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On a nonlinear parabolic equation involving Bessel's operator associated with a mixed inhomogeneous condition

by

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Abstract. *In this paper we consider the following nonlinear parabolic equation*

$$(*)\qquad \begin{cases} u_t - a(t)\left(u_{rr} + \frac{\gamma}{r}u_r\right) + F(r,u) = f(r,t), \; 0 < r < 1, 0 < t < T, \\ \left|\lim\limits_{r\to 0_+} r^{\gamma/2}u_r(r,t)\right| < +\infty, \; u_r(1,t) + h(t)\left(u(1,t) - \widetilde{u}_0\right) = 0, \\ u(r,0) = u_0(r), \end{cases}$$

where $\gamma > 0$, $\widetilde{u}_0$ *are given constants,* $a(t)$, $h(t)$, $F(r,u)$, $f(r,t)$ *are given functions. In section* III, *we use the Galerkin and compactness method in appropriate Sobolev spaces with weight to prove the existence of a unique weak solution of the problem* (*) *on* $(0,T)$, *for every* $T > 0$. *In section* IV, *we prove that if the initial condition is bounded, then so is the solution. In section* V, *we study asymptotic behavior of the solution as* $t \to +\infty$. *In section* VI *we give numerical results*.

Keywords: *Nonlinear parabolic equation, Galerkin method, Sobolev spaces with weight, Asymptotic behavior of the solution.*

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I. INTRODUCTION

In this paper we will consider the following initial and boundary value problem

$$(1.1) \qquad u_t - a(t)\left(u_{rr} + \tfrac{\gamma}{r} u_r \right) + F(r,u) = f(r,t),\ 0 < r < 1,\ 0 < t < T,$$

$$(1.2) \qquad \left| \lim_{r \to 0_+} r^{\gamma/2} u_r(r,t) \right| < +\infty,\ u_r(1,t) + h(t)\left(u(1,t) - \tilde{u}_0 \right) = 0,$$

$$(1.3) \qquad u(r,0) = u_0(r),$$

where $\gamma > 0$, $\tilde{u}_0$ are given constants, $a(t)$, $h(t)$, $F(r,u)$, $f(r,t)$, are given functions satisfying conditions specified later.
The equation (1.1) can be rewritten in the form

$$(1.4) \qquad \frac{\partial u}{\partial t} - \frac{a(t)}{r^\gamma} \frac{\partial}{\partial r}\left(r^\gamma \frac{\partial u}{\partial r} \right) + F(r,u) = f(r,t),\ 0 < r < 1,\ 0 < t < T.$$

For $\gamma = 1$ with $F = 0$, the problem describes the radial axisymmetric heat flow in a cylinder.
With $\gamma = 2$ and always $F = 0$, the problem (1.2)-(1.4) represents in polar coordinates in $\mathbb{R}^3$ the mass fraction of a liquid fuel droplet in the case of his evaporation inside an infinite vessel, the boundary condition (1.2) being associated to the Rankine-Hugoniot condition on the surface of the droplet after a changing of the scale [8].
In [6], Minasjan studied a special case of the problem (1.1), (1.2) associated with the following T-periodic condition

$$(1.5) \qquad u(r,0) = u(r,T),$$

with

$$(1.6) \qquad \gamma = 1,\ F(r,u) = 0,\ \tilde{u}_0 = 0.$$

and the functions $a(t)$, $h(t)$, $f(r,t)$ are T-periodic in time t. The physical interpretation of the problem (1.1), (1.2), (1.5), (1.6) is that of a periodic heat flow in an infinite cylinder with the assumption that the cylinder is subjected to convective heat transfer (periodic in time) at the boundary surface ($r = 1$) at zero temperature. Inside the cylinder, there are circular symmetric sources of heat that change periodically. Minasjan[6] gave for this problem a classical solution using Fourier transforms. This method leads to an infinite pseudoregular system of linear algebraic equations. However, the solvability of this system is not proved in detail in [6].
In [3] Lauerova has proved that with T-periodic data, the problem (1.1), (1.2), (1.5), (1.6) has a T-periodic weak solution in t.
In the case of

$$(1.7) \qquad \gamma = 1, \tilde{u}_0 = 0, f = 0, F = F(u), F \in C^1(\mathbb{R}), F'(u) \geq -\varepsilon,$$

($\varepsilon > 0$ small enough), we have proved [4] that the problem (1.1), (1.2), (1.5) has a T-periodic unique weak solution in appropriate Sobolev spaces with weight. Furthermore, the solution also depends continuously on the functions $a(t)$ and $h(t)$.

The paper consists of three sections. In section III, under appropriate conditions of $a(t)$, $h(t)$, $F(r,u)$, $f(r,t)$ we prove the existence of a unique solution on $(0,T)$, for every $T > 0$. Theses results generalize relatively the ones in [3, 4, 6].
In section IV, we shall show that if the initial condition is bounded, then so is the solution u. More precisely if $u_0 \in L^\infty((0,1))$ then the solution $u \in L^\infty((0,1) \times (0,T))$. This last result generalizes to the nonlinear case the same result obtained in the linear case [8]. In section V, we study asymptotic behavior of the solution as t tends to infinity: assuming some asymptotic exponential decay on the data, we show that $u(t)$ converges as $t \to +\infty$ to the solution u_∞ of the corresponding steady state equation, with an exponential decay to 0 of the difference $u(t) - u_\infty$.

The aim of this paper is mainly to get some integral inequalities via various assumptions on the nonlinear term $F(r,u)$ in order to have some a priori estimates for $u(t)$, $tu(t)$ and his respectives derivatives in appropriate Sobolev spaces with weight. The hypotheses on $F(r,u)$ are sufficiently large to include a class enough great of nonlinear problems. For instance if we consider $\gamma = 2$ (radial Laplace in polar coordinates in $\mathbb{R}^3$) all the functions F of the kind $F(u) = sgn(u)|u|^\alpha$, $\alpha \in (0,2)$. In section VI we give numerical results.

II. PRELIMINARY RESULTS, NOTATIONS, FUNCTION SPACES

We omit the definitions of the usual function spaces $C^m([0,1])$, $L^p(0,1)$, $H^m(0,1)$, $W^{m,p}(0,1)$. For any function $v \in C^0([0,1])$ we define $\|v\|_0$ as

$$(2.1) \qquad \|v\|_0 \equiv \|v\|_{0,\gamma} = \left(\int_0^1 r^\gamma v^2(r) dr \right)^{1/2}$$

and define the space V_0 as completion of the space $C^0([0,1])$ with respect to the norm $\|\bullet\|_0$. Similarly, for any function $v \in C^1([0,1])$ we define $\|v\|_1$ as

$$(2.2) \qquad \|v\|_1 \equiv \|v\|_{1,\gamma} = \left(\|v\|_0^2 + \|v'\|_0^2 \right)^{1/2}$$

and define the space V_1 as completion of the space $C^1([0,1])$ with respect to the norm $\|\bullet\|_1$.

Note that the norms $\|\bullet\|_0$ and $\|\bullet\|_1$can be defined, respectively, from the inner products

$$(2.3) \qquad \begin{aligned} &\langle u,v \rangle = \int_0^1 r^\gamma u(r)v(r)dr \quad \text{and} \\ &\langle u,v \rangle + \langle u',v' \rangle = \int_0^1 r^\gamma [u(r)v(r) + u'(r)v'(r)]dr. \end{aligned}$$

It is then easy to prove that V_0 and V_1 are Hilbert spaces, with V_1 continuously and densely embedded into V_0. Identifying V_0 with its dual V_0' we have $V_1 \hookrightarrow V_0 \equiv V_0' \hookrightarrow V_1'$. On the other hand, the notation $\langle \bullet, \bullet \rangle$ is used for the pairing between V_1 and V_1'.

We then have the following lemmas, the proofs of which can be found in [5].

Lemma 1. *For every* $v \in C^1([0,1])$, $\gamma > 0$, $\varepsilon > 0$, *and* $r \in [0,1]$ *we have*

$$(2.4) \qquad \|v\|_0^2 \leq \frac{1}{\gamma} \|v'\|_0^2 + v^2(1),$$

$$(2.5) \qquad |v(1)| \leq K_1 \|v\|_1,$$

$$(2.6) \qquad r^{\gamma/2}|v(r)| \leq K_2 \|v\|_1,$$

$$(2.7) \qquad v^2(1) \leq \varepsilon \|v'\|_0^2 + C_\varepsilon \|v\|_0^2,$$

where

$$(2.8) \qquad K_1 = \sqrt{\gamma+2}\,,\ K_2 = \sqrt{\gamma+3}\,,\ C_\varepsilon = 1+\gamma+1/\varepsilon.$$

Lemma 2. *The embedding* $V_1 \hookrightarrow V_0$ *is compact*.

Remark 1. The result of (2.4), (2.5) proves that $\left(v^2(1) + \|v'\|_0^2\right)^{1/2}$ and $\|v\|_1$ are two equivalent norms on V_1 and

$$(2.9) \qquad \frac{\gamma}{\gamma+1} \|v\|_1^2 \leq v^2(1) + \|v'\|_0^2 \leq (\gamma+3)\|v\|_1^2, \text{ for all } v \in V_1.$$

We also note that

$$(2.10) \qquad \lim_{r \to 0_+} r^{\gamma/2} v(r) = 0, \text{ for all } v \in V_1.$$

(See [1], Lemma 5.40, p.128).

On the other hand, by $H^1(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1])$, $0 < \varepsilon < 1$, and

$$(2.11) \qquad \varepsilon^{\gamma/2} \|v\|_{H^1(\varepsilon,1)} \leq \|v\|_1, \text{ for all } v \in V_1,\ 0 < \varepsilon < 1.$$

It follows that

$$(2.12) \qquad v|_{[\varepsilon,1]} \in C^0([\varepsilon,1]), \text{ for all } \varepsilon,\ 0 < \varepsilon < 1.$$

From (2.10), (2.12) we deduce that

$$(2.13) \qquad r^{\gamma/2} v \in C^0([0,1]), \text{ for all } v \in V_1.$$

We denote by $\|\bullet\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < +\infty, \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X, \text{ for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u_r(t) = \nabla u(t)$, $u_{rr}(t)$ denote $u(r,t)$, $\frac{\partial u}{\partial t}(r,t)$, $\frac{\partial u}{\partial r}(r,t)$, $\frac{\partial^2 u}{\partial r^2}(r,t)$, respectively.

III. THE EXISTENCE AND UNIQUENESS THEOREM

We form the following assumptions

(H_1) $\quad u_0 \in V_0$, $\tilde{u}_0 \in \mathbb{R}$;

(H_2) $\quad a, h \in W^{1,\infty}(0,T)$, $a(t) \geq a_0 > 0$;

(H_3) $\quad f \in L^2(0,T;V_0)$;

$F : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition, i.e.,

(F_1) $\quad F(\bullet, u)$ is measurable on $(0,1)$ for every $u \in \mathbb{R}$,
and $F(r,\bullet)$ is continuous on $\mathbb{R}$ for a.e., $r \in (0,1)$.

(F_2) $\quad$ There exist positive constants C_1, C_1', C_2 and p, $1 < p < 2 + 2/\gamma$ such that

(i) $\quad uF(r,u) \geq C_1|u|^p - C_1'$,

$(2i)$ $\quad |F(r,u)| \leq C_2(1 + |u|^{p-1})$.

The weak formulation of the initial and boundary value problem (1.1)-(1.3) can be make in the following manner:

Find $u(t)$ defined in the open set $(0,T)$ such that $u(t)$ satisfies the following variational problem

$$\begin{aligned}\frac{d}{dt}\langle u(t),v\rangle + a(t)\langle u_r(t),v_r\rangle + a(t)h(t)u(1,t)v(1) + \langle F(r,u(t)),v\rangle \\ = \langle f(t),v\rangle + \tilde{u}_0 a(t)h(t)v(1), \quad \text{for all } v \in V_1,\end{aligned} \tag{3.1}$$

and the initial condition

$$u(0) = u_0. \tag{3.2}$$

We then have the following theorem.

Theorem 1. *Let $T > 0$ and $(H_1) - (H_3)$, (F_1), (F_2) hold. Then, there exists a solution u of problem* (3.1)- (3.2) *such that*

$$\begin{aligned}&u \in L^2(0,T;V_1) \cap L^\infty(0,T;V_0), \quad r^{\gamma/p}u \in L^p(Q_T),\\ &tu \in L^\infty(0,T;V_1), \quad tu_t \in L^2(0,T;V_0).\end{aligned} \tag{3.3}$$

Furthermore, if F satisfies the following condition, in addition,

(F_3) $\quad (F(r,u)-F(r,v))(u-v) \geq -\varepsilon|u-v|^2$, *for all* $u,v \in \mathbb{R}$,
for $a.e.$, $r \in (0,1)$, *with* $\varepsilon > 0$ *sufficiently small,*

then the solution is unique.

Proof. The proof consists of several steps.

Step1. The Galerkin method. Denote by $\{w_j\}$, $j = 1,2,\ldots$ an orthonormal basis in the separable Hilbert space V_1. We find $u_m(t)$ of the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)w_j, \tag{3.4}$$

where c_{mj} satisfy the following system of nonlinear differential equations

$$\begin{aligned}\langle u_m'(t), w_j\rangle + a(t)\,\langle u_{mr}(t), w_{jr}\rangle + a(t)h(t)u_m(1,t)w_j(1) + \langle F(r,u_m(t)), w_j\rangle \\ = \langle f(t)), w_j\rangle + \tilde{u}_0 a(t)h(t)w_j(1),\ 1 \leq j \leq m,\end{aligned} \tag{3.5}$$

$$u_m(0) = u_{0m}, \tag{3.6}$$

where

$$u_{0m} \to u_0 \ \text{strongly in}\ V_0. \tag{3.7}$$

It is clear that for each m there exists a solution $u_m(t)$ in form (3.4) which satisfies (3.5) and (3.6) almost everywhere on $0 \leq t \leq T_m$ for some T_m, $0 < T_m \leq T$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. A priori estimates.

a) *The first estimate*. Multiplying the j^{th} equation of the system (3.5) by $c_{mj}(t)$ and summing up with respect to j, we have

$$\begin{aligned}\frac{d}{dt}\|u_m(t)\|_0^2 + 2a(t)\|u_{mr}(t)\|_0^2 + 2u_m^2(1,t) + 2\langle F(r,u_m(t)), u_m(t)\rangle \\ = 2(1-a(t)h(t))u_m^2(1,t) + 2\langle f(t), u_m(t)\rangle \\ +2\tilde{u}_0 a(t)h(t)u_m(1,t).\end{aligned} \tag{3.8}$$

By the assumptions (H_2), (F_2, i), and the inequalities (2.5), (2.7), (2.9), it follows from (3.8), that

$$\frac{d}{dt}\|u_m(t)\|_0^2 + 2C_3\|u_m(t)\|_1^2 + 2C_1\int_0^1 r^\gamma |u_m(r,t)|^p dr$$
$$\leq \frac{2C_1'}{\gamma+1} + \frac{1}{2\varepsilon_1}|\widetilde{u}_0|^2 K_1^2 \|ah\|^2_{L^\infty(0,T)} + \|f(t)\|_0^2 \tag{3.9}$$
$$+ 2\varepsilon_1\Big(2 + \|ah\|_{L^\infty(0,T)}\Big)\|u_m(t)\|_1^2$$
$$+ \Big(1 + 2C_{\varepsilon_1}(1 + \|ah\|_{L^\infty(0,T)})\Big)\|u_m(t)\|_0^2,$$

for all $\varepsilon_1 > 0$. Choosing $\varepsilon_1 > 0$ such that

$$2\varepsilon_1(2 + \|ah\|_{L^\infty(0,T)}) \leq \frac{\gamma}{\gamma+1}\min\{1, a_0\} \equiv C_3. \tag{3.10}$$

Hence, from (3.9), (3.10) we obtain

$$\frac{d}{dt}\|u_m(t)\|_0^2 + C_3\|u_m(t)\|_1^2 + 2C_1\int_0^1 r^\gamma |u_m(r,t)|^p dr$$
$$\leq \frac{2C_1'}{\gamma+1} + \frac{1}{2\varepsilon_1}|\widetilde{u}_0|^2 K_1^2 \|ah\|^2_{L^\infty(0,T)} + \|f(t)\|_0^2 \tag{3.11}$$
$$+\Big(1 + 2C_{\varepsilon_1}(1 + \|ah\|_{L^\infty(0,T)})\Big)\|u_m(t)\|_0^2.$$

Integrating (3.11) and by means of (3.7), we have

$$\|u_m(t)\|_0^2 + C_3\int_0^t \|u_m(s)\|_1^2 ds + 2C_1\int_0^t ds\int_0^1 r^\gamma |u_m(r,s)|^p dr$$
$$\leq M_T^{(2)} + M_T^{(1)}\int_0^t \|u_m(s)\|_0^2 ds, \tag{3.12}$$

where $M_T^{(1)}, M_T^{(2)}$ are the constants depending only on T, with

$$M_T^{(1)} = 1 + 2C_{\varepsilon_1}(1 + \|ah\|_{L^\infty(0,T)}),$$
$$M_T^{(2)} \geq \|u_{0m}\|_0^2 + \Big(\frac{2C_1'}{\gamma+1} + \frac{1}{2\varepsilon_1}|\widetilde{u}_0|^2 K_1^2 \|ah\|^2_{L^\infty(0,T)}\Big)T + \int_0^T \|f(s)\|_0^2 ds,$$

for all m.

By the Gronwall's lemma, we obtain from (3.12), that

$$\|u_m(t)\|_0^2 + C_3 \int_0^t \|u_m(s)\|_1^2 ds + 2C_1 \int_0^t ds \int_0^1 r^\gamma |u_m(r,s)|^p dr \tag{3.13}$$
$$\leq M_T^{(2)} \exp(tM_T^{(1)}) \leq M_T,$$

for all m, for all t, $0 \leq t \leq T_m \leq T$, i.e., $T_m = T$.

b) *The second estimate*. Multiplying the j^{th} equation of the system (3.5) by $t^2 c'_{mj}(t)$ and summing up with respect to j, we have

$$2\left\| tu'_m(t) \right\|_0^2 + \frac{d}{dt}\Big(a(t)\|tu_{mr}(t)\|_0^2 + a(t)h(t)t^2u_m^2(1,t)\Big)$$
$$+2\frac{d}{dt}\left(t^2 \int_0^1 r^\gamma \widehat{F}(r,u_m(r,t))dr\right)$$
$$= \|u_{mr}(t)\|_0^2 \frac{d}{dt}[t^2a(t)] + u_m^2(1,t)\frac{d}{dt}[t^2a(t)h(t)] \tag{3.14}$$
$$+4t \int_0^1 r^\gamma \widehat{F}(r,u_m(r,t))dr + 2\langle tf(t), tu'_m(t)\rangle$$
$$+ 2\tilde{u}_0 \frac{d}{dt}[t^2a(t)h(t)u_m(1,t)] - 2\tilde{u}_0 u_m(1,t)\frac{d}{dt}[t^2a(t)h(t)],$$

where

$$\widehat{F}(r,\lambda) = \int_0^\lambda F(r,s)ds. \tag{3.15}$$

Integrating (3.14) with respect to time variable from 0 to t, we shall have, after some rearrangements

$$2\int_0^t \left\| su'_m(s) \right\|_0^2 ds + a(t)\|tu_{mr}(t)\|_0^2 + t^2u_m^2(1,t) \tag{3.16}$$

$$= [1 - a(t)h(t)]t^2u_m^2(1,t) + \int_0^t [s^2a(s)]'\|u_{mr}(s)\|_0^2 ds$$
$$+ \int_0^t [s^2a(s)h(s)]'u_m^2(1,s)ds$$
$$+4\int_0^t sds\int_0^1 r^\gamma \widehat{F}(r,u_m(r,s))dr - 2t^2\int_0^1 r^\gamma \widehat{F}(r,u_m(r,t))dr$$
$$+2\int_0^t \langle sf(s), su_m'(s)\rangle ds + 2\tilde{u}_0 t^2 a(t)h(t)u_m(1,t)$$
$$-2\tilde{u}_0\int_0^t [s^2a(s)h(s)]'u_m(1,s)ds.$$

By means of the assumption (H_2) and the inequality (2.9), we have

$$(3.17)\qquad a(t)\|tu_{mr}(t)\|_0^2 + t^2u_m^2(1,t) \geq C_3\|tu_m(t)\|_1^2$$

for all $t \in [0,T]$, for all m, where C_3 is constant defined by (3.10).
Using the inequalities (2.5), (2.7), and with $\varepsilon_1 > 0$ as in (3.10), we estimate without difficulty the following terms in the right-hand side of (3.16) as follows

$$(3.18)\qquad [1 - a(t)h(t)]t^2u_m^2(1,t) \leq \left(1 + \|ah\|_{L^\infty(0,T)}\right)\left(\varepsilon_1\|tu_m(t)\|_1^2 + C_{\varepsilon_1}t^2M_T\right),$$

$$(3.19)\qquad \int_0^t [s^2a(s)]'\|u_{mr}(s)\|_0^2 ds + \int_0^t [s^2a(s)h(s)]'u_m^2(1,s)ds \leq \left(\|(t^2a)'\|_{L^\infty(0,T)} + K_1^2\|(t^2ah)'\|_{L^\infty(0,T)}\right)(M_T/C_3),$$

$$(3.20)\qquad 2\left|\tilde{u}_0\int_0^t [s^2a(s)h(s)]'u_m(1,s)ds\right| \leq 2|\tilde{u}_0|\|(t^2ah)'\|_{L^\infty(0,T)}K_1\sqrt{t}\,(M_T/C_3)^{1/2},$$

$$(3.21)\qquad 2|\tilde{u}_0t^2a(t)h(t)u_m(1,t)| \leq \varepsilon_1\|tu_m(t)\|_1^2 + \tfrac{1}{\varepsilon_1}\left(K_1\tilde{u}_0t\|ah\|_{L^\infty(0,T)}\right)^2,$$

$$(3.22)\qquad 2\left|\int_0^t \langle sf(s), su_m'(s)\rangle ds\right| \leq \int_0^t \|sf(s)\|_0^2 ds + \int_0^t \left\|su_m'(s)\right\|_0^2 ds.$$

On the other hand, from the assumptions (F_1), (F_2), we have

$$(3.23)\qquad -m_0 \equiv -\int_{-\lambda_0}^{\lambda_0} |F(r,s)|ds \leq \widehat{F}(r,\lambda) = \int_0^\lambda F(r,s)ds \leq C_2(|\lambda| + \tfrac{|\lambda|^p}{p}), \qquad \text{for all } \lambda \in \mathbb{R},$$

where $\lambda_0 = \left(C_1^{/}/C_1\right)^{1/p}$.

Using the inequalities (2.6), (3.13), (3.23), we obtain

$$(3.24)\qquad \begin{aligned} & 4\int_0^t sds\int_0^1 r^{\gamma}\widehat{F}(r,u_m(r,s))dr - 2t^2\int_0^1 r^{\gamma}\widehat{F}(r,u_m(r,t))dr \\ & \qquad \le \tfrac{4C_2K_2}{1+\gamma/2}\int_0^t \|su_m(s)\|_1 ds \\ & \qquad + \tfrac{4C_2t}{p}(M_T/2C_1) + \tfrac{2m_0t^2}{1+\gamma}. \end{aligned}$$

Hence, we deduce from (3.16) - (3.22), and (3.24) that

$$(3.25)\qquad \begin{aligned} & \int_0^t \left\|su_m^{/}(s)\right\|_0^2 ds + \tfrac{1}{2}C_3\|tu_m(t)\|_1^2 \\ & \qquad \le M_T^{(3)} + \tfrac{4C_2K_2}{1+\gamma/2}\int_0^t \|su_m(s)\|_1 ds \le M_T^{(4)} + \int_0^t \|su_m(s)\|_1^2 ds, \end{aligned}$$

where $M_T^{(3)}$, $M_T^{(4)}$ are the constants depending only on T.
By the Gronwall's lemma, we obtain from (3.25), that

$$(3.26)\qquad \int_0^t \left\|su_m^{/}(s)\right\|_0^2 ds + \tfrac{1}{2}C_3\|tu_m(t)\|_1^2 \le M_T^{(4)}\exp(t) \le M_T^{(5)}.$$

On the other hand, by using (3.13), and assumption (F_2) we have

$$(3.27)\qquad \begin{aligned} & \int_0^t ds\int_0^1 \left|r^{\gamma/p^{/}}F(r,u_m(r,s))\right|^{p^{/}} dr \\ & \qquad \le 2^{p^{/}-1}C_2^{p^{/}}\left(\tfrac{T}{1+\gamma} + \int_0^t ds\int_0^1 r^{\gamma}|u_m(r,s)|^p dr\right) \le M_T^{(6)}, \end{aligned}$$

where $M_T^{(6)}$ is a constant depending only on T.

Step 3. The limiting process.

By (3.13), (3.26), (3.27) we deduce that, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$ such that

$$(3.28) \qquad u_m \to u \qquad \text{in } L^\infty(0,T;V_0) \quad \text{weak*},$$

$$(3.29) \qquad u_m \to u \qquad \text{in } L^2(0,T;V_1) \quad \text{weak},$$

$$(3.30) \qquad r^{\gamma/p}u_m \to r^{\gamma/p}u \qquad \text{in } L^p(Q_T) \quad \text{weak},$$

$$(3.31) \qquad tu_m \to tu \qquad \text{in } L^\infty(0,T;V_1) \quad \text{weak*},$$

$$(3.32) \qquad (tu_m)' \to (tu)' \qquad \text{in } L^2(0,T;V_0) \quad \text{weak}.$$

Using a compactness lemma ([2], Lions, p.57) applied to (3.31), (3.32), we can extract from the sequence $\{u_m\}$ a subsequence still denotes by $\{u_m\}$, such that

$$(3.33) \qquad tu_m \to tu \qquad \text{strongly in } L^2(0,T;V_0).$$

By the Riesz- Fischer theorem, we can extract from $\{u_m\}$ a subsequence still denoted by $\{u_m\}$, such that

$$(3.34) \qquad u_m(r,t) \to u(r,t) \qquad \text{a.e. } (r,t) \text{ in } Q_T = (0,1)\times(0,T).$$

Because F is continuous, then

$$(3.35) \qquad F(r,u_m(r,t)) \to F(r,u(r,t)) \qquad \text{a.e. } (r,t) \text{ in } Q_T.$$

We shall now require the following lemma, the proof of which can be found in [2].
Lemma 3. *Let Q be a bounded open set of $\mathbb{R}^N$ and G_m, $G \in L^q(Q)$, $1 < q < \infty$, such that,*

$$\|G_m\|_{L^q(Q)} \le C, \text{ where } C \text{ is a constant independent of } m$$

and

$$G_m \to G \;\; a.e. \; (r,t) \text{ in } Q.$$

Then $G_m \to G$ in $L^q(Q)$ weakly.

Applying Lemma 3 with $N = 2$, $q = p'$, $G_m = r^{\gamma/p'}F(r,u_m)$, $G = r^{\gamma/p'}F(r,u)$, we deduce from (3.27), (3.35) that in

$$(3.36) \qquad r^{\gamma/p'}F(r,u_m) \to r^{\gamma/p'}F(r,u) \quad \text{in } L^{p'}(Q_T) \text{ weakly}.$$

Passing to the limit in (3.5), (3.6) by (3.7), (3.28), (3.29), (3.36) we have satisfying the equation

$$\frac{d}{dt}\langle u(t),v\rangle + a(t)\langle u_r(t),v_r\rangle + a(t)h(t)u(1,t)v(1) + \langle F(r,u(t)),v\rangle = \langle f(t),v\rangle + \tilde{u}_0 a(t)h(t)v(1),\ \text{for all}\ v\in V_1, \tag{3.37}$$

$$u(0) = u_0. \tag{3.38}$$

Step 4. Uniqueness of the solutions.

First, we shall need the following Lemma.

Lemma 4. *Let w be the weak solution of the following problem*

$$w_t - a(t)(w_{rr} + \tfrac{\gamma}{r}w_r) = \tilde{f}(r,t),\ 0<r<1,\ 0<t<T, \tag{3.39}$$

$$\left|\lim_{r\to 0_+} r^{\gamma/2}w_r(r,t)\right| < +\infty,\quad w_r(1,t) + h(t)w(1,t) = 0, \tag{3.40}$$

$$w(r,0) = 0, \tag{3.41}$$

$$\begin{aligned} &w\in L^2(0,T;V_1)\cap L^\infty(0,T;V_0),\ \ r^{\gamma/p}w\in L^p(Q_T),\\ &tw\in L^\infty(0,T;V_1),\ \ tw_t\in L^2(0,T;V_0). \end{aligned} \tag{3.42}$$

Then

$$\begin{aligned} &\tfrac{1}{2}\|w(t)\|_0^2 + \int_0^t a(s)[\|w_r(s)\|_0^2 + h(s)w^2(1,s)]ds\\ &\qquad - \int_0^t \langle \tilde{f}(s),w(s)\rangle ds = 0,\ a.e.\ t\in(0,T). \end{aligned} \tag{3.43}$$

The lemma 4 is a slight improvement of a lemma used in [8] (see also Lions's book [2]). Now, we will prove the uniqueness of the solutions. Let u and v be two weak solutions of (1.1)- (1.3). Then $w = u - v$ is a weak solution of the following problem (3.39)- (3.42) with the right hand side function replaced by $\tilde{f}(r,t) = -F(u) + F(v)$. Using Lemma 4 we have equality

$$\begin{aligned} &\tfrac{1}{2}\|w(t)\|_0^2 + \int_0^t a(s)[\|w_r(s)\|_0^2 + h(s)w^2(1,s)]ds\\ &\qquad = -\int_0^t \langle F(r,u) - F(r,v),w(s)\rangle ds. \end{aligned} \tag{3.44}$$

Using the monotonicity of $F(r,u) + \varepsilon u$, we obtain

$$(3.45) \qquad \int_0^t \langle F(r,u) - F(r,v), w(s)\rangle ds \geq -\varepsilon \int_0^t \|w(s)\|_0^2 ds.$$

It follows from (3.44), (3.45) and Gronwall's Lemma that $w = 0$.
Therefore, Theorem 1 is proved.■

IV. THE BOUNDEDNESS OF THE SOLUTION

Now we make the following assumptions

$(H_1^/)$ $\quad u_0 \in L^\infty(0,1),\ \tilde{u}_0 \in \mathbb{R},\ \max\{|u_0(r)|, |\tilde{u}_0|\} \leq M \quad a.e. \quad r \in (0,1).$

$(H_2^/)$ $\quad a, h \in W^{1,\infty}(0,\infty),\ a(t) \geq a_0 > 0,\ h(t) \geq h_0 > 0;$

$(H_3^/)$ $\quad f \in L^2(0,T;V_0), \quad f(r,t) \leq 0\ a.e. \quad (r,t) \in Q_T.$

$(F_1^/)$ $\quad uF(r,u) \geq 0\ \ \forall u \in \mathbb{R},\ |u| \geq \|u_0\|_{L^\infty(0,1)}$, for $a.e.$, $r \in (0,1)$.

We then have the following theorem.

Theorem 2. *Let* $(H_1^/) - (H_3^/)$, $(F_1) - (F_3)$, $(F_1^/)$ *hold. Then the unique weak solution of the initial and boundary value problem* (3.1) - (3.2), *as given by theorem 1, belongs to*

$L^\infty(Q_T)$.

Remark 3. Assumption $(H_1^/)$ is both physically and mathematically natural in the study of partial differential equation of the kind of (1.1)-(1.3), by means of the maximum principle.

Proof of Theorem 2. First, let us assume that $u_0(r) \leq M$ and $\tilde{u}_0$. Then $z = u - M$ satisfies the initial and boundary value

$$(4.1) \qquad z_t - a(t)(z_{rr} + \tfrac{\gamma}{r} z_r) + F(r, z+M) = f(r,t),\ 0 < r < 1,\ 0 < t < T,$$

$$(4.2) \qquad \begin{aligned} &\left|\lim_{r\to 0_+} r^{\gamma/2} z_r(r,t)\right| < +\infty, \\ &z_r(1,t) + h(t)\,(z(1,t) + M - \tilde{u}_0) = 0, \end{aligned}$$

$$(4.3) \qquad z(r,0) = u_0(r) - M.$$

Multiplying equation (4.1) by $r^\gamma v$, for $v \in V_1$ integrating by parts with respect to variable r and taking into account boundary condition (4.2), one has after some rearrangements

$$\int_0^1 r^\gamma z_t v dr + a(t)\int_0^1 r^\gamma z_r v_r dr + a(t)h(t)z(1,t)v(1)$$

$$(4.4) \qquad + \int_0^1 r^\gamma F(r, z+M) v dr$$

$$= \int_0^1 r^\gamma f v dr + (\widetilde{u}_0 - M)a(t)h(t)v(1), \text{ for all } v \in V_1.$$

Noticing from assumption $(H_1^{/})$ we deduce that the solution of the initial and boundary value problem (3.1) - (3.2) belongs to $L^2(0,T;V_1) \cap L^\infty(0,T;V_0)$, so that we are allowed to take $v = z^+ = \frac{1}{2}(|z| + z)$ in (4.4). Thus, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 r^\gamma |z^+|^2 dr + a(t)\int_0^1 r^\gamma |(z^+)_r|^2 dr + a(t)h(t)|z^+(1,t)|^2$$

$$(4.5) \qquad + \int_0^1 r^\gamma F(r, z^+ + M) z^+ dr$$

$$= \int_0^1 r^\gamma f z^+ dr + (\widetilde{u}_0 - M)a(t)h(t)z^+(1,t) \leq 0,$$

since

$$\int_0^1 r^\gamma z_t z^+ dr = \int_{0,\, z>0}^1 r^\gamma (z^+)_t z^+ dr = \frac{1}{2}\frac{d}{dt}\int_{0,\, z>0}^1 r^\gamma |z^+|^2 dr$$

$$= \frac{1}{2}\frac{d}{dt}\int_0^1 r^\gamma |z^+|^2 dr = \frac{1}{2}\frac{d}{dt}\|z^+(t)\|_0^2,$$

and on the domain $z > 0$ we have $z^+ = z$ and $z_r = (z^+)_r$.
On the other hand, by the assumption $(H_2^{/})$ and the inequality (2.9), we obtain

$$(4.6) \qquad a(t)\int_0^1 r^\gamma |(z^+)_r|^2 dr + a(t)h(t)|z^+(1,t)|^2$$

$$\geq \frac{\gamma a_0}{\gamma+1}\min\{1, h_0\}\|z^+(t)\|_1^2 = \widetilde{C}_0\|z^+(t)\|_1^2.$$

Using the monotonicity of $F(r,u) + \varepsilon u$ and $(F_1^{/})$ we obtain

$$(4.7)\qquad \int_0^1 r^\gamma F(r,z^+ + M)z^+ dr = \int_0^1 r^\gamma [F(r,z^+ + M) - F(r,M)]z^+ dr + \int_0^1 r^\gamma F(r,M)z^+ dr$$
$$\geq -\varepsilon \int_0^1 r^\gamma |z^+|^2 dr + \int_0^1 r^\gamma F(r,M)z^+ dr \geq -\varepsilon \|z^+(t)\|_0^2.$$

Hence, it follows from (4.5)-(4.7) that

$$(4.8)\qquad \frac{d}{dt}\|z^+(t)\|_0^2 + 2\widetilde{C}_0\|z^+(t)\|_1^2 \leq 2\varepsilon\|z^+(t)\|_0^2.$$

Integrating (4.8), we get

$$(4.9)\qquad \|z^+(t)\|_0^2 \leq \|z^+(0)\|_0^2 + 2\varepsilon \int_0^t \|z^+(s)\|_0^2 ds.$$

Since $z^+(0) = (u(r,0) - M)^+ = (u_0(r) - M)^+ = 0$, hence, using Gronwall's Lemma, we obtain $\|z^+(t)\|_0^2 = 0$. Thus $z^+ = 0$ and $u(r,t) \leq M$ for $a.e.\ (r,t) \in Q_T$.

The case $-M \leq u_0(r)$ and $-M \leq \widetilde{u}_0$ can be dealt with, in the same manner as above, by considering $z = u + M$ and $z^- = \frac{1}{2}(|z| - z)$, we also obtain $z^- = 0$ and hence $u(r,t) \geq -M$ for $a.e.\ (r,t) \in Q_T$.

From all above, one obtains $|u(r,t)| \leq M\ \ a.e.\ (r,t) \in Q_T$ and this ends the proof of Theorem 2.■

V. ASYMPTOTIC BEHAVIOR OF THE SOLUTION AS $t \to +\infty$.

In this part, let $T > 0$, $(H_1) - (H_3)$, and $(F_1) - (F_3)$ hold. Then, there exists a unique solution u of problem (3.1) - (3.2) such that

$$u \in L^2(0,T;V_1) \cap L^\infty(0,T;V_0),\ \ r^{\gamma/p}u \in L^p(Q_T),$$
$$tu \in L^\infty(0,T;V_1),\ tu' \in L^2(0,T;V_0).$$

We shall study asymptotic behavior of the solution $u(t)$ as $t \to +\infty$.

We make the following supplementary assumptions on the functions a, h, f.

$(H_3^{//})$ $\qquad f \in L^\infty(0,\infty;V_0)$;

(H_4) $\qquad$ There exist the positive constants C_a, C_h, C_f, γ_a, γ_h, γ_f, a_∞, h_∞ and a function $f_\infty \in V_0$ such that

$$\begin{array}{ll} (i) & |a(t)-a_\infty| \le C_a e^{-\gamma_a t},\ \ \forall t\ge 0,\\ (ii) & |h(t)-h_\infty| \le C_h e^{-\gamma_h t},\ \ \forall t\ge 0,\\ (iii) & \|f(t)-f_\infty\|_0 \le C_f e^{-\gamma_f t},\ \ \forall t\ge 0. \end{array}$$

First, we consider the following stationary problem

$$(5.1)\qquad -a_\infty\left(u_\infty^{//}(r)+\tfrac{\gamma}{r}u_\infty^{/}(r)\right)+F(r,u_\infty(r)) = f_\infty(r),\ 0<r<1,$$

$$(5.2)\qquad \left|\lim_{r\to 0_+} r^{\gamma/2}u_\infty^{/}(r)\right| < +\infty,\ \ u_\infty^{/}(1)+h_\infty u_\infty(1) = h_\infty \tilde{u}_0.$$

The weak solution of problem (5.1)-(5.2) is obtained from the following variational problem.

Find $u_\infty \in V_1$ such that

$$(5.3)\qquad \begin{aligned} &a_\infty\langle u_\infty^{/},v^{/}\rangle + a_\infty h_\infty u_\infty(1)v(1)+\langle F(r,u_\infty),v\rangle\\ &\qquad = \langle f_\infty,v\rangle+\tilde{u}_0 a_\infty h_\infty v(1),\ \text{ for all } v\in V_1. \end{aligned}$$

We then have the following theorem.

Theorem 3. *Let* (F_1), (F_2), (H_4) *hold. Then there exists a solution* u_∞ *of the variational problem* (5.3) *such that*

$$u_\infty \in V_1 \ \ \textit{and}\ \ r^{\gamma/p}u_\infty \in L^p(0,1).$$

Furthermore, if F satisfies the following condition, in addition,

$$(F_4)\qquad \begin{aligned} &F(r,u)+\varepsilon u \ \ \textit{is nondecreasing with respect to variable } u,\\ &\textit{with } 0<\varepsilon<\tfrac{\gamma a_\infty}{\gamma+1}\min\{1,h_\infty\}. \end{aligned}$$

Then the solution is unique.

Proof. Denote by $\{w_j\}$, $j=1,2,\ldots$ an orthonormal basis in the separable Hilbert space V_1. Put

$$(5.4)\qquad y_m = \sum_{j=1}^{m} d_{mj}w_j,$$

where d_{mj} satisfy the following nonlinear equation system:

$$(5.5)\qquad \begin{aligned} & a_\infty\langle y_m', w_j'\rangle + a_\infty h_\infty y_m(1) w_j(1) + \langle F(r, y_m), w_j\rangle \\ &\qquad = \langle f_\infty, w_j\rangle + \widetilde{u}_0 a_\infty h_\infty w_j(1),\ 1 \le j \le m. \end{aligned}$$

By the Brouwer's lemma(see Lions [2], Lemma 4.3, p.53), it follows from the hypotheses (F_1), (F_2), (H_4) that system (5.4), (5.5) has a solution y_m.
Multiplying the j^{th} equation of system (5.5) by d_{mj}, then summing up with respect to j, we have

$$(5.6)\qquad \begin{aligned} & a_\infty \left\| y_m' \right\|_0^2 + a_\infty h_\infty y_m^2(1) + \langle F(r, y_m), y_m\rangle \\ &\qquad = \langle f_\infty, y_m\rangle + \widetilde{u}_0 a_\infty h_\infty y_m(1). \end{aligned}$$

By using the inequalities (2.5), (2.9) and by the hypotheses (F_1), (H_4), we obtain

$$(5.7)\qquad \begin{aligned} & C_0 \| y_m \|_1^2 + C_1 \int_0^1 r^\gamma |y_m(r)|^p dr \\ &\qquad \le (\| f_\infty \|_0 + |\widetilde{u}_0| a_\infty h_\infty K_1) \| y_m \|_1 + \tfrac{C_1'}{\gamma+1}, \end{aligned}$$

where $C_0 = \frac{\gamma a_\infty}{\gamma+1} \min\{1, h_\infty\}$.

Hence, we deduce from (5.7) that

$$(5.8)\qquad \| y_m \|_1 \le C,$$

$$(5.9)\qquad \int_0^1 r^\gamma |y_m(r)|^p dr \le C,$$

C is a constant independent of m.
By means of (5.8), (5.9) and Lemma 2, the sequence $\{y_m\}$ has a subsequence still denoted by $\{y_m\}$ such that

$$(5.10)\qquad y_m \to u_\infty \quad \text{in } V_1 \text{ weakly,}$$

$$(5.11)\qquad y_m \to u_\infty \quad \text{in } V_0 \text{ strongly and } a.e. \text{ in } (0,1),$$

$$(5.12)\qquad r^{\gamma/p} y_m \to r^{\gamma/p} u_\infty \quad \text{in } L^p(0,1) \text{ weakly.}$$

On the other hand, by (5.11) and the hypothesis (F_1), (F_2) we have

$$(5.13) \qquad F(r,y_m) \to F(r,u_\infty) \text{ a.e. in } (0,1),$$

We also deduce from the hypothesis (F_2) and from (5.9) that

$$(5.14) \qquad \int_0^1 \left| r^{\gamma/p'} F(r,y_m(r)) \right|^{p'} dr \leq 2^{p'-1} C_2^{p'} [1 + \int_0^1 r^\gamma |y_m(r)|^p dr] \leq C,$$

where C is a constant independing of m.

Applying Lemma 3 with $N = 1$, $q = p'$, $G_m = r^{\gamma/p'} F(r,y_m)$, $G = r^{\gamma/p'} F(r,u_\infty)$, we deduce from (5.13), (5.14) that

$$(5.15) \qquad r^{\gamma/p'} F(r,y_m) \to r^{\gamma/p'} F(r,u_\infty) \text{ in } L^{p'}(0,1) \text{ weakly.}$$

Passing to the limit in Eq.(5.5), we find without difficulty from (5.10), (5.15) that u_∞ satisfies the equation

$$(5.16) \qquad a_\infty \langle u'_\infty, w'_j \rangle + a_\infty h_\infty u_\infty(1) w_j(1) + \langle F(r,u_\infty), w_j \rangle = \langle f_\infty, w_j \rangle + \tilde{u}_0 a_\infty h_\infty w_j(1).$$

Equation (5.16) holds for every $j = 1,2,\ldots$, i.e., (5.3) holds.
The solution of the problem (5.3) is unique; that can be showed using the same arguments as in the proof of Theorem 1.■

Remark 4. The result of Theorem 3 is similar to one in [7].

Now we consider asymptotic behavior of the solution $u(t)$ as $t \to +\infty$.

We then have the following theorem.

Theorem 4. *Let* (F_1), (F_2), (F_4), (H_1), (H'_2), (H''_3), (H_4) *hold. Then we have*

$$\|u(t) - u_\infty\|_0^2 \leq \left(\|u_0 - u_\infty\|_0^2 + \frac{\widetilde{C}_2}{2(\gamma_1 - \gamma_0)} \right) e^{-2\gamma_0 t}, \ \forall t \geq 0,$$

where

$$\widetilde{C}_2 = \tfrac{1}{\delta} (C^2 C_a^2 + C_f^2 + K_1^2 (|\tilde{u}_0| + CK_1)^2 (C_a \|h\|_\infty + C_h a_\infty)^2),$$

$$\delta = \tfrac{1}{4} \left(\tfrac{\gamma a_0}{\gamma+1} \min\{1,h_0\} - \varepsilon \right),$$

γ_0 *is a constant depending only on the constants* $\gamma_1 = \min\{\gamma_a, \gamma_h, \gamma_f\}$ *and* $\widetilde{C}_1 = \frac{\gamma a_0}{\gamma+1} \min\{1,h_0\} - \varepsilon$.

Proof. Put $Z_m(t) = u_m(t) - y_m$. Let us subtract (3.5) with (5.5) to obtain

$$(5.17)\qquad \begin{cases} \langle Z_m'(t), w_j\rangle + a(t)\langle Z_{mr}(t), w_{jr}\rangle + (a(t)-a_\infty)\langle y_{mr}, w_{jr}\rangle \\ \qquad + a(t)h(t)Z_m(1,t)w_j(1) + (a(t)h(t)-a_\infty h_\infty)y_m(1)w_j(1) \\ \qquad + \langle F(r,u_m(t)) - F(r,y_m), w_j\rangle \\ \qquad = \langle f(t)-f_\infty, w_j\rangle + \widetilde{u}_0(a(t)h(t)-a_\infty h_\infty)w_j(1),\ 1\le j\le m, \\ Z_m(0) = u_{0m} - y_m. \end{cases}$$

By multiplying (5.17) by $c_{mj}(t) - d_{mj}$ and summing up in j, we obtain

$$(5.18)\qquad \begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|Z_m(t)\|_0^2 + a(t)\|Z_{mr}(t)\|_0^2 + (a(t)-a_\infty)\langle y_{mr}, Z_{mr}(t)\rangle \\ &\qquad + a(t)h(t)Z_m^2(1,t) + (a(t)h(t)-a_\infty h_\infty)y_m(1)Z_m(1,t) \\ &\qquad + \langle F(r,u_m(t)) - F(r,y_m), Z_m\rangle \\ &\qquad = \langle f(t)-f_\infty, Z_m\rangle + \widetilde{u}_0(a(t)h(t)-a_\infty h_\infty)Z_m(1). \end{aligned}$$

From the assumption (H_2') and the inequality (2.9), it follows that

$$(5.19)\qquad a(t)\|Z_{mr}(t)\|_0^2 + a(t)h(t)Z_m^2(1,t) \ge \widetilde{C}_0\|Z_m(t)\|_1^2,$$

where $\widetilde{C}_0 = \frac{\gamma a_0}{\gamma+1}\min\{1,h_0\}$.

By (F_4), we get

$$(5.20)\qquad \langle F(r,u_m(t)) - F(r,y_m), Z_m\rangle \ge -\varepsilon\|Z_m(t)\|_0^2.$$

It follows from (5.18)-(5.20), and (2.5), that

$$(5.21)\qquad \begin{aligned} &\tfrac{d}{dt}\|Z_m(t)\|_0^2 + 2\widetilde{C}_0\|Z_m(t)\|_1^2 \le 2|a(t)-a_\infty|\|y_{mr}\|_0\|Z_{mr}(t)\|_0 \\ &\qquad + 2|a(t)h(t)-a_\infty h_\infty|K_1^2\|y_m\|_1\|Z_m(t)\|_1 + 2\varepsilon\|Z_m(t)\|_0^2 \\ &\qquad + 2\|f(t)-f_\infty\|_0\|Z_m(t)\|_0 + 2|\widetilde{u}_0||a(t)h(t)-a_\infty h_\infty|K_1\|Z_m(t)\|_1. \end{aligned}$$

Note that $\|y_m\|_1 \le C$, we obtain from (5.21) that

$$
(5.22) \qquad \begin{aligned}
&\frac{d}{dt}\|Z_m(t)\|_0^2 + 2\widetilde{C}_0\|Z_m(t)\|_1^2 \\
&\quad \leq 2C|a(t) - a_\infty|\|Z_m(t)\|_1 + 2\varepsilon\|Z_m(t)\|_1^2 + 2\|f(t) - f_\infty\|_0\|Z_m(t)\|_0 \\
&\quad +2K_1(|\widetilde{u}_0| + CK_1)|a(t)h(t) - a_\infty h_\infty|\|Z_m(t)\|_1.
\end{aligned}
$$

Choose $\delta > 0$ such that $3\delta < \widetilde{C}_0 - \varepsilon \equiv \widetilde{C}_1$, then we have from (5.22)

$$
(5.23) \qquad \begin{aligned}
&\frac{d}{dt}\|Z_m(t)\|_0^2 + \widetilde{C}_1\|Z_m(t)\|_1^2 \\
&\quad \leq \frac{1}{\delta}C^2|a(t) - a_\infty|^2 + \frac{1}{\delta}\|f(t) - f_\infty\|_0^2 \\
&\quad + \frac{1}{\delta}K_1^2(|\widetilde{u}_0| + CK_1)^2|a(t)h(t) - a_\infty h_\infty|^2.
\end{aligned}
$$

Put $\gamma_1 = \min\{\gamma_a, \gamma_h, \gamma_f\}$, we deduce from (5.23) and (H_4) that

$$
(5.24) \qquad \begin{aligned}
&\frac{d}{dt}\|Z_m(t)\|_0^2 + \widetilde{C}_1\|Z_m(t)\|_1^2 \\
&\quad \leq \frac{1}{\delta}(C^2C_a^2 + C_f^2 + K_1^2(|\widetilde{u}_0| + CK_1)^2(C_a\|h\|_\infty + C_h a_\infty)^2)e^{-2\gamma_1 t} \\
&\quad \equiv \widetilde{C}_2 e^{-2\gamma_1 t}, \quad \text{for all } t \geq 0.
\end{aligned}
$$

Put $\gamma_0 = \frac{1}{2}\min\{\gamma_1, \widetilde{C}_1\}$. Hence, we obtain from (5.24) that

$$
(5.25) \qquad \begin{aligned}
\|Z_m(t)\|_0^2 &\leq e^{-2\gamma_0 t}\|Z_{0m}\|_1^2 + \widetilde{C}_2 e^{-2\gamma_0 t}\int_0^t e^{-2(\gamma_1-\gamma_0)s}ds \\
&= e^{-2\gamma_0 t}\|Z_{0m}\|_1^2 + \frac{\widetilde{C}_2}{2(\gamma_1-\gamma_0)}e^{-2\gamma_0 t}(1 - e^{-2(\gamma_1-\gamma_0)t}) \\
&\leq \left(\|Z_{0m}\|_1^2 + \frac{\widetilde{C}_2}{2(\gamma_1-\gamma_0)}\right)e^{-2\gamma_0 t}.
\end{aligned}
$$

Letting $m \to +\infty$ in (5.25) we obtain

$$
(5.26) \qquad \begin{aligned}
\|u(t) - u_\infty\|_0^2 &\leq \liminf_{m\to+\infty} \|u_m(t) - y_m\|_0^2 \\
&\leq \left(\|u_0 - u_\infty\|_0^2 + \frac{\widetilde{C}_2}{2(\gamma_1-\gamma_0)}\right)e^{-2\gamma_0 t}, \ \text{for all } t \geq 0.
\end{aligned}
$$

This completes the proof of Theorem 4.■

VI. NUMERICAL RESULTS

First, we present some results of numerical comparison of the approximated representation of the solution of a nonlinear problem of the type (1.1)-(1.3) and the corresponding exact solution of this problem.

Let the problem

$$(6.1)\qquad u_t - \left(u_{rr} + \tfrac{2}{r} u_r \right) + F(u) = 0,$$

$$(6.2)\qquad u_r(1,t) + u(1,t) = 0,\ u_r(0,t) = 0,$$

$$(6.3)\qquad u(r,0) = 0,$$

where

$$f(r,t) = e^{-\alpha r}(1+\alpha r)\cos t + \alpha^2 e^{-\alpha r}\sin t(3-\alpha r) + e^{-\frac{3}{2}\alpha r}(1+\alpha r)^{3/2} sgn(\sin t),$$
$$F(u) = |u|^{3/2} sgn(u);$$
$$\alpha = \tfrac{1+\sqrt{5}}{2}, \text{ and the domain } D = \{(r,t) : 0 \le r \le 1,\ 0 \le t \le 1\}.$$

The exact solution of the problem (6.1)-(6.3) is $v(r,t) = e^{-\alpha r}(1+\alpha r)\sin t$.

To solve numerically the problem (6.1)-(6.3), we consider the nonlinear differential system for the unknowns $u_k(t) = u(r_k,t)$, $r_k = kh$, $h = 1/N$.

$$(6.4)\qquad \begin{cases} \frac{du_k}{dt} = \frac{1}{h^2}\left(1-\frac{2}{k}\right)u_{k-1} + \frac{2}{h^2}\left(\frac{1}{k}-1\right)u_k + \frac{u_{k+1}}{h^2} - F(u_k) + f(r_k,t), \\ u_1 = u_0,\ u_N = \frac{u_{N-1}}{h+1}, \\ u_k(0) = 0,\ k = 1,2,\ldots,N-1. \end{cases}$$

To solve the nonlinear differential (6.4) at the time t, we use the following linear recursive scheme generated by the nonlinear term $F(u_k)$:

$$(6.5)\qquad \begin{cases} \frac{du_{k,n}}{dt} = \frac{1}{h^2}\left(1-\frac{2}{k}\right)u_{k-1,n} + \frac{2}{h^2}\left(\frac{1}{k}-1\right)u_{k,n} + \frac{u_{k+1,n}}{h^2} - F(u_{k,n}) + f(r_k,t), \\ u_{k,n}(0) = 0,\ k = 1,2,\ldots,N-1. \end{cases}$$

The linear differential system (6.5) is solved by searching the associated eigenvalues and eigenfunctions. With a spatial step $h = \frac{1}{10}$ on the interval $[0,1]$ and for $t \in [0,2]$, we have drawn the corresponding approximate surface solution $(t,t) \to u(r,t)$ in figure 1, obtained by successive re-initializations in t with a time step $\Delta t = \frac{1}{50}$. For comparison in figure 2, we have also drawn the exact surface solution $(t,t) \to v(r,t)$.

Now consider the following problem

$$(6.6)\qquad u_t - \left(u_{rr} + \tfrac{2}{r} u_r \right) + |u|^{3/2} sgn(u) = 0,$$

$$(6.7)\qquad u_r(1,t) + u(1,t) = 0,\ u_r(0,t) = 0,$$

$$(6.8)\qquad u(r,0) = \tfrac{1}{4}.$$

Using the same method as previously we have drawn in figure 3 the approximate surface solution $(t,t) \longrightarrow u(r,t)$ which decreases exponentially to 0 as t tends to infinity, 0 being the unique solution ot the corresponding steady state problem

(6.9) $$u_{rr} + \frac{2}{r}u_r - |u|^{3/2}sgn(u) = 0,$$

(6.10) $$u_r(1) + u(1) = 0,\ u_r(0) = 0.$$

Notice, since the function $F(u) = |u|^{3/2}sgn(u)$ has a derivative positive the solution of the problem (6.6)-(6.8) is bounded and unique according section IV.

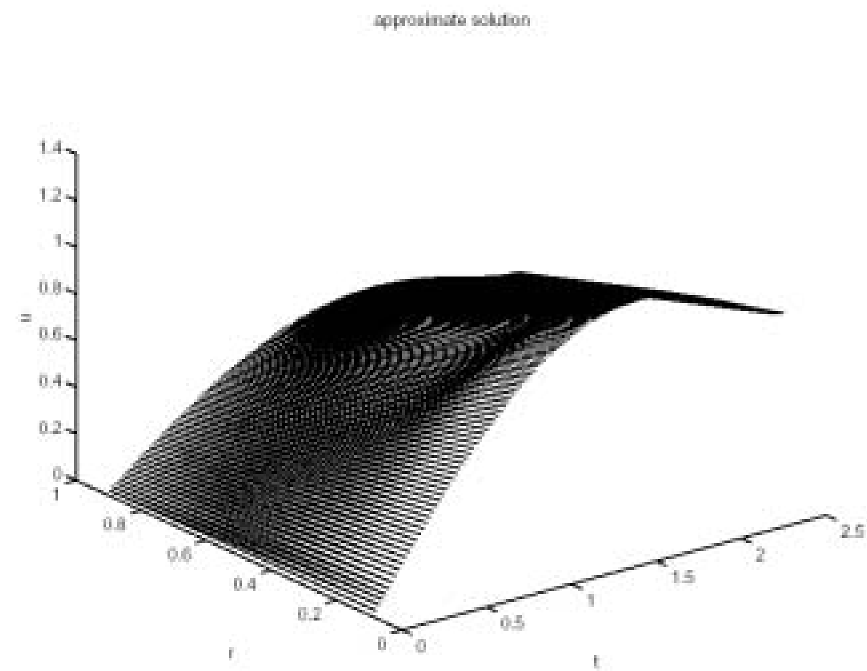

Figure 1. **Approximate solution**

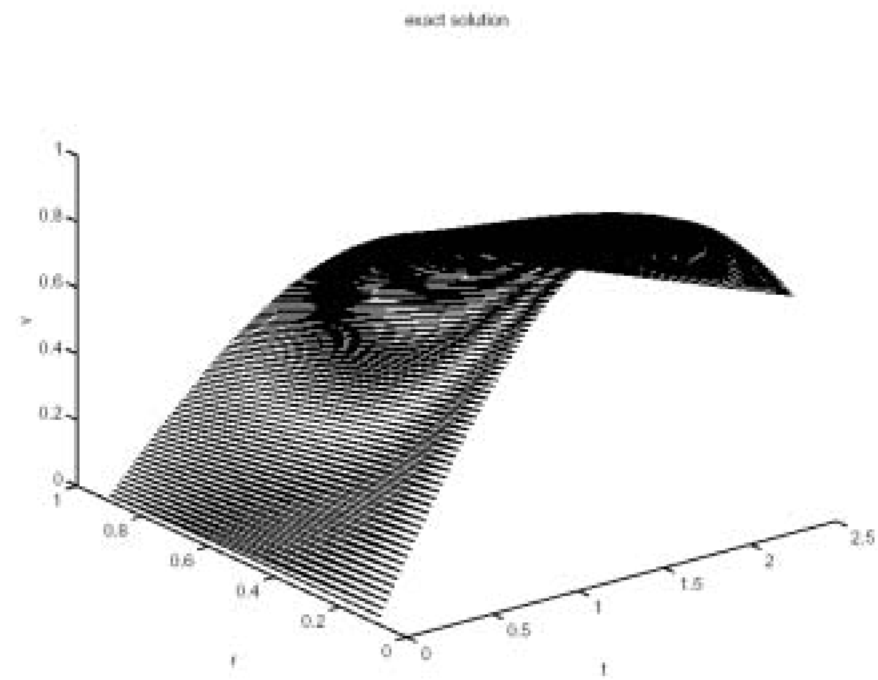

Figure 2. **Exact solution**

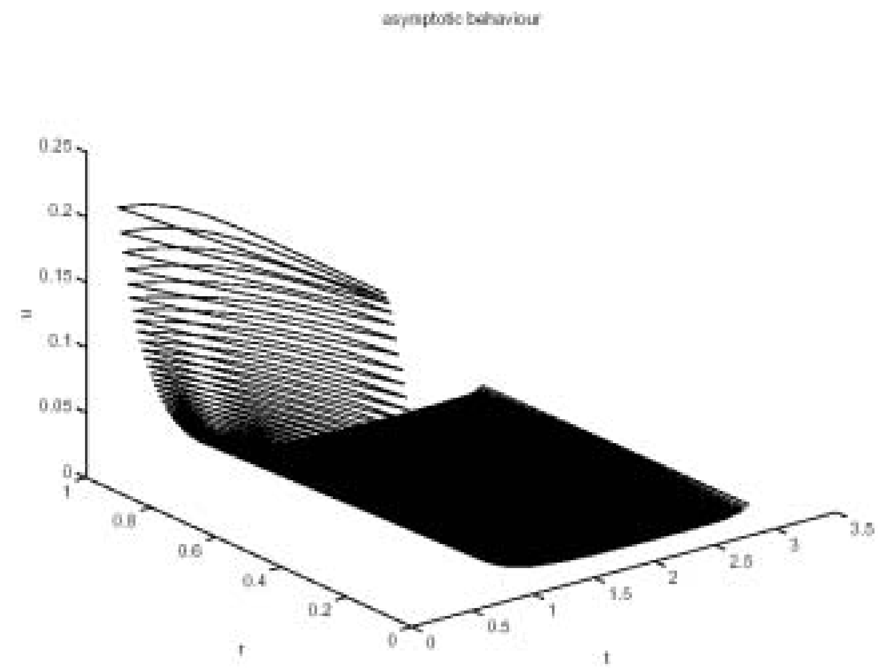

Figure 3. **Asymptotic behavior**

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