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INDECOMPOSABLE MODULES AND GELFAND RINGS

FRANÇOIS COUCHOT

Abstract. It is proved that a commutative ring is clean if and only if it is Gelfand with a totally disconnected maximal spectrum. It is shown that each indecomposable module over a commutative ring $R$ satisfies a finite condition if and only if $R_P$ is an artinian valuation ring for each maximal prime ideal $P$. Commutative rings for which each indecomposable module has a local endomorphism ring are studied. These rings are clean and elementary divisor rings. It is shown that each commutative ring $R$ with a Hausdorff and totally disconnected maximal spectrum is local-global. Moreover, if $R$ is arithmetic then $R$ is an elementary divisor ring.

In this paper $R$ is a commutative ring with unity and modules are unitary.

In [12, Proposition 2] Goodearl and Warfield proved that each zero-dimensional ring $R$ satisfies the second condition of our Theorem 1.1, and this condition plays a crucial role in their paper. In Section 1, we show that a ring $R$ enjoys this condition if and only if it is clean, if and only if it is Gelfand with a totally disconnected maximal spectrum. So we get a generalization of results obtained by Anderson and Camillo in [1] and by Samei in [21]. We deduce that every commutative ring $R$ with a Hausdorff and totally disconnected maximum prime spectrum is local-global, and moreover, $R$ is an elementary divisor ring if, in addition, $R$ is arithmetic. One can see in [8] that local-global rings have very interesting properties.

In Section 3 we give a characterization of commutative rings for which each indecomposable module satisfies a finite condition: finitely generated, finitely co-generated, cyclic, cocyclic, artinian, noetherian or of finite length. We deduce that a commutative ring is Von Neumann regular if and only if each indecomposable module is simple. This last result was already proved in [3]. We study commutative rings for which each indecomposable module has a local endomorphism ring. These rings are clean and elementary divisor rings. It remains to find valuation rings satisfying this property to give a complete characterization of these rings. We also give characterizations of Gelfand rings and clean rings by using properties of indecomposable modules. Similar results are obtained in Section 4 for commutative rings for which each prime ideal contains only one minimal prime ideal.

We denote respectively $\text{Spec } R$, $\text{Max } R$ and $\text{Min } R$, the space of prime ideals, maximal ideals, and minimal prime ideals of $R$, with the Zariski topology. If $A$ a subset of $R$, then we denote

$V(A) = \{ P \in \text{Spec } R \mid A \subseteq P \}$,
$D(A) = \{ P \in \text{Spec } R \mid A \nsubseteq P \}$,
$V_M(A) = V(A) \cap \text{Max } R$ and $D_M(A) = D(A) \cap \text{Max } R$.

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1. Local-global Gelfand rings

As in [13] we say that a commutative ring $R$ is Gelfand if each prime ideal is contained in only one maximal ideal. In this case, we put $\mu : \text{Spec } R \to \text{Max } R$ the map defined by $\mu(J)$ is the maximal ideal containing $J$ for each prime ideal $J$. Then $\mu$ is continuous and Max $R$ is Hausdorff by [6, Theorem 1.2].

In [2]. Goodearl and Warfield proved that every zero-Krull-dimensional commutative ring satisfies the second condition of the following theorem. This property is used to show cancellation, $n$-root and isomorphic refinement theorems for finitely generated modules over algebras over a commutative ring which is Von Neumann regular modulo its Jacobson radical. So, the following theorem allows us to extend these results to each ring with a Hausdorff and totally disconnected maximal spectrum. As in [21] we say that a ring $R$ is clean if each element of $R$ is the sum of a unit with an idempotent. In [21, Proposition 1.8 and Theorem 2.1] Nicholson proved that commutative clean rings are exactly the exchange rings defined by Warfield in [24]. In [21] Samei proved that the conditions (1), (3) and (4) are equivalent when $R$ is semiprimitive and in [11, Anderson and Camillo showed that each clean ring is Gelfand. We can also see [18, Theorem 3]. If $P$ is a prime ideal we denote by $0_P$ the kernel of the natural map $R \to R_P$.

**Theorem 1.1.** Let $R$ be a ring. The following conditions are equivalent:

1. $R$ is a Gelfand ring and $Max R$ is totally disconnected.
2. Each $R$-algebra $S$ satisfies this condition: let $f_1, \ldots, f_k$ be polynomials over $S$ in noncommuting variables $x_1, \ldots, x_m, y_1, \ldots, y_n$. Let $a_1, \ldots, a_m \in S$. Assume that $\forall P \in Max R$ there exists $b_1, \ldots, b_n \in S_P$ such that $f_j(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0 \forall i, 1 \leq i \leq k$. Then there exist $d_1, \ldots, d_n \in S$ such that $f_j(a_1, \ldots, a_m, d_1, \ldots, d_n) = 0 \forall i, 1 \leq i \leq k$.
3. $R$ is a clean ring.
4. $R$ is Gelfand and $\forall P \in Max R, 0_P$ is generated by a set of idempotents.

**Proof.** (1) $\Rightarrow$ (2). By [11, Theorem 16.17] Max $R$ has a base of clopen subsets.

Since $\mu$ is continuous, each clopen subset of Max $R$ is of the form $D_M(e)$ for some idempotent $e$. So we can do the same proof as in [13, Proposition 2] where we replace Spec $R$ with Max $R$.

(2) $\Rightarrow$ (3). Let $a \in R$. We consider the following equations: $x^2 = x$ and $y(a - x) = 1$. Since each local ring is clean, these equations have a solution in $R_P$ for each maximal ideal $P$. We conclude that there is also a solution in $R$ and that $R$ is clean.

(3) $\Rightarrow$ (1). Let $P, P' \in Max R, P \neq P'$. Then there exist $a \in P$ and $a' \in P'$ such that $a + a' = 1$. We have $a = u + e$ where $u$ is a unit and $e$ is an idempotent. Since $a \in P$ and $u \notin P$ we get that $e \notin P$. We have $a' = 1 - a = -u + (1 - e)$. So $1 - e \notin P'$. Consequently $P$ and $P'$ have disjoint clopen neighbourhoods. Since Max $R$ is quasi-compact, we deduce that this space is compact and totally disconnected. The equality $e(1 - e) = 0$ implies that $P \cap P'$ contains no prime ideal. Hence $R$ is Gelfand.

(1) $\Rightarrow$ (4). Let $P$ be a maximal ideal and $a \in 0_P$. Then there exists $s \in R \setminus P$ such that $sa = 0$. Since Max $R$ is totally disconnected there is a clopen subset $U$ such that $U \subseteq D_M(s)$. Because of $\mu$ is continuous, there exists an idempotent $e$ such that $P \subseteq D(e) = \mu^{-1}(U) \subseteq \mu^{-1}(D_M(s)) \subseteq D(s)$. Then $e \in Rs$, $ea = 0$, $a = a(1 - e)$ and $1 - e \in 0_P$. 

1.2 there exist a, b ∈ 0P \P'. Then there exists an idempotent e ∈ 0P \P'. Clearly 1 − e ∈ P. Consequently P and P' have disjoint clopen neighbourhoods.

We say that R is **local-global** if each polynomial over R in finitely many indeterminates which admits unit values locally, admits unit values. Recall that most of the results of [12, Theorem 1.2] and from Theorem 1.1 we deduce that

**Remark 1.3.** If R is the ring of algebraic integers, then R is local-global by [3] and semi-primitive. But this ring is not Gelfand.

2. Arithmetic Gelfand rings

We say that a module is uniserial if its set of submodules is totally ordered by inclusion, we say that a ring R is a **valuation ring** if it is uniserial as R-module and we say that R is arithmetic if R_P is a valuation ring for each maximal ideal P. Recall that R is a **Bézout ring** if each finitely generated ideal is principal and R is an **elementary divisor ring** if each finitely presented module is a direct sum of cyclic submodules.

**Theorem 2.1.** Let R be an arithmetic local-global ring. Then R is an elementary divisor ring. Moreover, for each a, b ∈ R, there exist d, a', b', c ∈ R such that a = a'd, b = b'd and a' + cb' is a unit of R.

**Proof.** Since every finitely generated ideal is locally principal R is Bézout by [3, Corollary 2.7]. Let a, b ∈ R. Then there exist a', b', d ∈ R such that a = a'd, b = b'd and Ra + Rb = Rd. Consider the following polynomial a' + b'T. If P is a maximal ideal, then we have aRP = dRP or bRP = dRP. So, either a' is a unit of RP and a' + b'r is a unit of RP for each r ∈ PnP, or b' is a unit of RP and a' + b'(1 − a'/b') is a unit of RP. We conclude that the last assertion holds. Now, let a, b, c ∈ R such that Ra + Rb + Rc = R. We set Rb + Rc = Rd. Let b', c', s and q such that b = b'd, c = c'd and b' + c'q and a + sd are units. Then (b' + c'q)(a + sd) = (b' + c'q)a + s(b + qc) is a unit. We conclude by [6, Theorem 6].

We deduce the following corollary which is a generalization of [4, Theorem III.6] and [3, Theorem 5.5].

**Corollary 2.2.** Let R be an arithmetic ring with a Hausdorff and totally disconnected maximal spectrum. Then R is an elementary divisor ring. Moreover, for each a, b ∈ R, there exist d, a', b', c ∈ R such that a = a'd, b = b'd and a' + cb' is a unit of R.
Corollary 2.3. Let \( R \) be an arithmetic Gelfand ring such that \( \text{Min } R \) is compact. Then \( R \) is an elementary divisor ring. Moreover, for each \( a, b \in R \), there exist \( d, a', b', c \in R \) such that \( a = a'd \), \( b = b'd \) and \( a' + cb' \) is a unit of \( R \).

Proof. Let \( \mu' \) be the restriction of \( \mu \) to \( \text{Min } R \). Since \( R \) is arithmetic each prime ideal contains only one minimal prime ideal. Then \( \mu' \) is bijective and it is an homeomorphism because \( \text{Min } R \) is compact. One can apply corollary 2.2 since \( \text{Min } R \) is totally disconnected.

Remark 2.4. In [9] there is an example of a Gelfand Bézout ring \( R \) which is not an elementary divisor ring. Consequently \( \text{Min } R \) is not compact.

3. Indecomposable modules and maximal ideals

In the two next propositions we give a characterization of Gelfand rings and clean rings by using properties of indecomposable modules.

Proposition 3.1. Let \( R \) be a ring. The following conditions are equivalent:

1. For each \( R \)-algebra \( S \) and for each left \( S \)-module \( M \) for which \( \text{End}_R(M) \) is local, \( \text{Supp } M \) contains only one maximal ideal.
2. \( R \) is a Gelfand ring.
3. \( \forall P \in \text{Max } R \) the natural map \( R \twoheadrightarrow R_P \) is surjective.

When these conditions are satisfied, \( M = M_P \) for each left \( S \)-module \( M \) for which \( \text{End}_R(M) \) is local, where \( P \) is the unique maximal ideal of \( \text{Supp } M \) and where \( S \) is an algebra over \( R \).

Proof. Assume that \( R \) is Gelfand. Let \( S \) be an \( R \)-algebra and let \( M \) be a left \( S \)-module such that \( \text{End}_S(M) \) is local. Let \( P \) be the prime ideal which is the inverse image of the maximal ideal of \( \text{End}_S(M) \) by the canonical map \( R \twoheadrightarrow \text{End}_R(M) \) and let \( Q = \mu(P) \). Since \( M \) is an \( R_P \)-module, \( 0_Q \subseteq \text{ann}_R(M) \). So, \( \text{Supp } M \subseteq V(0_Q) \) and \( Q \) is the only maximal ideal belonging to \( V(0_Q) \) since \( R \) is Gelfand.

Conversely, if \( P \) is a prime ideal then \( R_P = \text{End}_R(R_P) \). It follows that \( P \) is contained in only one maximal ideal.

By [9, Theorem 1.2] \( R \) is Gelfand if and only if, \( \forall P \in \text{Max } R \), \( P \) is the only maximal ideal containing \( 0_P \). This is equivalent to \( R/0_P \) is local, \( \forall P \in \text{Max } R \). It is obvious that \( R_P = R/0_P \) if \( R/0_P \) is local. (When \( R \) is semi-primitive we can see [9, Proposition 1.6.1]).

Recall that the diagonal map \( M \twoheadrightarrow \Pi_{P' \in \text{Max } R} M_{P'} \) is monic. Since \( R \) is Gelfand, we have \( M_P = M/0_P M \) where \( P \) is the only maximal ideal of \( \text{Supp } M \). Hence \( M = M_P \).

Proposition 3.2. Let \( R \) be a ring. The following conditions are equivalent:

1. For each \( R \)-algebra \( S \) and for each indecomposable left \( S \)-module \( M \), \( \text{Supp } M \) contains only one maximal ideal.
2. \( R \) is clean.

When these conditions are satisfied, \( M = M_P \) for each indecomposable left \( S \)-module \( M \), where \( P \) is the unique maximal ideal of \( \text{Supp } M \) and \( S \) is an \( R \)-algebra.

Proof. (2) \( \Rightarrow \) (1). By Theorem [9] \( \text{Max } R \) is totally disconnected. So, if \( P \) and \( P' \) are two different maximal ideals such that \( P \in \text{Supp } M \) then there exists an idempotent \( e \in P \setminus P' \) because \( \mu \) is continuous. Since \( (1 - e) \notin P \) and \( M_P \neq 0 \), we have \( (1 - e)M \neq 0 \). We deduce that \( eM = 0 \) and \( M_{P'} = 0 \).
Lemma 3.3. Let $R$ be a local ring which is not a valuation ring. Then there exists an indecomposable non-finitely generated $R$-module whose endomorphism ring is not local.

Proof. Since $R$ is not a valuation ring there exist $a, b \in R$ such that neither divides the other. By taking a suitable quotient ring, we may assume that $Ra \cap Rb = 0$ and $Pa = Pb = 0$. Let $F$ be a free module generated by $\{e_n \mid n \in \mathbb{N}\}$, let $K$ be the submodule generated by $\{ae_n - be_{n+1} \mid n \in \mathbb{N}\}$ and let $M = F/K$. Clearly $M/PM \cong F/FF$. We will show that $M$ is indecomposable and $S := \text{End}_R(M)$ is not local. Let us observe that $M$ is defined as in proof of \([3, \text{Theorem 2.3}]\). But, since $R$ is not necessarily artinian, we do a different proof to show that $M$ is indecomposable. We shall prove that $S$ contains no trivial idempotents. Let $s \in S$. Then $s$ is induced by an endomorphism $\tilde{s}$ of $F$ which satisfies $\tilde{s}(K) \subseteq K$. For each $n \in \mathbb{N}$ there exists a finite family $(\alpha_{p,n})$ of elements of $R$ such that:

(1) $\tilde{s}(e_n) = \sum_{p \in \mathbb{N}} \alpha_{p,n}e_p$

Since $\tilde{s}(K) \subseteq K$, $\forall n \in \mathbb{N}$, $\exists$ a finite family $(\beta_{p,n})$ of elements of $R$ such that:

(2) $\tilde{s}(ae_n - be_{n+1}) = \sum_{p \in \mathbb{N}} \beta_{p,n}(ae_p - be_{p+1})$

From (1) and (2) it follows that:

$$\sum_{p \in \mathbb{N}} (a\alpha_{p,n} - b\alpha_{p,n+1})e_p = a\beta_{0,n}e_0 + \sum_{p \in \mathbb{N}^*} (a\beta_{p,n} - b\beta_{p-1,n})e_p$$

Since $Pa = Pb = Ra \cap Rb = 0$ we deduce that

$$\alpha_{0,n+1} \equiv 0 [P], \alpha_{p,n} \equiv \beta_{p,n} [P] \quad \text{and} \quad \alpha_{p,n+1} \equiv \beta_{p-1,n} [P]$$

It follows that

(3) $(i) \alpha_{p,n} \equiv \alpha_{p+1,n+1} [P], \forall p, n \in \mathbb{N}$, and $(ii) \alpha_{p,p+k+1} \equiv 0 [P], \forall p, k \in \mathbb{N}$

Now we assume that $s$ is idempotent. Let $x_n = e_n + K$, $\forall n \in \mathbb{N}$. Let $\bar{s}$ be the endomorphism of $M/PM$ induced by $s$. If $L$ is an $R$-module and $x$ an element of $L$, we put $\bar{x} = x + PL$. From $s^2(x_0) = s(x_0)$ we get the following equality:

(4) $\sum_{n \in \mathbb{N}} (\sum_{p \in \mathbb{N}} \alpha_{n,p}\alpha_{p,0})x_n = \sum_{n \in \mathbb{N}} \alpha_{n,0}x_n$

Then $\bar{\alpha}_{0,0} = \sum_{p \in \mathbb{N}} \alpha_{0,p}\bar{\alpha}_{p,0} = \bar{\alpha}_{0,0}$, since $\bar{\alpha}_{0,p} = 0$ by (3)(ii), $\forall p > 0$. So, we have $\bar{\alpha}_{0,0} = 0$ or $\bar{\alpha}_{0,0} = 1$. If $\bar{\alpha}_{0,0} = 1$ then we replace $s$ with $1_M - s$. So we may assume that $\bar{\alpha}_{0,0} = 0$. By (3)(i) $\bar{\alpha}_{n,n} = 0$, $\forall n \in \mathbb{N}$. By using (3) and (3)(ii) we get that

$$\bar{\alpha}_{n,0} = \sum_{p=0}^{n-1} \bar{\alpha}_{n,p}\bar{\alpha}_{p,0} + \bar{\alpha}_{n,n}\bar{\alpha}_{n,0}$$
Hence, if \( \alpha_{p,0} = 0, \forall p < n \) then \( \alpha_{n,0} = 0 \) too. By induction we obtain that \( \alpha_{n,0} = 0, \forall n \in \mathbb{N} \). We deduce that

\[
\alpha_{p,n} \in P, \forall p, n \in \mathbb{N}
\]

Let \( A = \text{Im} \, s \), \( B = \text{Ker} \, s \) and let \( A' \) and \( B' \) be the inverse image of \( A \) and \( B \) by the natural map \( F \to M \). If \( x \in A' \) then \( \tilde{s}(x) = x + y \) for some \( y \in K \). By \( \tilde{s}(P_A) = P_b = 0 \) it follows that \( \tilde{s}(y) = 0 \) and \( \tilde{s}^2(x) = \tilde{s}(x) \). So \( \tilde{s}^2(x) = 0 \). We deduce that \( (\tilde{s}^2)^2 = \tilde{s}^2 \). Let \( C = \text{Im} \, \tilde{s}^2 \). Then \( C \) is projective and \( C = PC \) by \([3]\). By \([4]\) Proposition 2.7 \( C = 0 \). So \( s = 0 \) (or \( 1_M - s = 0 \)).

It remains to prove that \( S \) is not local. Let \( f, g \in S \) defined in the following way: \( f(x_n) = x_{n+1} \) and \( g(x_n) = x_n - x_{n+1}, \forall n \in \mathbb{N} \). We easily check that \( x_0 \notin \text{Im} \, f \cup \text{Im} \, g \). So \( f \) and \( g \) are not units of \( S \) and \( f + g = 1_M \) is a unit. Hence \( S \) is not local. \( \square \)

A module is **cocyclic** (respectively **finitely cogenerated**) if it is a submodule of the injective hull of a simple module (respectively of a finite direct sum of injective hulls of simple modules).

Now we give a characterization of commutative rings for which each indecomposable module satisfies a finite condition.

**Theorem 3.4.** Let \( R \) be a ring. The following conditions are equivalent:

1. Each indecomposable \( R \)-module is of finite length.
2. Each indecomposable \( R \)-module is noetherian.
3. Each indecomposable \( R \)-module is finitely generated.
4. Each indecomposable \( R \)-module is artinian.
5. Each indecomposable \( R \)-module is finitely cogenerated.
6. Each indecomposable \( R \)-module is cyclic.
7. Each indecomposable \( R \)-module is cocyclic.
8. For each maximal ideal \( P \), \( R_P \) is an artinian valuation ring.
9. \( R \) is an arithmetic ring of Krull-dimension 0 and its Jacobson ideal \( J \) is \( T \)-nilpotent.

**Proof.** The following implications are obvious: \((1) \Rightarrow (2) \Rightarrow (3), (1) \Rightarrow (4) \Rightarrow (5), (6) \Rightarrow (3) \) and \((7) \Rightarrow (5)\).

\((8) \Rightarrow (1), (6) \) and \((7)\). \( R \) has Krull dimension 0. Hence \( R \) is clean. So, if \( M \) is an indecomposable module, by proposition \([3]\) there is only one maximal ideal \( P \) such that \( M_P \neq 0 \). Moreover \( M \cong M_P \). We conclude by \([3]\) Theorem 4.3.

\((3) \Rightarrow (8)\). Let \( P \) be a maximal ideal and \( E \) the injective hull of \( R/P \). Then each submodule of \( E \) is indecomposable. It follows that \( E \) is a noetherian module. By \([3]\) Proposition 3 \( E \) is a module of finite length, and by \([3]\) Theorem 3 \( R_P \) is artinian. We conclude by \([3]\) Theorem 2.3 or Lemma \( 3.3 \).

\((5) \Rightarrow (8)\). Let \( P \) be a maximal ideal. Then each factor of \( R_P \) modulo an ideal of \( R_P \) is finitely cogenerated. It follows that \( R_P \) is artinian. We conclude as above.

\((8) \Rightarrow (9)\). There is a finite family of open sets \( D(s_{n_{p_1}}), \ldots, D(s_{n_{p_m}}) \) that cover \( \text{Spec} \, R \). We set \( n = \max\{n_{p_1}, \ldots, n_{p_m}\} \). Then \( x_1 \ldots x_n = 0 \).

\((9) \Rightarrow (8)\). \( \forall P \in \text{Max} \, R \), \( R_P \) is a valuation ring and \( PR_P \) is a nilideal. Then for every \( r \in P \) there exists \( s \in R \setminus P \) such that \( sr \) is nilpotent. So we get that \( PR_P = JR_P \), whence \( PR_P \) is \( T \)-nilpotent. We easily prove that \( R_P \) is artinian. \( \square \)
Corollary 3.5. Let $n$ be a positive integer, $R$ a ring and $J$ its Jacobson radical. Then the following conditions are equivalent:

1. Each indecomposable module has a length $\leq n$.
2. For each maximal ideal $P$, $R_P$ is a valuation ring and $(PR_P)^n = 0$.
3. $R$ is an arithmetic ring of Krull-dimension 0 and $J^n = 0$.

Corollary 3.6. A ring $R$ is Von Neumann regular if and only if every indecomposable module is simple.

The next theorem gives a partial characterization of commutative rings for which each indecomposable module has a local endomorphism ring.

Theorem 3.7. Let $R$ be a ring for which $\text{End}_R(M)$ is local for each indecomposable module $M$. Then $R$ is a clean elementary divisor ring.

Proof. Let $P$ be a prime ideal. Then $R/P = \text{End}_R(R/P)$ is local. Hence $R$ is Gelfand. We prove that $\text{Max } R$ is totally disconnected as in proof of proposition 3.2. If $P$ is a maximal ideal, each indecomposable $R_P$-module $M$ is also indecomposable over $R$ and $\text{End}_R(M) = \text{End}_{R_P}(M)$. By Lemma 3.3 $R_P$ is a valuation ring. □

Example 3.8. If $R$ is a ring satisfying the equivalent conditions of Theorem 3.4, then each indecomposable $R$-module has a local endomorphism ring. But, by [13, Corollary 2 p.52] and [22, Corollary 3.4], each complete discrete rank one valuation ring enjoys this property too. So, we consider a complete discrete rank one valuation ring $D$, $Q$ its ring of fractions and $R$ the subring of $Q^\infty$ defined as in [21, Example 1.7]: $x = (x_n)_{n \in \mathbb{N}} \in R$ if $\exists p \in \mathbb{N}$ and $s \in D$ such that $x_n = s_n, \forall n > p$. Since $D$ is local, $R$ is clean and semi-primitive by [13, Theorem 2]. We put $1 = (\delta_{n,p})_{n \in \mathbb{N}}$ and $\forall p \in \mathbb{N}, \; e_p = (\delta_{p,n})_{n \in \mathbb{N}}$ where $\delta_{n,p}$ is the Kronecker symbol. Let $J$ be the maximal ideal of $D$. If $P$ is a maximal ideal of $R$, then either $e_p \in P, \forall p \in \mathbb{N}$, whence $P = J1 + \oplus_{p \in \mathbb{N}} R e_p$ and $R_P \cong R/\oplus_{p \in \mathbb{N}} R e_p \cong D$, or $\exists p \in \mathbb{N}$ such that $e_p \notin P$, whence $P = R(1 - e_p)$ and $R_P \cong R/P \cong Q$. Thus $R$ is arithmetic and each indecomposable $R$-module has a local endomorphism ring. Observe that each indecomposable $R$-module is uniseriel and linearly compact and its endomorphism ring is commutative.

4. Indecomposable modules and minimal prime ideals

In this section we study rings $R$ for which each prime ideal contains only one minimal prime ideal. In this case, if $P \in \text{Spec } R$, let $\lambda(P)$ be the only minimal prime ideal contained in $P$. We shall see that $\lambda$ is continuous if and only if $\text{Min } R$ is compact. (See [16, Theorem 2] when $R$ is semi-prime). But, since $\lambda$ is surjective, the set of minimal primes can be endowed with the quotient topology induced by the Zariski topology of Spec $R$. We denote this topologic space by $\text{QMin } R$. Then we have the following:

Proposition 4.1. Let $R$ be a ring such that each prime ideal contains a unique minimal prime ideal and $N$ its nilradical. Then $\text{QMin } R$ is compact. Moreover, $\text{QMin } R$ and $\text{Min } R$ are homeomorphic if and only if $\text{Min } R$ is compact.

The following lemma is needed to prove this proposition. This lemma is a generalization of [5, Lemma 2.8]. We do a similar proof.
Lemma 4.2. Let $R$ be a ring, $N$ its nilradical and $a \in R \setminus N$. Let $P$ be a prime ideal such that $P/(N : a)$ is minimal in $R/(N : a)$. Then $P$ is a minimal prime ideal.

Proof. First we show that $a + (N : a)$ is a non-zerodivisor in $R/(N : a)$ and consequently $a \notin P$. Let $b \in R$ such that $ab \in (N : a)$. Then $a^2b \in N$. We easily deduce that $ab \in N$, whence $b \in (N : a)$. Let $r \in P$. Then there exist a positive integer $n$ and $s \in R \setminus P$ such that $sr^n \in (N : a)$. It follows that $asr^n \in N$. Since $as \notin P$ we deduce that $PR_P$ is a nilideal, whence $P$ is a minimal prime. □

Proof of proposition 4.1. Let $A$ and $B$ be two distinct minimal prime ideals. Since each maximal ideal contains only one minimal prime ideal, we have $A + B = R$. Therefore there exist $a \in A$ and $b \in B$ such that $a + b = 1$. Thus $a \notin B$ and $a \notin N$. But $a$ is a nilpotent element of $R_A$. Hence $(N : a) \not\subseteq A$. In the same way we show that $B \in D((N : b))$. We have $(N : a) \cap (N : b) = (N : Ra + Rb) = N$. So $D((N : a)) \cap D((N : b)) = \emptyset$. By Lemma 4.2, $D((N : a))$ and $D((N : b))$ are the inverse images of disjoint open subsets of $\text{QMin } R$ by $\lambda$. We conclude that this space is Hausdorff. Since Spec $R$ is quasi-compact, it follows that QMin $R$ is compact.

Let $\lambda'$ be the restriction of $\lambda$ to Min $R$. It is obvious that $(\lambda')^{-1}$ is continuous if and only if Min $R$ is compact. □

Remark 4.3. If we consider the set of D-components of Spec $R$, defined in [17], endowed with the quotient topology, we get a topologpic space $X$. Then $X$ is homeomorphic to Max $R$ (respectively QMin $R$) if $R$ is Gelfand (respectively every prime ideal contains only one minimal prime). But $X$ is not generally Hausdorff: see [17] Propositions 6.2 and 6.3.

Now we can show the two following propositions which are similar to Propositions 3.1 and 3.2. The proofs are similar too.

Proposition 4.4. Let $R$ be a ring. The following conditions are equivalent:

1. For each $R$-algebra $S$ and for each left $S$-module $M$ for which $\text{End}_S(M)$ is local, there exists only one minimal prime ideal $A$ such that $\text{Supp } M \subseteq V(A)$.
2. Every prime ideal contains only one minimal prime ideal.

Proof. (2) $\Rightarrow$ (1). Let $S$ be an $R$-algebra and let $M$ be a left $S$-module such that $\text{End}_S(M)$ is local. Let $P$ be the prime ideal which is the inverse image of the maximal ideal of $\text{End}_S(M)$ by the canonical map $R \to \text{End}_S(M)$, $A = \lambda(P)$ and $0_P$ the kernel of the natural map $R \to R_P$. Since $M$ is an $R_P$-module, $0_P \subseteq \text{ann}_R(M)$. It is obvious that $0_P \subseteq A$. On the other hand, $AR_P$ is the nilradical of $R_P$. It follows that $\text{rad}(0_P) = A$. Hence we get that $\text{Supp } M \subseteq V(\text{ann}_R(M)) \subseteq V(0_P) = V(A)$. If $B$ is another minimal prime, it is obvious that $V(A) \cap V(B) = \emptyset$. (1) $\Rightarrow$ (2). If $P$ is a prime ideal then $R_P = \text{End}_R(R_P)$. It follows that $P$ contains only one minimal prime ideal. □

Proposition 4.5. Let $R$ be a ring. The following conditions are equivalent:

1. For each $R$-algebra $S$ and for each indecomposable left $S$-module $M$, there is only one minimal prime ideal $A$ such that $\text{Supp } M \subseteq V(A)$.
Each prime ideal contains a unique minimal prime ideal and \( \text{QMin} \, R \) is totally disconnected.

**Proof.** (1) \( \Rightarrow \) (2). By proposition 4.4 each prime ideal contains a unique minimal prime ideal. Let \( P \in \text{QMin} \, R \) and \( C \) its connected component. There exists an ideal \( A \) such that \( V(A) = \lambda^{-1}(C) \). Then \( V(A) \) is connected. It follows that \( R/A \) is indecomposable. So \( V'(A) = V(P) \) and \( C = \{P\} \).

(2) \( \Rightarrow \) (1). Let \( S \) be an \( R \)-algebra and \( M \) be an indecomposable left \( S \)-module. Let \( P \in \text{Supp} \, M \), \( A = \lambda(P) \), \( P' \in \text{Spec} \, R \setminus V(A) \) and \( A' = \lambda(P') \). Since \( \text{QMin} \, R \) is totally disconnected, there exists an idempotent \( e \in A \setminus A' \). We easily deduce that \( e \in P \setminus P' \). Now we do as in the proof of Proposition 3.2 to conclude. \( \square \)

**References**


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