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# Mukai flops and deformations of symplectic resolutions

Baohua Fu

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## Abstract

We prove that two projective symplectic resolutions of  $\mathbb{C}^{2n}/G$  are connected by Mukai flops in codimension 2 for a finite sub-group  $G < \mathrm{Sp}(2n)$ . It is also shown that two projective symplectic resolutions of  $\mathbb{C}^4/G$  are deformation equivalent.

## 1 Introduction

A *symplectic variety* is a complex normal variety  $W$  with a holomorphic symplectic form on its smooth part which can be extended to a global holomorphic form on any resolution. A resolution  $Z \rightarrow W$  of  $W$  is called *symplectic* if the lifted holomorphic 2-form is again symplectic on  $Z$ .

Examples of symplectic varieties include the normalization of a nilpotent orbit closure in a semi-simple complex Lie algebra and the quotient of  $\mathbb{C}^{2n}$  by a finite subgroup  $G < \mathrm{Sp}(2n)$ . The purpose of this paper is to study projective symplectic resolutions of  $\mathbb{C}^{2n}/G$ .

One way of constructing a symplectic resolution from another is to perform Mukai flops. This process can be described as follows: let  $W$  be a symplectic variety and  $\pi : Z \rightarrow W$  a symplectic resolution. Assume that  $W$  contains a smooth closed subvariety  $Y$  and that  $\pi^{-1}(Y)$  contains a smooth subvariety  $P$  such that the restriction of  $\pi$  to  $P$  makes  $P$  into a  $\mathbb{P}^l$ -bundle over  $Y$ . If  $\mathrm{codim}(P) = l$ , we can blow up  $Z$  along  $P$  and then blow down along the other direction, which gives another (proper) symplectic resolution  $\pi^+ : Z^+ \rightarrow W$ . Notice that the resulting variety  $Z^+$  may be not algebraic.

Sometimes one needs to perform simultaneously several Mukai flops to obtain a projective morphism  $\pi^+$ . The diagram  $Z \rightarrow W \leftarrow Z^+$  is called a *Mukai flop over  $W$  with center  $P$* . A *Mukai flop in codimension 2* is a diagram which becomes a Mukai flop after removing subvarieties of codimension greater than 2.

As to the birational geometry in codimension 2 of projective symplectic resolutions, one has the following conjecture (due to Hu-Yau [HY]):

**Conjecture 1 (Hu-Yau).** *Any two projective symplectic resolutions of a symplectic variety are connected by Mukai flops in codimension 2.*

This conjecture is true for four-dimensional symplectic varieties by the work of Wierzba and Wiśniewski ([WW]) (partial results had previously been obtained in [BHL], see also [CMSB]) for the existence of flops and by the work of Matsuki [Mat] for the termination of flops. In [Fu], we have verified this conjecture for symplectic resolutions of nilpotent orbit closures. The first result of this note is the following theorem (partial but stronger results had been proven in [Fu]).

**Theorem 1.1.** *Let  $G < \mathrm{Sp}(2n)$  be a finite subgroup. Any two projective symplectic resolutions of  $\mathbb{C}^{2n}/G$  are connected by Mukai flops in codimension 2.*

The idea is to reduce the problem to dimension 4, and then apply [WW]. The main technics (see Proposition 2.1 and Lemma 2.4) for this reduction are already contained in [Ka1]. The proofs here are slightly different.

Then we study deformations of symplectic resolutions. Recall that a *deformation* of a variety  $X$  (usually not compact) is a flat morphism  $\mathcal{X} \xrightarrow{q} S$  from a variety  $\mathcal{X}$  to a pointed smooth connected curve  $0 \in S$  such that  $q^{-1}(0) \simeq X$ . A deformation of a proper morphism  $X \xrightarrow{f} Y$  is an  $S$ -morphism  $\mathcal{X} \xrightarrow{F} \mathcal{Y}$  such that  $\mathcal{X} \rightarrow S$  (resp.  $\mathcal{Y} \rightarrow S$ ) is a deformation of  $X$  (resp.  $Y$ ) and  $F_0 = f$ .

Let  $X \xrightarrow{f} Y \xleftarrow{f^+} X^+$  be two proper morphisms. One says that  $f$  and  $f^+$  are *deformation equivalent* if there exist deformations of  $f$  and  $f^+$ :  $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xleftarrow{F^+} \mathcal{X}^+$  such that for a general point  $s \in S$  the morphisms  $\mathcal{X}_s \xrightarrow{F_s} \mathcal{Y}_s \xleftarrow{F_s^+} \mathcal{X}_s^+$  are isomorphisms. Motivated by results of D. Huybrechts ([Huy2]), it is conjectured in [FN] (see also [Ka2]) that

**Conjecture 2.** *Any two symplectic resolutions of a symplectic variety are deformation equivalent.*

For nilpotent orbit closures of classical type, this conjecture is proved by Y. Namikawa in [Nam] (the case of  $\mathfrak{sl}(n)$  had previously been proved in [FN]). In [Ka2], D. Kaledin constructed the so-called *twister deformation* of a symplectic resolution (under mild assumptions). Combining this with a trick of D. Huybrechts ([Huy1]) and results in [WW], we prove the following

**Theorem 1.2.** *Let  $G < \mathrm{Sp}(4)$  be a finite subgroup. Any two projective symplectic resolutions of  $\mathbb{C}^4/G$  are deformation equivalent.*

This note ends with a study of symplectic resolutions of the wreath product  $W := (\mathbb{C}^2/\Gamma)^{(n)}$ , where  $\Gamma < \mathrm{SL}(2)$  is a finite subgroup. It is conjectured that any two projective symplectic resolutions of  $W$  are connected by Mukai flops with flop center contained in the fiber over  $0 \in W$ . In the case of  $\Gamma$  being of type  $A_k$  and  $n = 2$ , we give a way to describe all possible projective symplectic resolutions of  $W$ .

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## 2 Birational geometry in codimension 2

We begin with the following proposition, which is proved (as is Lemma 2.4 later) in the formal setting by D. Kaledin ([Ka1] Proposition 5.2).

**Proposition 2.1.** *Let  $W$  be a symplectic variety and  $\Delta^{2l}$  the open unit disk in  $\mathbb{C}^{2l}$ . Then any projective symplectic resolution of  $W \times \Delta^{2l}$  is of the form  $Z \times \Delta^{2l} \xrightarrow{\pi} W \times \Delta^{2l}$ , where  $Z \xrightarrow{\pi'} W$  is a symplectic resolution and  $\pi = \pi' \times \mathrm{id}$ .*

*Proof.* Suppose that we have a symplectic resolution  $X \xrightarrow{\pi} W \times \Delta^{2l}$ . For any non-zero vector  $v \in \Delta^{2l}$ , it defines a constant vector field  $\mathfrak{t}_v$  on the smooth part, say  $U$  of  $W \times \Delta^{2l}$ . Furthermore, on  $U$ , one has an isomorphism of sheaves  $\Omega^1 \simeq \mathcal{T}$ , under which the vector field  $\mathfrak{t}_v$  corresponds to a 1-form  $\alpha_v$ . It is easy to show that  $\alpha_v = p_2^* \beta$  for some 1-form  $\beta$  on  $\Delta^{2l}$ , where  $p_2 : W \times \Delta^{2l} \rightarrow \Delta^{2l}$  is the projection to the second factor. In particular,  $\alpha_v$  extends to a well-defined 1-form on the whole of  $W \times \Delta^{2l}$ . Let  $\mathfrak{t}'_v$  be the

vector field on  $X$  corresponding to the 1-form  $\pi^*\alpha_v$  under the isomorphism  $\Omega_X^1 \simeq \mathcal{T}_X$ . Then  $\mathfrak{t}'_v$  is the vector field lifting  $\mathfrak{t}_v$ . Furthermore  $\mathfrak{t}'_v$  vanishes nowhere on  $X$ , thus it defines a holomorphic flow  $\phi_v(t)$  on  $X$  (see the proof of Theorem 1.3 [Ka1]).

Let  $q : X \rightarrow \Delta^{2l}$  be the composition  $p_2 \circ \pi$  and  $Z = q^{-1}(0)$ . Let  $\pi' : Z \rightarrow W$  be the restriction of  $\pi$  to  $Z \rightarrow W \times \{0\}$ . Then the flow  $\phi_v(t)$  satisfies  $q(\phi_v(t)(z)) = tv$  for any  $z \in Z$ . We define a morphism  $Z \times \Delta^{2l} \rightarrow X$  as follows:  $(z, v) \mapsto \phi_v(1)(z)$ . One sees easily that this is an isomorphism. Moreover one has  $\pi(\phi_v(1)(z)) = (\pi'(z), v)$ .

In conclusion, we obtain a decomposition  $X = Z \times \Delta^{2l}$ , a map  $\pi' : Z \rightarrow W$  and an isomorphism  $\pi = \pi' \times \text{id}$ . That  $\pi$  is a symplectic resolution implies that  $Z$  is smooth and  $\pi'$  is a symplectic resolution of  $W$ .  $\square$

The same arguments hold if one replaces  $\Delta^{2l}$  by  $\mathbb{C}^{2l}$ . An immediate corollary is the following (which is also proved in [Ka1] Theorem 1.6):

**Corollary 2.2.** *Let  $V_i$  be a symplectic vector space and  $G_i < \text{Sp}(V_i)$  a finite subgroup,  $i \in \{1, 2\}$ . Then  $V_1/G_1 \times V_2/G_2$  admits a symplectic resolution if and only if  $V_1/G_1$  and  $V_2/G_2$  both admit symplectic resolutions.*

*Proof.* Take a smooth point  $v \in V_1/G_1$  and a neighborhood isomorphic to the unit disk  $\Delta$ . If the product admits a symplectic resolution, so does  $\Delta \times V_2/G_2$ . The precedent proposition then implies that  $V_2/G_2$  admits a symplectic resolution. Similarly  $V_1/G_1$  also admits a symplectic resolution.  $\square$

**Remark 2.3.** We do not know if every projective symplectic resolution of  $V_1/G_1 \times V_2/G_2$  is a product of resolutions of  $V_1/G_1$  and  $V_2/G_2$ . This is true if  $G_1$  or  $G_2$  is trivial by the precedent proposition.

From now on, let  $V$  be a  $2n$ -dimensional symplectic vector space and  $G < \text{Sp}(V)$  a finite subgroup. We denote by  $W$  the quotient space  $V/G$ . We have the rank stratification on  $W$  defined as  $V_k = \{v \in V \mid \text{codim } V^{G_v} = 2k\}$ . The quotient  $W_k = V_k/G$  is a smooth algebraic variety of dimension  $2n - 2k$  and  $W_0$  is the smooth part of  $W$ . Moreover, the projection  $V_k \rightarrow W_k$  is étale (Lemma 4.1 [Ka1]).

Take a component  $Y$  of  $W_k$  and a connected component  $V_Y$  of the preimage of  $Y$  in  $V_k$ . Let  $H$  be the stabilizer of a point in  $V_Y$ . Then  $H$  is independent of the choice of the point and  $V_Y$  is a Zariski open set in the  $H$ -fixed subspace  $V^H$ . Let  $N(H)$  be the normalizer of  $H$  in  $G$  and  $Q(H) = N(H)/H$

the quotient group. One shows that  $V^H$  is  $N(H)$ -invariant. Since  $H$  acts trivially on  $V^H$ , one obtains an action of  $Q(H)$  on  $V^H$ , which is a free action on  $V_Y$  and we have an isomorphism  $V_Y/Q(H) \simeq Y$ .

Let  $V_H$  be the annihilator of  $V^H$  with respect to the symplectic form  $\omega_0$  on  $V$ . Notice that  $\omega_0$  restricted to  $V^H$  is again symplectic, thus one has a decomposition  $V = V^H \oplus V_H$ , which is  $N(H)$ -invariant. Furthermore  $N(H)$  acts symplectically on  $V_H$ , i.e.,  $N(H) < \text{Sp}(V_H)$ . This decomposition induces a morphism  $\mu : (V^H \times V_H/H)/Q(H) \rightarrow V/G$  which maps  $V_Y/Q(H) \times \{0\}$  isomorphically to  $Y$  and  $\mu$  is étale in a Zariski open set containing  $V_Y/Q(H) \times \{0\}$ . For more details, see section 4 of [GK]. This implies (see also Lemma 4.2 [Kal]):

**Lemma 2.4.** *Any point in  $Y$  admits an analytical open neighborhood which is isomorphic to  $\Delta^{2l} \times D_H$ , where  $\Delta^{2l}$  is the unit disk of dimension  $\dim(V^H)$  and  $D_H$  is the image of the unit disk in  $V_H$  under the projection  $V_H \rightarrow V_H/H$ .*

**Remark 2.5.** In the case of  $\dim(V_H) = 4$ , any symplectic resolution of  $D_H$  extends to a symplectic resolution of  $S := V_H/H$ . In fact, outside the zero point,  $S$  has only *ADE* singularities and  $S - \{0\}$  admits a unique symplectic resolution  $\tilde{S} \rightarrow S - \{0\}$ . Now any symplectic resolution of  $D_H$  agrees automatically with  $\tilde{S}$  over  $D_H - \{0\}$ , thus it pastes with  $\tilde{S}$  to a symplectic resolution of  $S$ . If the resolution of  $D_H$  is projective, then the one obtained for  $V_H/H$  is again projective.

Suppose that we have two projective symplectic resolutions  $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$ . Let  $\phi$  be the rational map  $\pi^{-1} \circ \pi^+ : Z^+ \dashrightarrow Z$ .

**Lemma 2.6.** *The rational map  $\phi$  induces an isomorphism from  $(\pi^+)^{-1}(U)$  to  $\pi^{-1}(U)$ , where  $U = W_0 \cup W_1$ .*

*Proof.* By the lemma above, every point  $y \in W_1$  admits a neighborhood  $U_y$  isomorphic to  $\Delta^{2n-2} \times D_H$  for some finite subgroup  $H < \text{SL}(2)$ . By Proposition 2.1, every symplectic resolution of  $U_y$  is a product of  $\Delta^{2n-2}$  with a symplectic resolution of  $D_H$ , while  $D_H$  admits a unique symplectic resolution given by the minimal resolution, thus  $\phi$  is an isomorphism from  $(\pi^+)^{-1}(U_y)$  to  $\pi^{-1}(U_y)$ .  $\square$

**Theorem 2.7.** *Two projective symplectic resolutions of  $W$  are connected by Mukai flops over  $W$  in codimension 2.*

*Proof.* Let  $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$  be two projective symplectic resolutions. By the semi-smallness of symplectic resolutions (Prop. 4.4 [Ka1]),  $\pi^{-1}(\overline{W_3})$  (respectively  $(\pi^+)^{-1}(\overline{W_3})$ ) has codimension at least 3 in  $Z$  (resp.  $Z^+$ ). Since we are interested in the codimension 2 birational geometry, we can replace  $W$  by  $W_0 \cup W_1 \cup W_2$ . By the precedent lemma,  $\phi$  is already an isomorphism over  $W_0 \cup W_1$ .

Take a connected component  $Y$  in  $W_2$  and a point  $y \in Y$ . Then there exists an analytical neighborhood  $U_y$  of  $y$  isomorphic to  $\Delta^{2n-4} \times D_H$  for some finite subgroup  $H < \mathrm{Sp}(4)$ . By proposition 2.1, the projective symplectic resolution  $\pi^{-1}(U_y) \rightarrow U_y$  is isomorphic to the product  $\Delta^{2n-4} \times X \rightarrow U_y$ , where  $X \rightarrow D_H$  is a projective symplectic resolution. Similarly for  $\pi^+$ , one finds another projective symplectic resolution  $X^+ \rightarrow D_H$ . Since  $Y$  is connected,  $X, X^+$  and their morphisms to  $D_H$  are independent of the choice of  $y$ . By Remark 2.5, these two symplectic resolutions come in fact from symplectic resolutions of  $\mathbb{C}^4/H$ .

By [WW] and [Mat], the birational map  $X \dashrightarrow X^+$  is decomposed as a sequence of Mukai flops. Without any loss of generality, one may suppose that  $X \dashrightarrow X^+$  is a Mukai flop with flop center  $P \subset X$ . Since  $X$  is independent of the choice of  $y$ , one can find a subvariety  $E$  in  $Z$  which is a fibration over  $Y$  with fibers isomorphic to  $P$ . By the McKay correspondence (see [Ka3]), irreducible components of codimension 2 in  $\pi^{-1}(Y)$  correspond to dimension 2 components in the central fiber of  $X$ . The subvariety  $E$  is then the irreducible component of codimension 2 in the preimage of  $Y$  corresponding to  $P$ .

Now if we perform a Mukai flop in  $Z$  along  $E$ , one obtains another symplectic resolution  $X' \xrightarrow{\pi'} W$  such that the rational map  $X' \dashrightarrow X^+$  is an isomorphism between preimages of  $Y$ .

Now if we do the same operations for other components in  $W_2$ , one arrives finally at the resolution  $\pi^+$ .  $\square$

We end this section by the following proposition, whose proof is clear.

**Proposition 2.8.** *Let  $W := V/G$  be a quotient symplectic variety. Suppose that for every component  $Y$  in  $W_2$ , the corresponding 4-dimensional quotient  $\mathbb{C}^4/H_Y$  admits a unique projective symplectic resolution. Then any two projective symplectic resolutions of  $W$  are isomorphic in codimension 2.*

The following  $\mathbb{C}^4/G$  admit a unique projective symplectic resolution:

- (i)  $\mathbb{C}^2/G_1 \times \mathbb{C}^2/G_2$  where  $G_1, G_2$  are finite subgroups of  $\mathrm{SL}(2)$ ;

(ii)  $(T^*\mathbb{C}^2)/G$ , where  $G < \mathrm{GL}(2)$  such that  $\{g \mid \mathrm{Fix}(g) = 0\}$  form a single conjugacy class.

Case (i) follows from [WW] since the central fiber contains no copies of  $\mathbb{P}^2$ , while case (ii) is proved in [Fu] (Cor. 1.3).

### 3 Deformation equivalence

Let  $V$  be a  $2n$ -dimensional symplectic vector space and  $G < \mathrm{Sp}(V)$  a finite subgroup. Suppose that we have a projective symplectic resolution  $\pi : Z \rightarrow W := V/G$ . Take a  $\pi$ -ample line bundle  $L$  on  $Z$ . By [Ka2], there exists a twister deformation of  $\pi$  over the formal disk  $\mathrm{Spec}(\mathbb{C}[[x]])$ . Since  $W$  admits an expanding  $\mathbb{C}^*$ -action (i.e., positively weighted) which lifts to  $Z$  via  $\pi$ , this twister deformation extends to an actual deformation over  $S = \mathbb{C}$ , say  $\mathcal{Z} \xrightarrow{\Phi} \mathcal{W}$  (see Lemma A. 15 and Proposition 5.4 [GK]). Furthermore, for a generic  $s \in S$ , the morphism  $\Phi_s : \mathcal{Z}_s \rightarrow \mathcal{W}_s$  is an isomorphism. Moreover, by [Ka2], the Kodaira-Spencer class  $v$  of the deformation  $\mathcal{Z} \rightarrow S$  is nothing but  $c_1(L) \in H^1(Z, T_Z) \simeq H^1(Z, \Omega_Z)$ .

Let  $P \subset Z$  be a subvariety isomorphic to  $\mathbb{P}^n$ . Denote by  $\bar{v}$  the image of the Kodaira-Spencer class  $v$  under the morphism  $H^1(Z, \Omega_Z) \rightarrow H^1(P, \Omega_P)$ . The following lemma is a special case of Lemma 3.6 [Huy1]. We omit the proof here.

**Lemma 3.1.** *If  $\bar{v}$  is non-zero, then  $\mathcal{N}_{P|Z} \simeq \mathcal{O}_P(-1)^{\oplus n+1}$ .*

Let  $p : \tilde{\mathcal{Z}} \rightarrow \mathcal{Z}$  be the blow up of  $\mathcal{Z}$  along  $P$ . Under the assumption of the precedent lemma, the exceptional divisor  $E$  is isomorphic to  $\mathbb{P}(\mathcal{O}_P(-1)^{\oplus n+1}) = P \times P^*$ , where  $P^*$  is the dual of  $P$ , and the normal bundle  $\mathcal{N}_{E|\mathcal{Z}}$  is the tautological bundle. In particular, the restriction of  $\mathcal{O}_{\tilde{\mathcal{Z}}}(E)$  to any fiber of  $P \times P^* \rightarrow P^*$  is  $\mathcal{O}(-1)$ . By Nakano-Fujiki criterion, there exists a contraction  $\tilde{\mathcal{Z}} \rightarrow \mathcal{Z}^+$  which blows down  $E$  to  $P^*$ . Let  $Z^+$  be the Mukai flop of  $Z$  along  $P$ . Then  $\mathcal{Z}^+$  is a one-parameter deformation of  $Z^+$ . Let  $L^+$  be the strict transform of  $L$  under the rational map  $Z \dashrightarrow Z^+$ .

**Lemma 3.2.**  *$c_1(L^+)$  is the Kodaira-Spencer class of the deformation  $\mathcal{Z}^+ \rightarrow S$ .*

*Proof.* Let  $U = Z - P$ , isomorphic to  $U^+ := Z^+ - P^*$ . We denote by  $v|_U$  (resp.  $v^+|_{U^+}$ ) the image of the Kodaira-Spencer class under the map

$H^1(Z, T_Z) \rightarrow H^1(U, T_U)$  (resp.  $H^1(Z^+, T_{Z^+}) \rightarrow H^1(U^+, T_{U^+})$ ). Notice that we have an  $S$ -isomorphism  $\mathcal{Z} - P \simeq \mathcal{Z}^+ - P^*$ ; thus  $v|_U = v^+|_{U^+}$  via the isomorphism  $U \simeq U^+$ .

The map  $H^1(Z, T_Z) \rightarrow H^1(U, T_U)$  is injective since  $\text{codim}_Z P \geq 2$ . Furthermore  $v|_U = c_1(L)|_U = c_1(L^+)|_{U^+}$ , thus  $c_1(L^+)$  is the Kodaira-Spencer class of the deformation  $\mathcal{Z}^+ \rightarrow S$ .  $\square$

If furthermore  $P$  is mapped to a point by  $\pi$ , then one has another symplectic resolution  $Z^+ \rightarrow W$  which admits a deformation  $\mathcal{Z}^+ \rightarrow \mathcal{W}$ . The deformations one wants to construct in the following theorem are based on this.

**Theorem 3.3.** *Let  $V$  be a four-dimensional symplectic vector space and  $G < \text{Sp}(V)$  a finite subgroup. Then any two projective symplectic resolutions of  $V/G$  are deformation equivalent.*

*Proof.* Let  $W = V/G$  and  $Z \xrightarrow{\pi} W \xleftarrow{\pi^+} Z^+$  two projective symplectic resolutions. Take a  $\pi^+$ -ample line bundle  $L^+$  on  $Z^+$ . Then we have a deformation of  $\pi^+$ :  $\mathcal{Z}^+ \rightarrow \mathcal{W}$  such that  $c_1(L^+)$  is the Kodaira-Spencer class of  $\mathcal{Z}^+ \rightarrow S$ . Let  $L$  be the strict transform to  $Z$  of  $L^+$ . Then  $L$  is  $\pi$ -big. If  $L$  is  $\pi$ -nef, then the two resolutions  $\pi$  and  $\pi^+$  are isomorphic (see [FN] Theorem 2.2).

If  $L$  is not  $\pi$ -nef, we can find a  $(Z, \epsilon L)$ -extremal ray  $R$  for small  $\epsilon > 0$  (see [KMM]). The locus  $E$  of  $R$  in  $Z$  is contained in  $\pi^{-1}(0)$  by Lemma 2.6 and the contraction of  $R$  gives a small contraction since  $\dim(\pi^{-1}(0)) \leq 2$  by the semi-smallness of symplectic resolutions. By [WW],  $E$  is a disjoint union of copies isomorphic to  $\mathbb{P}^2$ . Furthermore  $L$  is negative on every curve in  $E$ . We can perform a Mukai flop along  $E$  to obtain  $\pi_1 : Z_1 \rightarrow W$ . The strict transform  $L_1$  of  $L$  is then positive on all curves of  $E^*$ . If  $L_1$  is not  $\pi_1$ -nef, then we can continue this process. After finitely many steps, say  $Z \dashrightarrow Z_1 \dashrightarrow \cdots \rightarrow Z_{l+1}$  one arrives to  $\pi_{l+1} = \pi^+$ .

Let  $L_i$  be the strict transform of  $L$  to  $Z_i$  and  $E_i$  the flop center of  $Z_i \dashrightarrow Z_{i+1}$ . Then  $L_{i+1}$  is positive on curves in  $E_i^*$  for  $i = 1, \dots, l$ . By Lemma 3.1, the normal bundle  $N_{E_i^*|Z_i}$  is isomorphic to  $\mathcal{O}_{E_i^*}(-1)^{\oplus 3}$ . Thus we can blow up  $\mathcal{Z}^+$  at  $E_i^*$  then blow down along the other direction to obtain a deformation of  $\pi_l: \mathcal{Z}_l \rightarrow \mathcal{W}$ . By Lemma 3.2 and Lemma 3.1, one can perform the same process to  $E_{l-1}^*$  in  $\mathcal{Z}_l$  and so on. Finally one obtains a deformation of  $\pi: \mathcal{Z} \rightarrow \mathcal{W}$ . Then the two deformations  $\mathcal{Z} \rightarrow \mathcal{W} \leftarrow \mathcal{Z}^+$  give the equivalence.  $\square$

## 4 Wreath product and Hilbert schemes

Let  $\Gamma < \mathrm{SL}(2)$  be a finite subgroup and  $W = (\mathbb{C}^2/\Gamma)^{(n)}$  the  $n$ -th symmetric product of  $\mathbb{C}^2/\Gamma$ . Then  $W$  is the quotient of  $\mathbb{C}^{2n}$  by the wreath product  $\Gamma_n = \Gamma \sim \mathcal{S}_n$ . Explicitly,  $\Gamma_n = \{(g, \sigma) | g \in \Gamma^n, \sigma \in \mathcal{S}_n\}$  with the multiplication  $(g, \sigma) \cdot (h, \tau) = (g\sigma(h), \sigma\tau)$ , where  $\sigma(h) = (h_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(n)})$ .

Let  $S \rightarrow \mathbb{C}^2/\Gamma$  be the minimal resolution. Then the composition

$$\mathrm{Hilb}^n(S) \xrightarrow{\tau} S^{(n)} \rightarrow (\mathbb{C}^2/\Gamma)^{(n)}$$

gives a projective symplectic resolution  $\mathrm{Hilb}^n(S) \xrightarrow{\pi} W$  (see also [Wan]). When  $\Gamma$  is trivial, this is the unique projective symplectic resolution of  $W$  (cf. [FN]). However it is not true for a non-trivial  $\Gamma$ . The following problem is open for  $n \geq 3$ .

**Problem 1.** *Find out all projective (resp. proper) symplectic resolutions of  $W = \mathbb{C}^{2n}/\Gamma_n$ .*

Let  $C_i, i \in \{1, \dots, k\}$  be the irreducible components in the exceptional divisor  $S \rightarrow \mathbb{C}^2/\Gamma$ . Then in the central fiber  $\pi^{-1}(0)$  there are  $k$  disjoint copies of  $\mathbb{P}^n$ , given by the strict transforms of  $C_i^{(n)}$  via  $\tau$ . In particular, we can perform Mukai flops to obtain many different symplectic resolutions of  $W$ . However it is not clear if these resolutions are still projective. An answer to Problem 1 is expected from the following

**Conjecture 3.** *Any two projective symplectic resolutions of  $W$  are connected by a sequence of Mukai flops with flop centers contained in the fiber over  $0 \in W$ . In particular, they are isomorphic over  $W - 0$ .*

It is not totally unlikely that the precedent conjecture holds for any quotient variety  $V/G$  which is not a product of quotient varieties. For 4-dimensional quotients, this is true thanks to the results in [WW].

A positive answer to this conjecture may imply that Conjecture 2 is valid for projective symplectic resolutions of  $W$ , by the arguments of the precedent section and results in [CMSB].

There exists another natural symplectic resolution of  $W$  that is constructed as follows (constructed in [Wan]): let  $N = |\Gamma|$  be the order of  $\Gamma$ . The action of  $\Gamma$  on  $\mathbb{C}^2$  extends to  $\mathrm{Hilb}^{nN}(\mathbb{C}^2)$  and  $(\mathbb{C}^2)^{(nN)}$ . Thus the Hilbert-Chow morphism induces a morphism between  $\Gamma$ -fixed points  $\mathrm{Hilb}^{nN, \Gamma}(\mathbb{C}^2) \rightarrow (\mathbb{C}^2)^{(nN), \Gamma}$ . Notice that  $(\mathbb{C}^2)^{(nN), \Gamma}$  is naturally identified with

$W = (\mathbb{C}^2/\Gamma)^{(n)}$ . Let  $Z_{\Gamma,n}$  be the closure in  $\text{Hilb}^{nN,\Gamma}(\mathbb{C}^2)$  of unordered  $n$ -tuple of distinct  $\Gamma$ -orbits in  $\mathbb{C}^2 - 0$ . It is shown in [Wan] that  $Z_{\Gamma,n}$  is a connected component of  $\text{Hilb}^{nN,\Gamma}(\mathbb{C}^2)$ , thus it is smooth and symplectic. Moreover, the morphism  $Z_{\Gamma,n} \xrightarrow{\pi^+} (\mathbb{C}^2)^{(nN),\Gamma} \simeq W$  is an isomorphism over  $W_{reg}$ , thus it gives a projective symplectic resolution of  $W$ .

**Problem 2.** *Connect the two resolutions  $\pi, \pi^+$  by Mukai flops.*

**Remark 4.1.**  $\pi$  and  $\pi^+$  are in general non-isomorphic. In the case of  $\Gamma = \{\pm 1\}$  and  $n = 2$ ,  $\pi$  and  $\pi^+$  are the two non-isomorphic projective symplectic resolutions that  $(\mathbb{C}^2/\pm 1)^{(2)}$  can have (see [FN], Example 2.7).

In the following we give a way to describe all possible projective symplectic resolutions of  $W = (\mathbb{C}^2/\Gamma)^{(2)}$ . The irreducible components in  $\pi^{-1}(0)$  can be described as follows:

(i)  $P_{i,i}$  ( $1 \leq i \leq k$ ): the strict transform of  $C_i^{(2)}$  via  $\tau$ . They are isomorphic to  $\mathbb{P}^2$ ;

(ii)  $P_{i,j}$  ( $1 \leq i < j \leq k$ ): the strict transform via  $\tau$  of the image of  $C_i \times C_j$  under the morphism  $S^2 \rightarrow S^{(2)}$ . If  $C_i \cap C_j = \emptyset$ , then  $P_{i,j}$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . If  $C_i \cap C_j = \{x\}$ , then  $P_{i,j}$  is isomorphic to the one point blow up of  $\mathbb{P}^1 \times \mathbb{P}^1$ ;

(iii)  $Q_i$  ( $1 \leq i \leq k$ ): the preimage  $\tau^{-1}(\Delta_{C_i})$ , where  $\Delta_{C_i}$  is the diagonal embedding of  $C_i$  in  $S^{(2)}$ . It is isomorphic to  $\mathbb{P}(T_S|_{C_i}) \simeq \mathbb{P}(\mathcal{O}_{C_i}(2) \oplus \mathcal{O}_{C_i}(-2))$ , thus a Hirzebruch surface  $F_4$ .

**Lemma 4.2.** *The strict transform of  $Q_i$  under any sequence of Mukai flops along components in  $\pi^{-1}(0)$  is not isomorphic to  $\mathbb{P}^2$ .*

*Proof.* To simplify the presentation, we will only prove the lemma for  $\Gamma$  being of type  $A_k$ , i.e.,  $\Gamma$  is a cyclic subgroup in  $\text{SL}(2)$  of order  $k + 1$ . Let  $C_i \cap C_{i+1} = \{x_i\}$  for  $i = 1, \dots, k$ . One checks that  $l_i := Q_i \cap P_{i,i}$  is a conic in  $P_{i,i}$  and a negative section in  $Q_i$ . If we perform a Mukai flop along  $P_{i,i}$ , then  $l_i$  is transformed to a conic in  $P_{i,i}^*$ , which is still called the strict transform of  $l_i$ . The strict transform of  $Q_i$  is isomorphic to  $Q_i$ . Among  $P_{i,j}$ , only  $P_{i-1,i}$  and  $P_{i,i+1}$  intersect  $l_i$ , both at one point (with multiplicity 2).

One way to make the self-intersection of the strict transform of  $l_i$  positive is to flop  $P_{i-1,i}$  or  $P_{i,i+1}$ . To do so, one needs to flop  $P_{i,i}$  first. After the flop along  $P_{i,i}$ ,  $P'_{i-1,i}$  intersects  $P_{i,i}^*$  at one point (which lies on the strict transform of  $l_i$ ). By this, one sees that the self-intersection of the strict transform of  $l_i$  is always negative.  $\square$

Thus to construct Mukai flops, one only needs to consider the components  $P_{i,j}$ . In the following we will assume that  $\Gamma$  is of type  $A_k$  (with minor changes, analogue results can be obtained for types  $D_k, E_l$ ). The configuration of  $P_{i,j}$  will be represented in  $\mathbb{N}^2$  as follows:  $P_{i,j}$  is placed at the position  $(i, j)$ , represented by a rectangle (resp. an ellipse, a  $\oplus$ , a circle) if  $P_{i,j}$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  (resp. one point blow-up of  $\mathbb{P}^1 \times \mathbb{P}^1$ , Hirzebruch surface  $F_1, \mathbb{P}^2$ ). These are the vertices of the graph. It is easy to see that the intersection of components of  $P_{i,j}$  is either one point or a  $\mathbb{P}^1$  if not empty. Two vertices are joined by a solid line (resp. dotted line) if their intersection is a  $\mathbb{P}^1$  (resp. a point).

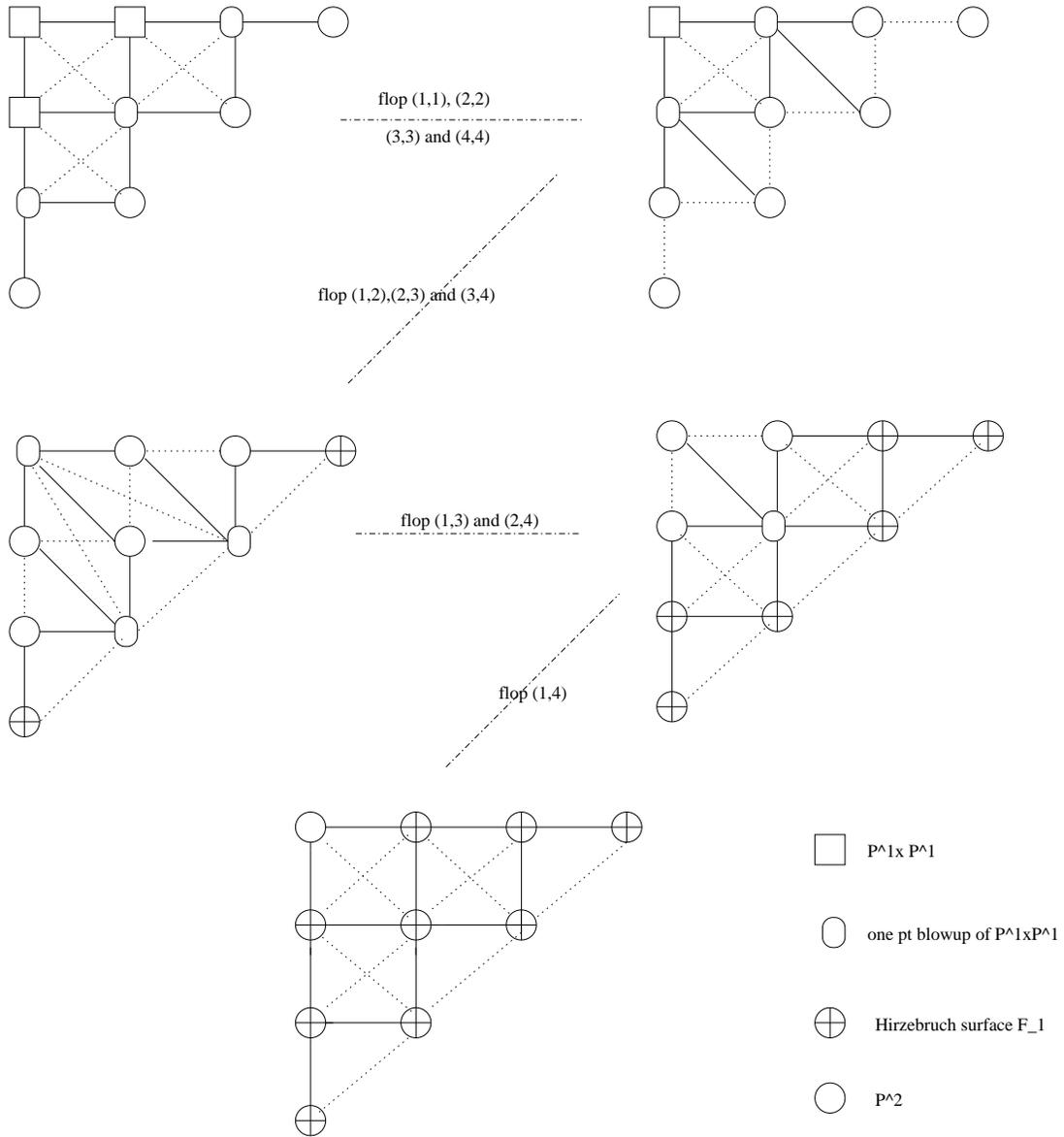
When we perform a Mukai flop at a vertex say  $P_{i,j}$ , the solid line (resp. dotted line) joining this vertex is replaced by a dotted line (resp. solid line). Other lines are untouched except the following case: the vertex  $P_{i,j}$  is joined to two vertices  $P_1, P_2$  by dotted lines. Then after the flop, the two dotted lines are replaced by solid lines, and furthermore  $P_1$  and  $P_2$  are joined by a dotted line. Surely this process is symmetric, i.e., if  $P_{i,j}$  is joined to  $P_1, P_2$  by solid lines and  $P_1, P_2$  are joined by dotted line, then after the flop along  $P_{i,j}$ , the dotted line between  $P_1$  and  $P_2$  should be removed, and the solid lines joining  $P_{i,j}$  to  $P_1, P_2$  are replaced by dotted ones.

Now we describe how the vertex labels change. Since the process is symmetric, we only describe the changes when  $P$  is a vertex joined to  $P_{i,j}$  by a solid line. Suppose that  $P$  is labeled by an ellipse (i.e., a one point blow-up of  $\mathbb{P}^1 \times \mathbb{P}^1$ ). There are two cases:

- (i) the solid line comes from a dotted line, i.e., this line corresponds to the exceptional fiber of the one point blow-up of  $\mathbb{P}^1 \times \mathbb{P}^1$ , then the label at  $P$  is changed to a square (i.e.,  $\mathbb{P}^1 \times \mathbb{P}^1$ );
- (ii) otherwise, the label at  $P$  is changed to be  $\oplus$  (i.e.,  $F_1$ ).

If  $P$  is labeled by a  $\oplus$ , then it is changed to a circle. The following pictures are examples of symplectic resolutions of  $\mathbb{C}^4/\Gamma_2$  with  $\Gamma$  of type  $A_4$ .

Any projective symplectic resolutions of  $W$  is obtained in this way. However, it is not clear (and it may be not true) that any sequence of Mukai flops gives a projective symplectic resolution. Sometimes one needs to flop simultaneously several disjoint  $\mathbb{P}^2$  to obtain a projective resolution.



Example of symplectic resolutions of  $W$ , type  $A_4$

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