove a  $\lambda$ -lemma for a normally hyperbolic annulus, i.e. for a normally hyperbolic compact manifold with boundaries which is a subset of a non-compact boundaryless normally hyperbolic manifold. We can also use the same result to prove a  $\lambda$ -lemma for compact invariant manifolds that cannot be described by a single coordinate chart.

The proof of the Smale-Birkhoff theorem as well as its applications for diffusion in Hamiltonian systems will be studied in a forthcoming paper [9].

## 2. A $\lambda$ -LEMMA FOR NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

We first define the norms that we will use throughout this paper. Essentially, we will only require two norms; one for vectors and one for matrices. All our vectors can be

viewed as elements of  $\mathbb{R}^n$  (for some appropriate  $n$ ) and our matrices will consist of real entries. As a vector norm we will use the sup norm on  $\mathbb{R}^n$ , denoted by  $|\cdot|$ . Let  $\mathcal{M}_{m \times n}$  denote the set of  $m \times n$  matrices over  $\mathbb{R}$ ,  $n \geq 1$ ,  $m \geq 1$ . An element of  $\mathcal{M}_{m \times n}$  has the form  $A = (a_{i,j})_{i=1,\dots,m, j=1,\dots,n} \in \mathcal{M}_{m \times n}$ . We define the norm of  $A \in \mathcal{M}_{m \times n}$  by  $\|A\| = \sup_i \sum_j |a_{i,j}|$ .

**2.1. Normally hyperbolic invariant manifolds.** Let  $N$  be a  $n$ -dimensional smooth manifold,  $n \geq 3$ , and  $f : N \rightarrow N$  be a  $C^\ell$  diffeomorphism,  $\ell \geq 1$ . Let  $M$  be a boundaryless  $m$ -dimensional submanifold (compact or non compact) of class  $C^\ell$  of  $N$ ,  $m < n$ , invariant under  $f$ , such that:

i)  $M$  is normally hyperbolic,

ii)  $M$  has a  $m + n_s$ -dimensional stable manifold  $W^s(M)$  and a  $m + n_u$ -dimensional unstable manifold  $W^u(M)$ , with  $m + n_s + n_u = n$ .

Let  $p \in N$ , we denote by  $Df_p$  the derivative of  $f$  at  $p$ . Let  $T_M N$  be the tangent bundle of  $N$  over  $M$ . As  $M$  is normally hyperbolic, there exists a  $Df$ -invariant splitting  $T_M N = E^s \oplus E^u \oplus TM$  such that  $E^s \oplus TM$  is tangent to  $W^s(M)$  at  $M$  and  $E^u \oplus TM$  is tangent to  $W^u(M)$  at  $M$ .

**2.2. Normal form.** We assume in the following that there exist a  $C^\ell$  coordinate systems  $(s, u, x) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_u} \times M$  in a neighbourhood  $U$  of  $M$  such that  $f$  takes the form:

$$(1) \quad f(s, u, x) = (A_s(x)s, A_u(x)u, g(x)) + r(s, u, x),$$

where  $r$  is the remainder,  $r = (r_s(s, u, x), r_u(s, u, x), r_x(s, u, x))$  with  $A_s, A_u, g$  and  $r$  satisfying

**a) (invariance of  $M$ )**  $r_s(0, 0, x) = r_u(0, 0, x) = r_x(0, 0, x) = 0$  for all  $x \in M$ .

As a consequence, the set  $M$  is given in this coordinates system by

$$(2) \quad M = \{(s, u, x) \in U \mid s = u = 0\},$$

and  $U$  can be chosen of the form

$$(3) \quad U = B_\rho \times M,$$

with  $\rho > 0$  and  $B_\rho$  is the open ball defined by  $B_\rho = \{(s, u) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}; |(s, u)| < \rho\}$ .

Let  $\rho > 0$ , we denote by  $B_\rho^s$  (resp.  $B_\rho^u$ ) the open ball of size  $\rho$  in  $\mathbb{R}^{n_s}$  (resp.  $\mathbb{R}^{n_u}$ ) around 0.

As  $M$  is normally hyperbolic, for  $\rho > 0$  sufficiently small, the stable manifold theorem (see [15],[28]) ensures that the stable and unstable manifolds can be represented as graphs, i.e. there exist two  $C^\ell$  functions  $G^s(s, x)$  and  $G^u(u, x)$  such that

$$(4) \quad \begin{aligned} W^s(M) \cap U &= \{(s, x) \in B_\rho^s \times M \mid u = G^s(s, x)\}, \\ W^u(M) \cap U &= \{(u, x) \in B_\rho^u \times M \mid s = G^u(u, x)\}, \end{aligned}$$

with

$$(5) \quad G^s(0, x) = 0, \quad \partial_s G^s(0, x) = 0, \quad \partial_x G^s(0, x) = 0,$$

and

$$(6) \quad G^u(0, x) = 0, \quad \partial_u G^u(0, x) = 0, \quad \partial_x G^u(0, x) = 0,$$

which reflect the tangency of  $W^s(M)$  and  $W^u(M)$  to  $E^s$  and  $E^u$  over  $M$  respectively.

Using these functions, we can find a coordinate system for which the stable and unstable manifolds are “straightened”, i.e.

- b) (straightening of the stable manifold)**  $r_u(s, 0, x) = 0$  for all  $(s, 0, x) \in U$ ,
- c) (straightening of the unstable manifold)**  $r_s(0, u, x) = 0$  for all  $(0, u, x) \in U$ .

As a consequence, the stable and unstable manifolds of  $M$  are given by

$$(7) \quad \begin{aligned} W^s(M) &= \{(s, u, x) \in U \mid u = 0\}, \\ W^u(M) &= \{(s, u, x) \in U \mid s = 0\}. \end{aligned}$$

Indeed, following the classical work of Palis-deMelo [23], the change of variables

$$(8) \quad \Phi: \begin{array}{ccc} U & \longrightarrow & U \\ (s, u, x) & \longmapsto & (s - G^u(u, x), u - G^s(s, x), x), \end{array}$$

realizes the straightening:

For all  $P \in W^s(M)$ , we denote by  $\Phi(P) = (s', u', x')$ . Then,  $P \in W^s(M)$  if and only if  $\Phi(P) = (s', 0, x')$  and  $P \in W^u(M)$  if and only if  $\Phi(P) = (0, u', x')$ .

**d) (conjugacy on the stable and unstable manifold)** We assume that  $r_x(0, u, x) = 0$  and  $r_x(s, 0, x) = 0$  for all  $s \in B_\rho^s$ ,  $u \in B_\rho^u$  and  $x \in M$ .

In many examples of importance this condition is satisfied. It tells us that the dynamics on the stable and unstable manifolds in the invariant manifold direction is given by the

dynamics on  $M$ . We refer to Graff [14] for such an example of rigidity in an analytic context.

**e) (hyperbolicity)**  $\|A_s(x)\| \leq \lambda < 1$ ,  $\|A_u(x)^{-1}\| \leq \lambda < 1$ .

These results lead us to introduce the following definition of a normal form for diffeomorphisms near a normally hyperbolic invariant manifold:

**Definition 1** (Normal form). *Let  $N$  be a smooth manifold and  $f$  a  $C^\ell$  diffeomorphism of  $N$ ,  $\ell \geq 2$ . Let  $M$  be a compact normally hyperbolic invariant manifold of  $f$ . The diffeomorphism is said to be in normal form if there exist a neighbourhood  $U$  of  $M$  and a  $C^\ell$  coordinate system on  $U$  such that  $f$  takes the form (1) and satisfies conditions a)-e).*

Standard results on normal form theory can be used to prove in some case that we have a diffeomorphism in normal form. We refer to ([3],p.332) for a general normal form theorem. In particular, we derive such a normal form in a Hamiltonian setting near a normally hyperbolic cylinder.

Moreover, general normal form results for normally hyperbolic manifolds already imply that our assumptions are general, at least if we restrict the regularity assumption on the coordinates system to  $C^1$ . Indeed, we have the following result due to M. Gidea and R. De Llave [13]:

**Theorem 1.** *Let  $N$  be a smooth manifold and  $f$  a  $C^\ell$  diffeomorphism of  $N$ ,  $\ell \geq 2$ . Let  $M$  be a normally hyperbolic invariant manifold of  $f$  (compact or non compact). There exists a neighbourhood  $U$  of  $M$  and a  $C^1$  coordinate system on  $U$  such that  $f$  is in normal form.*

We refer to ([13],§.5.1) for a proof.

Of course, such a result is not sufficient for our purposes as we need some control on the second order derivatives of  $g$  and  $r$ . However, it proves that our assumptions are general. In the same paper, M. Gidea and R. De Llave [13] proves that we can take  $r_x = 0$ , i.e. that we have a decoupling between the center dynamics and the hyperbolic dynamics.

**2.3. The  $\lambda$ -lemma.** We have the following generalization of the toral  $\lambda$ -lemma of S. Wiggins [27]:

**Theorem 2** ( $\lambda$ -lemma). *Let  $N$  be a smooth manifold and  $M$  be a  $C^\ell$  submanifold of  $N$ , normally hyperbolic, invariant under a  $C^\ell$  diffeomorphism  $f$ ,  $\ell \geq 2$ , in normal form in a*

given neighbourhood  $U$  of  $M$  and such that

i) There exists  $C > 0$  such that

$$(9) \quad \sup \{ \| \partial_{\sigma, \sigma'}^2 r_i(z) \|, z \in U, \sigma \in \{s, u, x\}, \sigma' \in \{u, x\}, i \in \{s, x\} \} \leq C.$$

ii) There exists  $\tilde{C} > 0$  such that

$$(10) \quad \sup \{ \| \partial_{\sigma, x}^2 g(z) \|, z \in U, \sigma \in \{s, u, x\} \} \leq \tilde{C}.$$

iii) There exists  $D > 0$  such that for all  $x \in M$ ,  $\| \partial_x A_s(x) \| \leq D$ .

Let  $\Delta$  be an  $m + n_u$  dimensional manifold intersecting  $W^s(M)$  transversally and let  $\Delta_k = f^k(\Delta) \cap U$  be the connected component of  $f^k(\Delta) \cap U$  intersecting  $W^s(M)$ . Then for  $\epsilon > 0$ , there exists a positive integer  $K$  such that for  $k \geq K$ ,  $\Delta_k$  is  $C^1$   $\epsilon$ -close to  $W^u(M)$ .

The proof follows essentially the same line as in ([27], p.324-329) and is given in section 4.

### 3. HAMILTONIAN SYSTEMS AND NORMALLY HYPERBOLIC INVARIANT CYLINDERS AND ANNULI

Normally hyperbolic invariant annuli or cylinders are the basic pieces of all geometric mechanisms for diffusion in Hamiltonian systems. This may seem to be an unusual statement in light of the fact that the classical “transition chain” is a series of heteroclinic connections of stable and unstable manifolds of “nearby” lower dimensional tori. However, these lower dimensional tori are contained in normally hyperbolic invariant annuli and cylinders which have their own stable and unstable manifolds (which, in turn, contain the stable and unstable manifolds of the lower dimensional tori used to construct Arnold’s transition chains). The importance of normally hyperbolic invariant annuli or cylinders can be clearly seen in the papers of Z. Xia [32], R. Moeckel [21], and A. Delshams, R. De Llave and T. Seara [11] where normally hyperbolic annuli are a fundamental tool.

In this section, we prove a  $\lambda$ -lemma for Hamiltonian systems possessing, in a fixed energy manifold, a normally hyperbolic manifold of the form  $\mathbb{T} \times I$ , where  $I$  is a given compact interval, which belong to a non-compact boundaryless invariant manifold  $\mathbb{T} \times \mathbb{R}$ . We will see that the fact that a  $\lambda$ -lemma can be proven in this case is related to the fact that the boundaries are partially hyperbolic invariant tori for which an analogue of the  $\lambda$ -lemma for which a different type of  $\lambda$ -lemma has already been proven. Moreover, we will also construct a class of three degree of freedom Hamiltonian systems which satisfy our assumptions.



**3.1. Main result.** All of our results will be stated in terms of discrete time systems, or maps. However, many of the applications we have in mind will be for continuous time Hamiltonian systems. Our results will apply in this setting by considering an appropriate Poincaré map for the continuous time system. It is important to keep this reduction from continuous time Hamiltonian system to discrete time Poincaré map firmly in mind from the point of view of considering the dimensions of the relevant invariant manifolds and transversal intersection in the two systems. Finally, we note that even though the specific applications we consider here are for Hamiltonian systems, as is true of most hyperbolic phenomena, a Hamiltonian structure is not generally required for their validity. We now describe the setting and our hypotheses for the applications of interest.

Let  $H$  be a  $C^r$  Hamiltonian  $H(I, \theta)$ ,  $(I, \theta) \in \mathbb{R}^3 \times \mathbb{T}^3$ ,  $r \geq 3$ . The Hamiltonian defines a  $C^{r-1}$  Hamiltonian vector field for which we make the following assumptions:

i) the Hamiltonian vector field possesses an invariant normally hyperbolic manifold  $\bar{\Lambda}$  which is diffeomorphic to  $\mathbb{T}^2 \times \mathbb{R}^2$ , with 5-dimensional stable and unstable manifold  $W^s(\bar{\Lambda})$  and  $W^u(\bar{\Lambda})$  respectively (note: these invariant manifolds are not isoenergetic).

ii) There exists a Poincaré section in a tubular neighbourhood of  $\bar{\Lambda}$  and a  $C^2$  coordinate system of the form  $(v, w, s, u) \in S^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  such that the Poincaré map is in normal form near  $\bar{\Lambda}$ , i.e. it takes the form

$$(11) \quad f(s, u, v, w) = (A_s(z) s, A_u(z) u, g(v), w) + r(s, u, v, w),$$

where  $r$  is the remainder,  $r = (r_s(s, u, v, w), r_u(s, u, v, w), r_x(s, u, v, w))$  with  $A_s$ ,  $A_u$ ,  $g$  and  $r$  satisfying assumptions a)-e) of section 2.2. We denote the intersection of  $\bar{\Lambda}$ ,  $W^s(\bar{\Lambda})$ , and  $W^u(\bar{\Lambda})$  with the Poincaré section by  $\Lambda$ ,  $W^s(\Lambda)$ , and  $W^u(\Lambda)$ , respectively. The 4-dimensional Poincaré section is chosen such that the intersection of these manifolds with the Poincaré section is isoenergetic, and  $\Lambda$  is 2-dimensional,  $W^s(\Lambda)$  is 3-dimensional, and  $W^u(\Lambda)$  is 3-dimensional.

iii) There exist two circles  $C_0$  and  $C_1$  belonging to  $\Lambda$  and invariant under  $f$ , possessing 2 dimensional stable and unstable manifolds.

iv) We assume that a  $\lambda$ -lemma is valid for the partially hyperbolic invariant circles  $C_0$  and  $C_1$ .

Assumptions iv) can be made more precise as there already exists many versions of the  $\lambda$ -lemma for partially hyperbolic tori. We refer in particular to [6],[7],[18] and [12], which

is the most general.

We denote by  $A$  the invariant normally hyperbolic annulus whose boundaries are  $C_0$  and  $C_1$ .  $W^s(A) \subset W^s(\Lambda)$  is the stable of  $A$  and  $W^u(A) \subset W^u(\Lambda)$  is the unstable manifold of  $A$  respectively. We define the boundary of  $W^s(A)$  to be  $W^s(C_0)$  and  $W^s(C_1)$  and the boundary of  $W^u(A)$  to be  $W^u(C_0)$  and  $W^u(C_1)$ . The important point here is that even though  $A$  has a boundary, it is still invariant with respect to both directions of time. This is because its boundary is an invariant manifold. Similarly,  $W^s(A)$  and  $W^u(A)$  are also invariant manifolds (cf. with inflowing and outflowing invariant manifolds with boundary described in [28]).

Our main result is the following:

**Theorem 3** ( $\lambda$ -lemma for normally hyperbolic annuli). *Let  $H$  denote the Hamiltonian for a three degree of freedom Hamiltonian system satisfying assumptions i)-iv). Let  $\Delta$  be a 3 dimensional manifold intersecting  $W^s(\Lambda)$  transversally. We assume that there exists a subset  $\tilde{\Delta} \subset \Delta$ , such that  $\tilde{\Delta}$  intersects  $W^s(A)$  transversally, and such that the boundaries  $\partial\tilde{\Delta}_0$  and  $\partial\tilde{\Delta}_1$  intersect transversally the stable manifolds of  $C_0$  and  $C_1$  respectively. Then, for all  $\epsilon > 0$ , there exists a positive integer  $K$  such that for all  $k \geq K$ ,  $f^k(\Delta)$  is  $C^1$   $\epsilon$ -close to  $W^u(A)$ .*

The proof follows from our previous Theorem 2 for the noncompact case and the  $\lambda$ -lemma for partially hyperbolic tori to control the boundaries of  $\tilde{\Delta}$ . In considering the proof of Theorem 2 one sees that the difficulty arising for invariant manifolds is that iterates of points may leave the manifold by crossing the boundary. This is dealt with here by choosing the boundary of the manifold to also be (lower dimensional) invariant manifold(s).

*Proof.* Since the boundary of  $W^s(A)$  is invariant, we know that under iteration by  $f$   $\tilde{\Delta} \cap W^s(A)$  is always contained in  $W^s(A)$ . Hence we can use the  $\lambda$ -lemma proven in Theorem 2 for non-compact normally hyperbolic invariant manifolds to conclude that for all  $\epsilon > 0$ , there exists a positive integer  $K$  such that for all  $k \geq K$ ,  $f^k(\tilde{\Delta})$  is  $C^1$   $\epsilon$ -close to  $W^u(A)$ .

We also need to show that the boundaries of  $\tilde{\Delta}$  correctly accumulate on the boundaries of  $W^u(A)$  which are formed by  $W^u(C_0)$  and  $W^u(C_1)$ . This follows by applying the  $\lambda$ -lemma for partially hyperbolic tori of Fontich and Martin [12] to  $C_0$  and  $C_1$ . It follows from this lemma that for all  $\epsilon > 0$ , there exists a positive integer  $K'$  such that for all  $k \geq K'$ ,  $f^k(\partial\tilde{\Delta}_i)$  is  $C^1$   $\epsilon$ -close to  $W^u(C_i)$  for  $i = 0, 1$ . This concludes the proof.  $\square$

We remark that if one is in a neighborhood of  $\Lambda$  where the manifolds are “straightened” it is likely that one can construct  $\tilde{\Delta}$  as a graph over  $W^u(A)$  such that it intersects  $W^s(A)$  transversally, and such that the boundaries  $\partial\tilde{\Delta}_0$  and  $\partial\tilde{\Delta}_1$  intersect transversally the stable manifolds of  $C_0$  and  $C_1$ , respectively,

In the following section we prove that our assumptions are satisfied in a large class of near integrable Hamiltonian systems.

**3.2. An example.** In order to construct a normally hyperbolic cylinder for a near integrable Hamiltonian system, we follow the same geometrical set-up as in the seminal paper of Arnold [1]. We emphasize that the results described above are not perturbative in nature, but we have in mind near-integrable systems as a setting where our results provide a key ingredient for proving the existence of symbolic dynamics an instability mechanism.

**3.2.1. Variation around the Arnold example.** We consider the near integrable Hamiltonian system

$$(12) \quad H_{\epsilon,\mu}(p, I, J, q, \theta, \phi) = \frac{1}{2}p^2 + \frac{1}{2}I^2 + J + \epsilon(\cos q - 1) + \epsilon f(\theta, \phi) + \mu(\sin q)^{\alpha(\nu,\sigma)} g(\theta, \phi),$$

where as usual,  $(I, J, \theta, \phi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}$  are action-angle variables,  $(p, q) \in \mathbb{R} \times \mathbb{T}$ ,  $0 < \epsilon \ll 1$  is a small parameter and  $\mu$  is such that  $0 < \mu \ll \epsilon$ , and  $f$  and  $g$  are two given smooth functions, and  $\nu \in \mathbb{N}$  is a parameter controlling the order of contact between  $H_{\epsilon,\mu}$  and  $H_{\epsilon,0}$  via the function  $\alpha(\nu, \sigma) = 2 \left[ \frac{\log \nu}{4\sigma} + 1 \right]$  introduced in ([16], equation (2.5)),  $\sigma > 0$  and  $\nu \geq \nu_\sigma$  where  $\nu_\sigma$  is the smaller positive integer such that  $\alpha(\nu) = 2$  and  $\alpha(\nu, \sigma) \geq 2$  for  $\nu \geq \nu_\sigma$ .

For  $\epsilon = \mu = 0$  the system is completely integrable and the set

$$(13) \quad \bar{\Lambda} = \{(p, I, J, q, \theta, \phi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T} \times \mathbb{T} \mid p = q = 0\},$$

is invariant under the flow and normally hyperbolic. The dynamics on  $\bar{\Lambda}$  is given by the completely integrable Hamiltonian system

$$(14) \quad H(I, \theta) = \frac{1}{2}I^2 + J.$$

As a consequence, the set  $\bar{\Lambda}$  is foliated by invariant 2-tori.

For  $\mu = 0$  and  $\epsilon \neq 0$ , the set  $\bar{\Lambda}$  persists, but the dynamics on  $\bar{\Lambda}$  is no longer integrable. In particular, it is not foliated by invariant two tori. However, the KAM theorem applies and we have a Cantor set of invariant two tori whose measure tends to the full measure when  $\epsilon$  goes to zero. As a consequence, we are in a situation where the “large gaps”

problem arises ([17]), contrary to the well known example of Arnold [1] where all the foliation by invariant two tori is preserved under the perturbation.

Let  $h$  be a given real number. We denote by  $\mathcal{H}_{\epsilon,0} = H_{\epsilon,0}^{-1}(h)$  the energy manifold. There exists a global cross-section to the flow denoted by  $S$  and defined by

$$(15) \quad S = \{(p, I, J, q, \theta, \phi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T} \times \mathbb{T} \mid \phi = 0\}.$$

We denote by  $\Lambda$  the intersection of  $\bar{\Lambda}$  with  $S \cap \mathcal{H}_{\epsilon,0}$ . We can find a symplectic analytic coordinate system on  $S \cap \mathcal{H}_{\epsilon,0}$ , denoted by  $(x, y, s, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  such that the set  $\Lambda$  is defined by

$$(16) \quad \Lambda = \{(x, y, s, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}; s = u = 0\}.$$

Geometrically,  $\Lambda$  is a cylinder. This cylinder is a normally hyperbolic boundaryless manifold, but not compact.

We can introduce the compact counterpart, which is a normally hyperbolic annulus, but now with boundaries. Let  $T_0$  and  $T_1$  be two invariant 2-dimensional partially hyperbolic tori belonging to  $\bar{\Lambda}$ . We denote by  $C_0$  and  $C_1$  the intersection of  $T_0$  and  $T_1$  with  $S \cap \mathcal{H}_{\epsilon,0}$ . The invariant circles  $C_0$  and  $C_1$  are defined as

$$(17) \quad C_i = \{(x, y, s, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}; y = y_i, s = u = 0\},$$

for  $y_i \in \mathbb{R}$  well chosen,  $i = 0, 1$ . We assume in the following that  $y_0 < y_1$ .

The compact counterpart of  $\Lambda$  is then defined as

$$(18) \quad A = \{(x, y, s, u) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}; y_0 \leq y \leq y_1, s = u = 0\}.$$

Let  $P$  be the Poincaré first return map associated to  $S \cap \mathcal{H}_{\epsilon,0}$ . The dynamics on  $A$  is given by an  $\epsilon$  perturbation of an analytic twist map, i.e. that  $P|_A$  is defined by

$$(19) \quad P|_A(x, y) = (x + \omega(y), y) + \epsilon r(x, y),$$

where  $\omega'(y) \neq 0$  and  $r(x, y_i) = 0$  for  $i = 0, 1$ .

When  $\epsilon \neq 0$  and  $\mu \neq 0$ , then the set  $\Lambda$  persists since the perturbation vanishes on  $\Lambda$ . However, the stable and unstable manifold of  $\Lambda$  intersect transversally for a well chosen perturbation  $g$ .

We then obtain an example of a three degrees of freedom Hamiltonian system satisfying the geometrical assumptions i), iii) and iv) of section 3.1. We prove in the following section that the analytic assumption ii) is satisfied.

**3.2.2. Poincaré section and normal form.** The main problem is to prove that the diffeomorphism of the cross-section to the flow defined in a neighbourhood of  $\Lambda$  is in normal form with respect to definition 1.

The basic theorem which we use to obtain a smooth normal form near a normally hyperbolic manifold is a generalized version of the *Sternberg linearization theorem*. In the compact case, this result has been obtained by Bronstein and Kopanskii ([3], theorem 2.3,p.334). The non compact case has been proven by P. Lochak and J-P. Marco [16]. This theorem, which can be stated for flows or maps, ensures that we can obtain a conjugacy as smooth as we want between  $\phi_{\epsilon,\mu}$  and  $\phi_{\epsilon,0}$  by choosing the two flows with a sufficiently high order of contact (see [3],p.334). A key remark in our case is that the flows  $\phi_{\epsilon,0}$  and  $\phi_{\epsilon,\mu}$  generated by  $H_{\epsilon,0}$  and  $H_{\epsilon,\mu}$ , respectively, have contact of order  $\alpha(\nu,\sigma)$  on  $\bar{\Lambda}$ . Moreover, we can obtain an arbitrary order of contact between the two flows by choosing the parameter  $\nu$  sufficiently large. As a consequence, we will always be able to realize the assumptions of the Sternberg linearization theorem for normally hyperbolic manifolds and as a consequence, to obtain a normal form as smooth as we desire.

Before stating the normal form theorem, which is only a minor modification of the result of Lochak-Marco (see [16], Theorem D), we introduce some notation:

As a general notation, for any manifold  $M$ , we denote by  $M_\rho$  a tubular neighbourhood of  $M$  of radius  $\rho$ . We let  $f_\nu$  and  $f_*$  denote the Poincaré maps defined in a neighborhood of  $\Lambda$  obtained from the flows generated by the Hamiltonians  $H_{\epsilon,\mu}$  and  $H_{\epsilon,0}$ , respectively.

Using the Sternberg linearization theorem for normally hyperbolic manifolds proved in [16], we obtain the following theorem:

**Theorem 4** (Normal form). *For  $\nu_0$  large enough, there exist  $\rho, \rho'$  with  $0 < \rho' < \rho$  such that for all  $\nu \geq \nu_0$ ,  $f_\nu(\Lambda_{2\rho'}) \subset \Lambda_\rho$  and there exists a  $C^k$  diffeomorphism  $\phi_\nu$  ( $k \geq 1$ ) satisfying  $\Lambda_{\rho'} \subset \phi_\nu(\Lambda_{2\rho'}) \subset \Lambda_\rho$  and  $\phi_\nu \circ f_* = f_\nu \circ \phi_\nu$  on  $\Lambda_{\rho'}$ . Moreover, there exists a constant  $a$  ( $0 < a < 1$ ) such that  $\|\phi_\nu^{\pm 1} - Id\|_{C^k} \leq a^{\alpha(\nu)}$ , where  $\|\cdot\|_{C^k}$  denotes the  $C^k$  norm on  $\Lambda_{\rho'}$ .*

This theorem is a direct application of a result of Lochak-Marco [16], and we refer to their paper for more details (in particular Theorem D).

Theorem 4 implies that the  $\lambda$ -lemma proven in Theorem 3 applies to the three degree-of-freedom Hamiltonian systems defined by (12).

In this case one can, moreover, prove an analogue of the Smale-Birkhoff theorem using the fact that symbolic dynamics is stable under small  $C^1$  perturbations and taking for the Poincaré map near the normally hyperbolic invariant annulus the linear mapping. A complete study of this problem will be done in [9] in the context of homoclinic normally hyperbolic invariant manifolds.

#### 4. PROOF OF THE $\lambda$ -LEMMA

**4.1. Preliminaries.** In this section we develop the set-up for the proof of the  $\lambda$ -lemma. First we discuss some useful consequences of the normal form assumptions for the diffeomorphism  $f$ .

- We denote by  $Dr(p)$  the differential of  $r$  at point  $p \in U$ . By invariance of  $M$  (assumption a) in section 2.2) we have  $Dr(p) = 0$  for all  $p \in M$ . Let  $0 < k < 1$ , since  $r$  is a  $C^1$  function, we have for  $U$  sufficiently small and for all  $p \in U$ ,

$$(20) \quad \|Dr(p)\| \leq k.$$

This implies in particular that for all  $p \in U$ , the partial derivatives  $\partial_i r_j(p)$ ,  $i, j \in \{s, u, x\}$  satisfy  $\|\partial_i r_j(p)\| \leq k$ .

We take  $U$  sufficiently small in order to have the following inequalities satisfied for  $k$ :

$$(21) \quad 0 < \lambda + k < 1,$$

$$(22) \quad \lambda^{-1} - k > 1.$$

- The straightening conditions b) and c) imply that:

$$(23) \quad \begin{aligned} \forall p \in W^s(M) \cap U, \quad \partial_s r_u(p) &= \partial_x r_u(p) = 0, \\ \forall p \in W^u(M) \cap U, \quad \partial_u r_s(p) &= \partial_x r_s(p) = 0. \end{aligned}$$

- The conjugacy assumption d) implies that:

$$(24) \quad \begin{aligned} \forall p \in W^s(M) \cap U, \quad \partial_s r_x(p) = \partial_x r_x(p) = 0, \\ \forall p \in W^u(M) \cap U, \quad \partial_u r_x(p) = \partial_x r_x(p) = 0. \end{aligned}$$

**4.2. Notation.** Let  $v_0$  be a unit vector in the tangent bundle to  $\Delta_n$ . We denote the components of  $v_0$  as  $v_0 = (v_0^s, v_0^u, v_0^x)$  where  $v_0^s \in E^s$ ,  $v_0^u \in E^u$  and  $v_0^x \in TM$ . We denote by  $v_n$  the iterate of  $v_0$  under  $Df^n$ , i.e.  $v_n = Df^n(v_0)$  and  $v_n = (v_n^s, v_n^u, v_n^x)$ . Let  $p = (s, u, x)$ , we denote by  $p_n = f^n(p)$  the  $n$ -th iterate of  $p$  and  $p_n = (s_n, u_n, x_n)$  its components. We identify  $p_0$  and  $p$ .

Let  $p \in U$ , then using (1) we have

$$(25) \quad Df_p = \begin{pmatrix} A_s(x) + \partial_s r_s(p) & \partial_u r_s(p) & \partial_x r_s(p) + \partial_x A_s(x) s \\ \partial_s r_u & A_u(x) + \partial_u r_u(p) & \partial_x r_u(p) + \partial_x A_u(x) u \\ \partial_s r_x(p) & \partial_u r_x(p) & \partial_x g(x) + \partial_x r_x(p) \end{pmatrix}.$$

We are going to prove that for all  $v_0 \in T\Delta_n$ , and for  $n$  sufficiently large, we have

$$(26) \quad \lim_{n \rightarrow +\infty} \sup \{I_n^x, I_n^s\} = 0,$$

where the *inclinations* are defined as:

$$(27) \quad I_n^x \equiv \frac{|v_n^x|}{|v_n^u|}, \quad I_n^s \equiv \frac{|v_n^s|}{|v_n^u|},$$

and

$$(28) \quad \frac{|v_{n+1}^u|}{|v_n^u|} > 1.$$

(26) implies that under iteration arbitrary tangent vectors align with tangent vectors to the unstable manifold and (28) implies that these tangent vectors also grow in length.

The proof essentially involves three steps. First we prove (28) for tangent vectors in  $\Delta_n \cap W^s(M)$ . Next we extend this result to tangent vectors in  $\Delta_n$ . Finally, we prove (28).

**4.3. Inclinations for tangent vectors in the stable manifold.** We first prove (26) for  $v_0$  in the tangent bundle of  $\Delta_n \cap W^s(M)$ , denoted  $T_{W^s(M)}\Delta_n$ . Let  $p \in W^s(M)$ , then by the invariance property of the stable manifold (25) simplifies to:

$$(29) \quad Df_p = \begin{pmatrix} A_s(x) + \partial_s r_s(p) & \partial_u r_s(p) & \partial_x r_s(p) + \partial_x A_s(x) s \\ 0 & A_u(x) + \partial_u r_u(p) & 0 \\ 0 & \partial_u r_x(p) & \partial_x g(x) \end{pmatrix}.$$

Acting on the tangent vector  $(v_n^s, v_n^u, v_n^x)$ , we obtain the following relations

$$(30) \quad v_{n+1}^s = (A_s(x_n) + \partial_s r_s(p_n))v_n^s + \partial_u r_s(p_n)v_n^u + (\partial_x r_s(p_n) + \partial_x A_s(x_n) s_n)v_n^x,$$

$$(31) \quad v_{n+1}^u = (A_u(x_n) + \partial_u r_u(p_n))v_n^u,$$

$$(32) \quad v_{n+1}^x = \partial_x r_u(p_n)v_n^u + \partial_x g(x_n)v_n^x.$$

Using these expressions, along with the estimates (20), (21), and (22), we then obtain

$$(33) \quad |v_{n+1}^s| \leq (\lambda + k) |v_n^s| + \|\partial_u r_s(p_n)\| |v_n^u| + (k + \|\partial_x A_s(x_n)\| |s_n|) |v_n^x|,$$

$$(34) \quad |v_{n+1}^u| \geq (\lambda^{-1} - k) |v_n^u|,$$

$$(35) \quad |v_{n+1}^x| \leq \|\partial_u r_x(p_n)\| |v_n^u| + k |v_n^x|.$$

4.3.1. *Inclination in the tangential direction.* Using (35), (34) and (22) gives:

$$(36) \quad \frac{|v_{n+1}^x|}{|v_{n+1}^u|} \leq \frac{k}{\lambda^{-1} - k} \frac{|v_n^x|}{|v_n^u|} + \|\partial_u r_x(p_n)\|.$$

Using the estimate(9) on the second derivatives with the mean value inequality gives:

$$(37) \quad \frac{|v_{n+1}^x|}{|v_{n+1}^u|} \leq \frac{k}{\lambda^{-1} - k} \frac{|v_n^x|}{|v_n^u|} + C |s_n|.$$

Let  $p = (s, 0, x) \in W^s(M) \cap U$  and  $p_n = f^n(p) = (s_n, 0, x_n)$ . By definition, we have  $s_{n+1} = A_s(x_n)s_n + r_s(s_n, 0, x_n)$ . Estimating this expression using assumption d) of section 1, as well as the mean value inequality with (20) gives:

$$(38) \quad |s_{n+1}| \leq (\lambda + k) |s_n|,$$

from which it follows that:

$$(39) \quad |s_n| \leq (\lambda + k)^n |s|.$$

Replacing  $|s_n|$  by this expression in (37), we obtain

$$(40) \quad \frac{|v_{n+1}^x|}{|v_{n+1}^u|} \leq \frac{k}{\lambda^{-1} - k} \frac{|v_n^x|}{|v_n^u|} + C |s| (\lambda + k)^n.$$

As a consequence, we have

$$(41) \quad \frac{|v_n^x|}{|v_n^u|} \leq \left( \frac{k}{\lambda^{-1} - k} \right)^n \frac{|v_0^x|}{|v_0^u|} + C |s| \sum_{i=0}^{n-1} \left( \frac{k}{\lambda^{-1} - k} \right)^i (\lambda + k)^{n-1-i}.$$



From (21) and (22) it follows that:

$$(42) \quad \frac{k}{\lambda^{-1} - k} < k + \lambda,$$

Using this we obtain

$$(43) \quad \frac{|v_n^x|}{|v_n^u|} \leq \left( \frac{k}{\lambda^{-1} - k} \right)^n \frac{|v_0^x|}{|v_0^u|} + C |s| n(\lambda + k)^{n-1}.$$

From (21) we have  $\lambda + k < 1$ , and therefore:

$$(44) \quad \lim_{n \rightarrow +\infty} \frac{|v_n^x|}{|v_n^u|} = 0.$$

4.3.2. *Inclination in the stable direction.* Using (33) and (34), we obtain

$$(45) \quad \frac{|v_{n+1}^s|}{|v_{n+1}^u|} \leq \left( \frac{\lambda + k}{\lambda^{-1} - k} \right) \frac{|v_n^s|}{|v_n^u|} + \|\partial_u r_s(p_n)\| + (k + \|\partial_x A_s(x_n)\| |s_n|) \frac{|v_n^x|}{|v_n^u|}.$$

Using the assumption (9), the mean value inequality, and (39) gives:

$$(46) \quad \|\partial_u r_s(s_n, 0, x_n)\| \leq C |s_n| \leq C |s| (\lambda + k)^n.$$

Moreover, recall that from assumption iii) in the statement of the  $\lambda$ -lemma we have:

$$(47) \quad \|\partial_x A_s(x_n)\| \leq D.$$

Using these two estimates, (45) becomes:

$$(48) \quad \frac{|v_{n+1}^s|}{|v_{n+1}^u|} \leq \left( \frac{\lambda + k}{\lambda^{-1} - k} \right) \frac{|v_n^s|}{|v_n^u|} + C |s| (\lambda + k)^n + (k + D |s| (\lambda + k)^n) \frac{|v_n^x|}{|v_n^u|}.$$

As a preliminary step to estimating (48), we first estimate the third term on the right-hand-side of (48) using (43):

$$(49) \quad (k + D |s| (\lambda + k)^n) \frac{|v_n^x|}{|v_n^u|} \leq (k + D |s| (\lambda + k)^n) \left( \left( \frac{k}{\lambda^{-1} - k} \right)^n \frac{|v_0^x|}{|v_0^u|} + C |s| n(\lambda + k)^{n-1} \right)$$

Now for  $n$  sufficiently large we have:

$$(50) \quad k + D |s| (\lambda + k)^n < 1,$$

and by assumption (22) we have  $\lambda^{-1} - k > 1$ , and therefore:

$$(51) \quad (k + D |s| (\lambda + k)^n) \frac{|v_n^x|}{|v_n^u|} \leq k^n \frac{|v_0^x|}{|v_0^u|} + C |s| n (\lambda + k)^{n-1}.$$

Substituting this expression into (48) gives:

$$(52) \quad \begin{aligned} \frac{|v_{n+1}^s|}{|v_{n+1}^u|} &\leq \left( \frac{\lambda + k}{\lambda^{-1} - k} \right) \frac{|v_n^s|}{|v_n^u|} + C |s| (\lambda + k)^n + C |s| n (\lambda + k)^{n-1} + k^n \frac{|v_0^x|}{|v_0^u|}, \\ &\leq \left( \frac{\lambda + k}{\lambda^{-1} - k} \right) \frac{|v_n^s|}{|v_n^u|} + (\lambda + k)^{n-1} \left( C |s| (1 + n) + \frac{|v_0^x|}{|v_0^u|} \right) \end{aligned}$$

From this expression we obtain:

$$(53) \quad \begin{aligned} \frac{|v_{n+1}^s|}{|v_{n+1}^u|} &\leq \left( \frac{\lambda + k}{\lambda^{-1} - k} \right)^n \frac{|v_0^s|}{|v_0^u|} + \sum_{i=0}^{n-1} \left( \frac{\lambda + k}{\lambda^{-1} - k} \right)^i \left( (\lambda + k)^{n-i-2} \left( C |s| (n - i) + \frac{|v_0^x|}{|v_0^u|} \right) \right), \\ &\leq \left( \frac{\lambda + k}{\lambda^{-1} - k} \right)^n \frac{|v_0^s|}{|v_0^u|} + (\lambda + k)^{n-2} n \left( C |s| + \frac{|v_0^x|}{|v_0^u|} \right) \end{aligned}$$

Consequently, as  $n$  goes to infinity, we have

$$(54) \quad \lim_{n \rightarrow \infty} \frac{|v_n^s|}{|v_n^u|} = 0.$$

**4.4. Extending the estimates to tangent vectors not in  $T_{W^s(M)}\Delta_n$ .** We have shown that for  $n$  sufficiently large,  $U$  sufficiently small, for all  $v \in T_{W^s(M)}\Delta_n$ ,  $|v| = 1$ ,

$$(55) \quad I(v) \leq \epsilon.$$

By continuity of the tangent plane, there exists  $\tilde{\Delta}_n \subset \Delta_n$  such that

$$(56) \quad \forall v \in T\tilde{\Delta}_n, \quad I(v) \leq 2\epsilon.$$

For  $n$  sufficiently large  $\Delta_n$  is very close to  $W^u(M)$ . We choose a neighbourhood  $V_{\epsilon_s}$  of  $W^u(M)$  of the form

$$(57) \quad V_{\epsilon_s} = \{(s, u, x) \in B_{\epsilon_s}^s \times B_\rho^u \times M\},$$

for  $0 < \epsilon_s < 1$ . Recall the estimates

$$(58) \quad \forall p \in U, \quad \begin{aligned} \|\partial_u r_x(p)\| &\leq \delta, & \|\partial_x r_x(p)\| &\leq \delta, \\ \|\partial_u r_s(p)\| &\leq \delta, & \|\partial_x r_s(p)\| &\leq \delta. \end{aligned}$$

Using the mean value theorem and (9), we choose  $\epsilon_s$  such that:

$$(59) \quad \delta \geq C\epsilon_s.$$

As a consequence, we can take

$$(60) \quad \delta = (C + 1)\epsilon_s.$$

Using (25), analogously to the estimates above we obtain the following:

$$(61) \quad |v_{n+1}^s| \leq (\lambda + k) |v_n^s| + \delta |v_n^u| + (C\epsilon_s + \delta) |v_n^x|,$$

$$(62) \quad |v_{n+1}^u| \geq (\lambda^{-1} - k) |v_n^u| - k |v_n^s| - (k + \rho C) |v_n^x|,$$

$$(63) \quad |v_{n+1}^x| \leq k |v_n^s| + \delta |v_n^u| + (\|\partial_x g(x_n)\| + \delta) |v_n^x|.$$

We use these expressions to obtain the following estimates of the inclinations:

$$(64) \quad I_{n+1}^x \leq \frac{1}{\lambda^{-1} - k} [kI_n^s + \delta + (\|\partial_x g(x_n)\| + \delta)I_n^x] \mu,$$

$$(65) \quad I_{n+1}^s \leq \frac{1}{\lambda^{-1} - k} [(\lambda + k)I_n^s + \delta + (C\epsilon_s + \delta)I_n^x] \mu,$$

where

$$(66) \quad \mu^{-1} = 1 - \frac{k}{\lambda^{-1} - k} I_n^s - \frac{k + C\rho}{\lambda^{-1} - k} I_n^x.$$

Since  $\delta = (C + 1)\epsilon_s$ , and  $\tilde{C} \geq \sup\{\|\partial_{i,x}^2 g(z)\|, z \in U_\rho\}$ , we obtain the following estimates:

$$(67) \quad (\|\partial_x g(x_n)\| + \delta)I_n^x \leq \epsilon_s(\tilde{C} + C + 1)I_n^x,$$

$$(68) \quad \delta + (C\epsilon_s + \delta)I_n^x \leq \epsilon_s[C + 1 + I_n^x(2C + 1)].$$

We substitute these estimates into (64) and (65), and *assuming* that

$$(69) \quad I_n^x \leq \epsilon \quad \text{and} \quad I_n^s \leq \epsilon,$$

we obtain:

$$(70) \quad I_{n+1}^x \leq \frac{\mu_*}{\lambda^{-1} - k} (k\epsilon + (C + 1)\epsilon_s + \epsilon\epsilon_s(M + C + 1)),$$

$$(71) \quad I_{n+1}^s \leq \frac{\mu_*}{\lambda^{-1} - k} ((\lambda + k)\epsilon + \epsilon_s(C + 1 + (2C + 1)\epsilon)),$$

where  $\mu_*^{-1} = 1 - \frac{2k + C\rho}{\lambda^{-1} - k}\epsilon$ .

Now if we choose  $k$  small enough such that:

$$(72) \quad \frac{\lambda + k}{\lambda^{-1} - k} \mu_* < 1,$$

and  $\epsilon_s$  satisfies:

$$(73) \quad \epsilon_s \leq \inf \begin{cases} \epsilon \left( \frac{\lambda^{-1} - k}{\mu_*} \right) \left( 1 - \frac{k \mu_*}{\lambda^{-1} - k} \right) \left( \frac{1}{C+1+\epsilon(M+C+1)} \right), \\ \epsilon \left( 1 - \frac{\lambda+k}{\lambda^{-1}-k} \mu_* \right) \frac{1}{C+1+(2C+1)\epsilon} \end{cases}$$

then

$$(74) \quad I_{n+1}^x \leq \epsilon,$$

$$(75) \quad I_{n+1}^s \leq \epsilon.$$

Hence, we have shown that for  $k$  and  $\epsilon_s$  sufficiently small,  $I_{n+1}^x \leq \epsilon$  and  $I_{n+1}^s \leq \epsilon$ . Therefore the estimates  $I_n^s \leq \epsilon$  and  $I_n^x \leq \epsilon$  are maintained under iteration.

As  $\epsilon$  can be chosen as small as we want, the inclinations  $I_n^s$  and  $I_n^x$  is as small as we want for  $n$  sufficiently large.

**4.5. Stretching along the unstable manifold.** We want to show that  $f^n(\Delta) \cap U$  is stretched in the direction  $W^u(M)$ . In order to see this we compare the norm of a tangent vector in  $\tilde{\Delta}_n$  with its image under  $Df$ :

$$(76) \quad \sqrt{\frac{|v_{n+1}^s|^2 + |v_{n+1}^u|^2 + |v_{n+1}^x|^2}{|v_n^s|^2 + |v_n^u|^2 + |v_n^x|^2}} = \frac{|v_{n+1}^u|}{|v_n^u|} \sqrt{\frac{1 + (I_{n+1}^s)^2 + (I_{n+1}^x)^2}{1 + (I_n^s)^2 + (I_n^x)^2}}.$$

Using (62) we obtain:

$$(77) \quad \frac{|v_{n+1}^u|}{|v_n^u|} \geq \lambda^{-1} - k - k\epsilon - (k + C\rho)\epsilon.$$

Since  $\epsilon$  can be chosen arbitrarily small we have:

$$(78) \quad \lambda^{-1} - k - k\epsilon - (k + C\rho)\epsilon \geq \lambda^{-1} - 2k > 1.$$

Since the inclinations are arbitrarily small, it follows that the norms of (nonzero) vectors in  $\tilde{\Delta}_n$  are growing by a ratio that approaches  $\lambda^{-1} - 2k > 1$ . Therefore the diameter of  $\tilde{\Delta}_n$  is increasing. Putting this together with the fact that the tangent spaces have uniformly small slope implies that there exists  $\tilde{n}$  such that for all  $n \geq \tilde{n}$   $\tilde{\Delta}_n$  is  $C^1$   $\epsilon$ -close to  $W^u(M)$ .

This concludes the proof.

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UNIVERSITÉ DE FRANCHE-COMTÉ, ÉQUIPE DE MATHÉMATIQUES DE BESANÇON, CNRS-UMR 6623,  
16 ROUTE DE GRAY, 25030 BESANÇON CEDEX, FRANCE.  
*E-mail address:* `cresson@math.univ-fcomte.fr`

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, UNIVERSITY WALK, BRISTOL, BS8 1TW,  
UNITED KINGDOM.  
*E-mail address:* `S.Wiggins@bristol.ac.uk`