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Nguyen Thanh Long, Alain Pham Ngoc Dinh

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by

Nguyen Thanh Long(1), Alain Pham Ngoc Dinh(2)

(1) Department of Mathematics and Computer Science, College of Natural Science, Vietnam National University HoChiMinh City, 227 Nguyen Van Cu Str., Dist.5, HoChiMinh City, Vietnam.
E-mail: longnt@hcmc.netnam.vn
(2) MAPMO, UMR 6628, bâtiment de Mathématiques, Université d’Orléans, BP 6759 Orléans Cedex 2, France.
E-mail: alain.pham@univ-orleans.fr

Abstract. In this paper we consider the following nonlinear parabolic equation

\[
\begin{aligned}
&u_t - a(t) \left( u_{rr} + \frac{r}{r} u_r \right) + F(r,u) = f(r,t), \quad 0 < r < 1, \quad 0 < t < T, \\
&\lim_{r \to 0^+} r^{\gamma/2} u_r(r,t) < +\infty, \quad u_r(1,t) + h(t) \left( u(1,t) - \tilde{u}_0 \right) = 0, \\
u(r,0) = u_0(r),
\end{aligned}
\]

where \( \gamma > 0, \tilde{u}_0 \) are given constants, \( a(t), h(t), F(r,u), f(r,t) \) are given functions. In section III, we use the Galerkin and compactness method in appropriate Sobolev spaces with weight to prove the existence of a unique weak solution of the problem (*) on \((0, T)\), for every \( T > 0 \). In section IV, we prove that if the initial condition is bounded, then so is the solution. In section V, we study asymptotic behavior of the solution as \( t \to +\infty \). In section VI we give numerical results.

Keywords: Nonlinear parabolic equation, Galerkin method, Sobolev spaces with weight, Asymptotic behavior of the solution.

Address for correspondence: Alain Pham Ngoc Dinh.
1. INTRODUCTION

In this paper we will consider the following initial and boundary value problem

\[(1.1) \quad u_t - a(t) \left( u_{rr} + \frac{r}{r} u_r \right) + F(r, u) = f(r, t), \quad 0 < r < 1, \quad 0 < t < T, \]

\[(1.2) \quad \lim_{r \to 0} \frac{r^{\gamma/2} u_r(r, t)}{r^{\gamma/2}} = +\infty, \quad u_r(1, t) + h(t) \left( u(1, t) - \bar{u}_0 \right) = 0, \]

\[(1.3) \quad u(r, 0) = u_0(r), \]

where \(\gamma > 0, \bar{u}_0\) are given constants, \(a(t), h(t), F(r, u), f(r, t)\), are given functions satisfying conditions specified later.

The equation (1.1) can be rewritten in the form

\[(1.4) \quad \frac{\partial u}{\partial t} - \frac{a(t)}{r^\gamma} \frac{\partial}{\partial r} \left( r^{\gamma} \frac{\partial u}{\partial r} \right) + F(r, u) = f(r, t), \quad 0 < r < 1, \quad 0 < t < T. \]

For \(\gamma = 1\) with \(F = 0\), the problem describes the radial axisymmetric heat flow in a cylinder.

With \(\gamma = 2\) and always \(F = 0\), the problem (1.2)-(1.4) represents in polar coordinates in \(\mathbb{R}^3\) the mass fraction of a liquid fuel droplet in the case of his evaporation inside an infinite vessel, the boundary condition (1.2) being associated to the Rankine-Hugoniot condition on the surface of the droplet after a changing of the scale [8].

In [6], Minasjan studied a special case of the problem (1.1), (1.2) associated with the following \(T\)-periodic condition

\[(1.5) \quad u(r, 0) = u(r, T), \]

with

\[(1.6) \quad \gamma = 1, \quad F(r, u) = 0, \quad \bar{u}_0 = 0. \]

and the functions \(a(t), h(t), f(r, t)\) are \(T\)-periodic in time \(t\). The physical interpretation of the problem (1.1), (1.2), (1.5), (1.6) is that of a periodic heat flow in an infinite cylinder with the assumption that the cylinder is subjected to convective heat transfer (periodic in time) at the boundary surface \((r = 1)\) at zero temperature. Inside the cylinder, there are circular symmetric sources of heat that change periodically. Minasjan[6] gave for this problem a classical solution using Fourier transforms. This method leads to an infinite pseudoregular system of linear algebraic equations. However, the solvability of this system is not proved in detail in [6].

In [3] Lauerova has proved that with \(T\)-periodic data, the problem (1.1), (1.2), (1.5), (1.6) has a \(T\)-periodic weak solution in \(t\).

In the case of
\[ (1.7) \quad \gamma = 1, \, \bar{u}_0 = 0, \, f = 0, \, F = F(u), \, F' \in C^1(\mathbb{R}), \, F'(u) \geq -\varepsilon, \]

(\varepsilon > 0 \text{ small enough}), we have proved [4] that the problem (1.1), (1.2), (1.5) has a \( T \)-periodic unique weak solution in appropriate Sobolev spaces with weight. Furthermore, the solution also depends continuously on the functions \( a(t) \) and \( h(t) \).

The paper consists of three sections. In section III, under appropriate conditions of \( a(t), h(t), F(r,u), f(r,t) \) we prove the existence of a unique solution on \((0,T)\), for every \( T > 0 \). Theses results generalize relatively the ones in [3, 4, 6].

In section IV, we shall show that if the initial condition is bounded, then so is the solution \( u \). More precisely if \( u_0 \in L^\infty((0,1)) \) then the solution \( u \in L^\infty((0,1) \times (0,T)) \). This last result generalizes to the nonlinear case the same result obtained in the linear case [8]. In section V, we study asymptotic behavior of the solution as \( t \) tends to infinity: assuming some asymptotic exponential decay on the data, we show that \( u(t) \) converges as \( t \to +\infty \) to the solution \( u_\infty \) of the corresponding steady state equation, with an exponential decay to 0 of the difference \( u(t) - u_\infty \).

The aim of this paper is mainly to get some integral inequalities via various assumptions on the nonlinear term \( F(r,u) \) in order to have some a priori estimates for \( u(t), tu(t) \) and his respective derivatives in appropriate Sobolev spaces with weight. The hypotheses on \( F(r,u) \) are sufficiently large to include a class enough great of nonlinear problems. For instance if we consider \( \gamma = 2 \) (radial Laplace in polar coordinates in \( \mathbb{R}^3 \)) all the functions \( F \) of the kind \( F(u) = \text{sgn}(u)|u|^\alpha, \, \alpha \in (0,2) \). In section VI we give numerical results.

II. PRELIMINARY RESULTS, NOTATIONS, FUNCTION SPACES

We omit the definitions of the usual function spaces \( C^m([0,1]), L^p(0,1), H^m(0,1), W^{m,p}(0,1) \). For any function \( v \in C^0([0,1]) \) we define \( \|v\|_0 \) as

\[ (2.1) \quad \|v\|_0 = \|v\|_{0,\gamma} = \left( \int_0^1 r^\gamma v^2(r)dr \right)^{1/2} \]

and define the space \( V_0 \) as completion of the space \( C^0([0,1]) \) with respect to the norm \( \|\cdot\|_0 \).

Similarly, for any function \( v \in C^1([0,1]) \) we define \( \|v\|_1 \) as

\[ (2.2) \quad \|v\|_1 = \|v\|_{1,\gamma} = \left( \|v\|_0^2 + \|v'\|_0^2 \right)^{1/2} \]

and define the space \( V_1 \) as completion of the space \( C^1([0,1]) \) with respect to the norm \( \|\cdot\|_1 \).

Note that the norms \( \|\cdot\|_0 \) and \( \|\cdot\|_1 \) can be defined, respectively, from the inner products

\[ \langle u, v \rangle = \int_0^1 r^\gamma u(r)v(r)dr \quad \text{and} \]

\[ (2.3) \quad \langle u, v \rangle + \langle u', v' \rangle = \int_0^1 r^\gamma [u(r)v(r) + u'(r)v'(r)]dr. \]

It is then easy to prove that \( V_0 \) and \( V_1 \) are Hilbert spaces, with \( V_1 \) continuously and densely embedded into \( V_0 \). Identifying \( V_0 \) with its dual \( V'_0 \) we have \( V_1 \hookrightarrow V_0 \equiv V'_0 \hookrightarrow V'_1 \). On the other
hand, the notation $\langle \cdot, \cdot \rangle$ is used for the pairing between $V_1$ and $V'_1$.

We then have the following lemmas, the proofs of which can be found in [5].

**Lemma 1.** For every $v \in C^1([0,1]), \gamma > 0, \varepsilon > 0, \text{and } r \in [0,1]$ we have

\begin{align*}
(2.4) \quad & \|v\|^2_0 \leq \frac{1}{\gamma} \|v'\|^2_0 + v^2(1), \\
(2.5) \quad & |v(1)| \leq K_1 \|v\|_1, \\
(2.6) \quad & r^{\gamma/2}|v(r)| \leq K_2 \|v\|_1, \\
(2.7) \quad & v^2(1) \leq \varepsilon \|v'\|^2_0 + C_\varepsilon \|v\|^2_0,
\end{align*}

where

\begin{align*}
(2.8) \quad & K_1 = \sqrt{\gamma + 2}, \quad K_2 = \sqrt{\gamma + 3}, \quad C_\varepsilon = 1 + \gamma + 1/\varepsilon.
\end{align*}

**Lemma 2.** The embedding $V_1 \hookrightarrow V_0$ is compact.

**Remark 1.** The result of (2.4), (2.5) proves that $\left(v^2(1) + \|v'\|^2_0\right)^{1/2}$ and $\|v\|_1$ are two equivalent norms on $V_1$ and

\begin{align*}
(2.9) \quad & \frac{\gamma}{\gamma + 1} \|v\|^2_1 \leq v^2(1) + \|v'\|^2_0 \leq (\gamma + 3)\|v\|^2_1, \text{ for all } v \in V_1.
\end{align*}

We also note that

\begin{align*}
(2.10) \quad & \lim_{r \to 0} r^{\gamma/2}v(r) = 0, \text{ for all } v \in V_1.
\end{align*}

(See [1], Lemma 5.40, p.128).

On the other hand, by $H^1(\varepsilon,1) \hookrightarrow C^0([\varepsilon,1]), 0 < \varepsilon < 1, \text{ and }$

\begin{align*}
(2.11) \quad & \varepsilon^{\gamma/2} \|v\|_{H^1(\varepsilon,1)} \leq \|v\|_1, \quad \text{for all } v \in V_1, 0 < \varepsilon < 1.
\end{align*}

It follows that

\begin{align*}
(2.12) \quad & v|_{[\varepsilon,1]} \in C^0([\varepsilon,1]), \quad \text{for all } \varepsilon, 0 < \varepsilon < 1.
\end{align*}

From (2.10), (2.12) we deduce that

\begin{align*}
(2.13) \quad & r^{\gamma/2}v \in C^0([0,1]), \quad \text{for all } v \in V_1.
\end{align*}

We denote by $\| \cdot \|_X$ the norm in the Banach space $X$. We call $X'$ the dual space of $X$. We denote by $L^p(0,T;X), 1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

\begin{align*}
\|u\|_{L^p(0,T;X)} = \left( \int_0^T \|u(t)\|^p_X dt \right)^{1/p} < +\infty, \quad \text{for } 1 \leq p < \infty,
\end{align*}

and
\[ \|u\|_{L^\infty(0,T;X)} = \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_X, \text{ for } p = \infty. \]

Let \( u(t), u'(t) = u_r(t), u_{rr}(t) \) denote \( u(r,t), \frac{\partial u}{\partial t}(r,t), \frac{\partial u}{\partial r}(r,t), \frac{\partial^2 u}{\partial r^2}(r,t) \), respectively.

**III. THE EXISTENCE AND UNIQUENESS THEOREM**

We form the following assumptions

\((H_1)\) \quad u_0 \in V_0, \quad \bar{u}_0 \in \mathbb{R};
\(\quad (H_2)\) \quad a, h \in W^{1,\infty}(0,T), \quad a(t) \geq a_0 > 0;
\(\quad (H_3)\) \quad f \in L^2(0,T;V_0);

\[ F : (0,1) \times \mathbb{R} \to \mathbb{R} \text{ satisfies the Caratheodory condition, i.e.,} \]
\(\quad (F_1)\) \quad \text{is measurable on } (0,1) \text{ for every } u \in \mathbb{R},
\quad \text{and } F(r,\cdot) \text{ is continuous on } \mathbb{R} \text{ for a.e. } r \in (0,1).
\(\quad (F_2)\) \quad \text{There exist positive constants } C_1, C_1', C_2 \text{ and } p, 1 < p < 2 + 2/\gamma \text{ such that}

\[\begin{align*}
(i) & \quad uF(r,u) \geq C_1|u|^p - C_1', \\
(2i) & \quad |F(r,u)| \leq C_2(1 + |u|^{p-1}).
\end{align*}\]

The weak formulation of the initial and boundary value problem (1.1)-(1.3) can be made in the following manner:

Find \( u(t) \) defined in the open set \((0,T)\) such that \( u(t) \) satisfies the following variational problem

\[\begin{align*}
\frac{d}{dt} \langle u(t), v \rangle + a(t) \langle u_r(t), v_r \rangle + a(t)h(t)u(1,t)v(1) + \langle F(r,u(t)), v \rangle = \langle f(t), v \rangle + \bar{u}_0 a(t)h(t)v(1), \text{ for all } v \in V_1,
\end{align*}\]

and the initial condition

\[ u(0) = u_0. \]

We then have the following theorem.

**Theorem 1.** Let \( T > 0 \) and \((H_1) - (H_3), (F_1), (F_2)\) hold. Then, there exists a solution \( u \) of problem (3.1)- (3.2) such that

\[\begin{align*}
(3.3) & \quad u \in L^2(0,T;V_1) \cap L^\infty(0,T;V_0), \quad r^{\gamma/p} u \in L^p(Q_T), \\
tu \in L^\infty(0,T;V_1), \quad tu_t \in L^2(0,T;V_0).
\end{align*}\]
Furthermore, if $F$ satisfies the following condition, in addition,

\[(F_3) \quad (F(r,u) - F(r,v))(u - v) \geq -\varepsilon|u - v|^2, \text{ for all } u,v \in \mathbb{R},\]

for a.e., $r \in (0,1)$, with $\varepsilon > 0$ sufficiently small,

then the solution is unique.

**Proof.** The proof consists of several steps.

**Step 1. The Galerkin method.** Denote by $\{w_j\}, j = 1,2,\ldots$ an orthonormal basis in the separable Hilbert space $V_1$. We find $u_m(t)$ of the form

\[(3.4) \quad u_m(t) = \sum_{j=1}^{m} c_{mj}(t)w_j,\]

where $c_{mj}$ satisfy the following system of nonlinear differential equations

\[(3.5) \quad \langle u_m(t), w_j \rangle + a(t) \langle u_{mr}(t), w_{jr} \rangle + a(t)h(t)u_m(1,t)w_j(1) + \langle F(r,u_m(t)), w_j \rangle = \langle f(t), w_j \rangle + \bar{a}_0a(t)h(t)w_j(1), 1 \leq j \leq m,\]

(3.6) \quad $u_m(0) = u_{0m},$

where

(3.7) \quad $u_{0m} \rightarrow u_0$ strongly in $V_0.$

It is clear that for each $m$ there exists a solution $u_m(t)$ in form (3.4) which satisfies (3.5) and (3.6) almost everywhere on $0 \leq t \leq T_m$ for some $T_m$, $0 \leq T_m \leq T$. The following estimates allow one to take $T_m = T$ for all $m$.

**Step 2. A priori estimates.**

a) **The first estimate.** Multiplying the $j^{th}$ equation of the system (3.5) by $c_{mj}(t)$ and summing up with respect to $j$, we have

\[(3.8) \quad \frac{d}{dt} \|u_m(t)\|_0^2 + 2a(t)\|u_{mr}(t)\|_0^2 + 2u_m^2(1,t) + 2\langle F(r,u_m(t)), u_m(t) \rangle \]

\[= 2(1 - a(t)h(t))u_m^2(1,t) + 2\langle f(t), u_m(t) \rangle + 2\bar{a}_0a(t)h(t)u_m(1,t).\]

By the assumptions $(H_2)$, $(F_2,i)$, and the inequalities (2.5), (2.7), (2.9), it follows from (3.8), that
\[
\frac{d}{dt} \| u_m(t) \|^2_0 + 2C_3 \| u_m(t) \|^2_1 + 2C_1 \int_0^1 r^\gamma |u_m(r, t)|^p dr \\
\leq \frac{2C_1}{\gamma + 1} + \frac{1}{2e_1} |\bar{u}_0|^2 K_1^2 \| ah \|^2_{L^\infty(0, T)} + \| f(t) \|^2_0 \\
+ 2e_1 \left( 2 + \| ah \|^2_{L^\infty(0, T)} \right) \| u_m(t) \|^2_1 \\
+ \left( 1 + 2C_1 \left( 1 + \| ah \|^2_{L^\infty(0, T)} \right) \right) \| u_m(t) \|^2_0.
\]

(3.9)

for all \( \varepsilon > 0 \). Choosing \( \varepsilon > 0 \) such that

\[
2e_1 \left( 2 + \| ah \|^2_{L^\infty(0, T)} \right) \leq \frac{2}{\gamma + 1} \min \{ 1, a_0 \} = C_3.
\]

(3.10)

Hence, from (3.9), (3.10) we obtain

\[
\frac{d}{dt} \| u_m(t) \|^2_0 + C_3 \| u_m(t) \|^2_1 + 2C_1 \int_0^1 r^\gamma |u_m(r, t)|^p dr \\
\leq \frac{2C_1}{\gamma + 1} + \frac{1}{2e_1} |\bar{u}_0|^2 K_1^2 \| ah \|^2_{L^\infty(0, T)} + \| f(t) \|^2_0 \\
+ \left( 1 + 2C_1 \left( 1 + \| ah \|^2_{L^\infty(0, T)} \right) \right) \| u_m(t) \|^2_0.
\]

(3.11)

Integrating (3.11) and by means of (3.7), we have

\[
\| u_m(t) \|^2_0 + C_3 \int_0^t \| u_m(s) \|^2_1 ds + 2C_1 \int_0^t ds \int_0^1 r^\gamma |u_m(r, s)|^p dr \\
\leq M_T^{(2)} + M_T^{(1)} \int_0^t \| u_m(s) \|^2_0 ds,
\]

(3.12)

where \( M_T^{(1)}, M_T^{(2)} \) are the constants depending only on \( T \), with

\[
M_T^{(1)} = 1 + 2C_1 \left( 1 + \| ah \|^2_{L^\infty(0, T)} \right),
\]

\[
M_T^{(2)} \geq \| u_{0m} \|^2_0 + \left( \frac{2C_1}{\gamma + 1} + \frac{1}{2e_1} |\bar{u}_0|^2 K_1^2 \| ah \|^2_{L^\infty(0, T)} \right) T + \int_0^T \| f(s) \|^2_0 ds,
\]

for all \( m \).

By the Gronwall’s lemma, we obtain from (3.12), that
\[
\|u_m(t)\|^2_0 + C_3 \int_0^t \|u_m(s)\|^2 ds + 2C_1 \int_0^1 ds \int_0^1 |u_m(r,s)|^p dr \\
\leq M_{T}^{(2)} \exp(tM_{T}^{(1)}) \leq M_T,
\]

for all \(m\), for all \(t, 0 \leq t \leq T_m \leq T\), i.e., \(T_m = T\).

b) \textit{The second estimate.} Multiplying the \(j\)th equation of the system (3.5) by \(t^2 c_{my}^j(t)\) and summing up with respect to \(j\), we have

\[
2 \left\| tu_m^j(t) \right\|^2_0 + \frac{d}{dt} \left( a(t) \| tu_m(t) \|^2_0 + a(t)h(t)t^2 u_m^2(1,t) \right) \\
+ 2 \frac{d}{dt} \left( r^2 \hat{F}(r, u_m(r,t)) dr \right) \\
= \| u_{mr}(t) \|^2_0 \frac{d}{dr} \left[ t^2 a(t) \right] + u_m^2(1,t) \frac{d}{dr} \left[ t^2 a(t)h(t) \right] \\
+ 4t \int_0^1 r^2 \hat{F}(r, u_m(r,t)) dr + 2 \langle tf(t), tu_m(t) \rangle \\
+ 2u_0 \frac{d}{dt} \left[ t^2 a(t)h(t)u_m(1,t) \right] - 2u_0u_m(1,t) \frac{d}{dt} \left[ t^2 a(t)h(t) \right],
\]

where

\[
\hat{F}(r, \lambda) = \int_0^\lambda F(r,s) ds.
\]

Integrating (3.14) with respect to time variable from 0 to \(t\), we shall have, after some rearrangements

\[
2 \int_0^t \left\| su_m^j(s) \right\|^2_0 ds + a(t) \| tu_m(t) \|^2_0 + t^2 u_m^2(1,t)
\]
\[ [1 - a(t)h(t)]t^2u_m^2(1, t) + \int_0^t \int_0^r \frac{\partial}{\partial t} \frac{\partial}{\partial s} F(r, u_m(s), t) ds dr \]

By means of the assumption \((H_2)\) and the inequality \((2.9)\), we have
\[ a(t)\|tu_m(t)\|^2_0 + t^2u_m^2(1, t) \geq C_3\|tu_m(t)\|^2_1 \]
for all \(t \in [0, T]\), for all \(m\), where \(C_3\) is constant defined by \((3.10)\).

Using the inequalities \((2.5), (2.7)\), and with \(\varepsilon_1 > 0\) as in \((3.10)\), we estimate without difficulty the following terms in the right-hand side of \((3.16)\) as follows
\[ (3.17) \quad a(t)\|tu_m(t)\|^2_0 + t^2u_m^2(1, t) \geq C_3\|tu_m(t)\|^2_1 \]

\[ (3.18) \quad \int_0^t \int_0^r \frac{\partial}{\partial t} \frac{\partial}{\partial s} F(r, u_m(s), t) ds dr \]

\[ (3.19) \quad a(t)\|tu_m(t)\|^2_0 + t^2u_m^2(1, t) \leq \left(1 + \|ah\|_{L^\infty(0, T)}\right)\left(\varepsilon_1\|tu_m(t)\|^2_1 + C_1, t^2M_T\right), \]

\[ (3.20) \quad 2\|Uh_0\|_{L^\infty(0, T)}+K_1\|t^2ah\|_{L^\infty(0, T)}(M_T/C_3), \]

\[ (3.21) \quad 2|\sum a(t)h(t)u_m(1, t)| \leq \varepsilon_1\|tu_m(t)\|^2_1 + \varepsilon_1\left(K_1\|u_0\|_{L^\infty(0, T)}\right)^2, \]

\[ (3.22) \quad \int_0^t \int_0^r \frac{\partial}{\partial t} \frac{\partial}{\partial s} F(r, u_m(s), t) ds dr \]

On the other hand, from the assumptions \((F_1), (F_2)\), we have
\[ (3.23) \quad -m_0 = -\int_{\lambda_0}^{\lambda_0} |F(r, s)| ds \leq \int_{\lambda_0}^{\lambda_0} F(r, \lambda) ds \leq C_2(1 + \frac{\|\lambda\|}{p}), \quad \text{for all } \lambda \in \mathbb{R}, \]
where \( \lambda_0 = \left( \frac{C_2}{C_1} \right)^{1/p} \).

Using the inequalities (2.6), (3.13), (3.23), we obtain

\[
4 \int_0^1 \int_0^1 r^{-p} \hat{F}(r, u_m(r, s)) dr - 2t^2 \int_0^1 r^{-p} \hat{F}(r, u_m(r, t)) dr \\
\leq \frac{4C_2K_2}{1+\gamma/2} \int_0^t \|su_m(s)\|_1 ds \\
+ \frac{4C_2t}{p} (MT/2C_1) + \frac{2mt^2}{1+\gamma}.
\]

(3.24)

Hence, we deduce from (3.16) - (3.22), and (3.24) that

\[
\int_0^t \|su'_m(s)\|_0^2 ds + \frac{1}{2} C_3 \|tu_m(t)\|_1^2 \\
\leq M_T^{(3)} + \frac{4C_2K_2}{1+\gamma/2} \int_0^t \|su_m(s)\|_1 ds \leq M_T^{(4)} + \int_0^t \|su_m(s)\|_1^2 ds,
\]

(3.25)

where \( M_T^{(3)} \), \( M_T^{(4)} \) are the constants depending only on \( T \).

By the Gronwall’s lemma, we obtain from (3.25), that

\[
\int_0^t \|su'_m(s)\|_0^2 ds + \frac{1}{2} C_3 \|tu_m(t)\|_1^2 \leq M_T^{(4)} \exp(t) \leq M_T^{(5)}.
\]

(3.26)

On the other hand, by using (3.13), and assumption \((F_2)\) we have

\[
\int_0^t \int_0^1 r^{-p} F(r, u_m(r, s)) dr \\
\leq 2^{p-1} C_2^{p'} \left( \frac{T}{1+\gamma} + \int_0^t \int_0^1 r^{-p} u_m(r, s) |u_m(r, s)|^p dr \right) \leq M_T^{(6)}.
\]

(3.27)

where \( M_T^{(6)} \) is a constant depending only on \( T \).

**Step 3. The limiting process.**

By (3.13), (3.26), (3.27) we deduce that, there exists a subsequence of \( \{u_m\} \), still denoted by \( \{u_m\} \) such that
(3.28) \( u_m \to u \) in \( L^\infty(0,T;V_0) \) weak*.

(3.29) \( u_m \to u \) in \( L^2(0,T;V_1) \) weak.

(3.30) \( r^{p/q} u_m \to r^{p/q} u \) in \( L^p(Q_T) \) weak.

(3.31) \( tu_m \to tu \) in \( L^\infty(0,T;V_1) \) weak*.

(3.32) \( (tu_m)' \to (tu)' \) in \( L^2(0,T;V_0) \) weak.

Using a compactness lemma ([2], Lions, p.57) applied to (3.31), (3.32), we can extract from the sequence \( \{u_m\} \) a subsequence still denotes by \( \{u_m\} \), such that

(3.33) \( tu_m \to tu \) strongly in \( L^2(0,T;V_0) \).

By the Riesz- Fischer theorem, we can extract from \( \{u_m\} \) a subsequence still denoted by \( \{u_m\} \), such that

(3.34) \( u_m(r,t) \to u(r,t) \) a.e. \((r,t)\) in \( Q_T = (0,1) \times (0,T) \).

Because \( F \) is continuous, then

(3.35) \( F(r,u_m(r,t)) \to F(r,u(r,t)) \) a.e. \((r,t)\) in \( Q_T \).

We shall now require the following lemma, the proof of which can be found in [2].

**Lemma 3.** Let \( Q \) be a bounded open set of \( \mathbb{R}^N \) and \( G_m, G \in L^q(Q), 1 < q < \infty \), such that,

\[ \|G_m\|_{L^q(Q)} \leq C \]

where \( C \) is a constant independent of \( m \)

and

\[ G_m \to G \text{ a.e. } (r,t) \text{ in } Q. \]

Then \( G_m \to G \) in \( L^q(Q) \) weakly.

Applying Lemma 3 with \( N = 2, q = p'/q, G_m = r^{p/q} F(r,u_m), G = r^{p/q} F(r,u) \), we deduce from (3.27), (3.35) that in

(3.36) \( r^{p/q} F(r,u_m) \to r^{p/q} F(r,u) \) in \( L^{p'/q}(Q_T) \) weakly.

Passing to the limit in (3.5), (3.6) by (3.7), (3.28), (3.29), (3.36) we have satisfying the equation
\[ \frac{d}{dt} \langle u(t), v \rangle + a(t) \langle u_r(t), v_r \rangle + a(t) h(t) u(1, t) v(1) + \langle F(r, u(t)), v \rangle \\
= \langle f(t), v \rangle + \bar{u}_0 a(t) h(t) v(1), \text{ for all } v \in V_1, \]

(3.38) \[ u(0) = u_0. \]

Step 4. Uniqueness of the solutions.

First, we shall need the following Lemma.

Lemma 4. Let \( w \) be the weak solution of the following problem

(3.39) \[ w_t - a(t)(w_{rr} + \frac{r}{r} w_r) = \tilde{f}(r,t), \quad 0 < r < 1, \quad 0 < t < T, \]

(3.40) \[ \lim_{r \to 0} r^{\gamma/2} w_r(r, t) < +\infty, \quad w_r(1, t) + h(t) w(1, t) = 0, \]

(3.41) \[ w(r, 0) = 0, \]

(3.42) \[ w \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0), \quad r^{\gamma/2} w \in L^p(Q_T), \]

Then

\[ \frac{1}{2} \| w(t) \|^2_0 + \int_0^t a(s) \left[ \| w_r(s) \|^2_0 + h(s) w^2(1, s) \right] ds \]

(3.43) \[ - \int_0^t \langle \tilde{f}(s), w(s) \rangle ds = 0, \quad a.e. \; t \in (0, T). \]

The lemma 4 is a slight improvement of a lemma used in [8] (see also Lions’s book [2]). Now, we will prove the uniqueness of the solutions. Let \( u \) and \( v \) be two weak solutions of (1.1)-(1.3). Then \( w = u - v \) is a weak solution of the following problem (3.39)-(3.42) with the right hand side function replaced by \( \tilde{f}(r, t) = -F(u) + F(v) \). Using Lemma 4 we have equality

\[ \frac{1}{2} \| w(t) \|^2_0 + \int_0^t a(s) \left[ \| w_r(s) \|^2_0 + h(s) w^2(1, s) \right] ds \]

(3.44) \[ = - \int_0^t \langle F(r, u) - F(r, v), w(s) \rangle ds. \]

Using the monotonicity of \( F(r, u) + \varepsilon u \), we obtain
It follows from (3.44), (3.45) and Gronwall’s Lemma that \( w = 0 \). Therefore, Theorem 1 is proved.

**IV. THE BOUNDEDNESS OF THE SOLUTION**

Now we make the following assumptions

\((H_1')\) \hspace{1cm} u_0 \in L^\infty(0,1), \overline{u}_0 \in \mathbb{R}, \max\{|u_0(r)|,|\overline{u}_0|\} \leq M \ a.e. \ r \in (0,1).

\((H_2')\) \hspace{1cm} a, h \in W^{1,\infty}(0,\infty), \ a(t) \geq a_0 > 0, \ h(t) \geq h_0 > 0;

\((H_3')\) \hspace{1cm} f \in L^2(0,T;V_0), \ f(r,t) \leq 0 \ a.e. \ (r,t) \in Q_T.

\((F_1')\) \hspace{1cm} uF(r,u) \geq 0 \ \forall u \in \mathbb{R}, \ |u| \geq \|u_0\|_{L^\infty(0,1)}, \ for \ a.e., \ r \in (0,1).

We then have the following theorem.

**Theorem 2.** Let \((H_1') - (H_3'), (F_1') - (F_3'), (F_1')\) hold. Then the unique weak solution of the initial and boundary value problem (3.1) - (3.2), as given by theorem 1, belongs to \( L^\infty(Q_T) \).

**Remark 3.** Assumption \((H_1')\) is both physically and mathematically natural in the study of partial differential equation of the kind of (1.1)-(1.3), by means of the maximum principle.

**Proof of Theorem 2.** First, let us assume that \( u_0(r) \leq M \) and \( \overline{u}_0 \). Then \( z = u - M \) satisfies the initial and boundary value

\begin{align}
(4.1) & \quad z_t - a(t)(z_{rr} + \frac{r}{r}z_r) + F(r,z + M) = f(r,t), \ 0 < r < 1, \ 0 < t < T, \\
(4.2) & \quad \lim_{r \to 0} r^{p/2}z_r(r,t) < +\infty, \\
(4.3) & \quad z(1,t) + h(t)(z(1,t) + M - \overline{u}_0) = 0, \\
(4.4) & \quad z(r,0) = u_0(r) - M.
\end{align}

Multiplying equation (4.1) by \( r^pv \), for \( v \in V_1 \) integrating by parts with respect to variable \( r \) and taking into account boundary condition (4.2), one has after some rearrangements
\[
\begin{align*}
\int_0^1 r^\gamma z_t v dr + a(t) \int_0^1 r^\gamma z_r v dr + a(t)h(t)z(1, t)v(1) \\
+ \int_0^1 r^\gamma F(r, z + M)v dr \\
= \int_0^1 r^\gamma f v dr + (\bar{a}_0 - M)a(t)h(t)v(1), \text{ for all } v \in V_1.
\end{align*}
\]

Noticing from assumption \((H'_1)\) we deduce that the solution of the initial and boundary value problem (3.1) - (3.2) belongs to \(L^2(0, T; V_1) \cap L^\infty(0, T; V_0)\), so that we are allowed to take \(v = z^+ = \frac{1}{2}(|z| + z)\) in (4.4). Thus, it follows that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_0^1 r^\gamma |z^+|^2 dr + a(t) \int_0^1 r^\gamma |(z^+)_r|^2 dr + a(t)h(t)|z^+(1, t)|^2 \\
+ \int_0^1 r^\gamma F(r, z^+ + M)z^+ dr \\
= \int_0^1 r^\gamma fz^+ dr + (\bar{a}_0 - M)a(t)h(t)z^+(1, t) \leq 0,
\end{align*}
\]

since

\[
\begin{align*}
\int_0^1 r^\gamma z_t z^+ dr &= \int_{0, z>0} r^\gamma (z^+)_t z^+ dr = \frac{1}{2} \frac{d}{dt} \int_{0, z>0} r^\gamma |z^+|^2 dr \\
&= \frac{1}{2} \frac{d}{dt} \int_0^1 r^\gamma |z^+|^2 dr = \frac{1}{2} \frac{d}{dt} \|z^+(t)\|_0^2,
\end{align*}
\]

and on the domain \(z > 0\) we have \(z^+ = z\) and \((z^+)_r = (z^+)_r\).

On the other hand, by the assumption \((H'_2)\) and the inequality (2.9), we obtain

\[
\begin{align*}
a(t) \int_0^1 r^\gamma |(z^+)_r|^2 dr + a(t)h(t)|z^+(1, t)|^2 \\
\geq \frac{\gamma a_0}{\gamma + 1} \min\{1, h_0\} \|z^+(t)\|_1^2 = \overline{C}_0 \|z^+(t)\|_1^2.
\end{align*}
\]

Using the monotonicity of \(F(r, u) + \varepsilon u\) and \((F'_1)\) we obtain
\[ \int_0^1 r^\gamma F(r, z^+ + M) z^+ dr = \int_0^1 r^\gamma [F(r, z^+ + M) - F(r, M)] z^+ dr + \int_0^1 r^\gamma F(r, M) z^+ dr \]
\[ \geq -\varepsilon \int_0^1 r^\gamma |z^+|^2 dr + \int_0^1 r^\gamma F(r, M) z^+ dr \geq -\varepsilon \|z^+(t)\|^2. \]

Hence, it follows from (4.5)-(4.7) that
\[ \frac{d}{dt} \|z^+(t)\|^2 + 2\bar{C}_0 \|z^+(t)\|^2 \leq 2\varepsilon \|z^+(t)\|^2. \]

Integrating (4.8), we get
\[ \|z^+(t)\|^2 \leq \|z^+(0)\|^2 + 2\varepsilon \int_0^t \|z^+(s)\|^2 ds. \]

Since \( z^+(0) = (u(r, 0) - M)^+ = (u_0(r) - M)^+ = 0 \), hence, using Gronwall’s Lemma, we obtain
\[ \|z^+(t)\|^2 = 0. \]
Thus \( z^+ = 0 \) and \( u(r, t) \leq M \) for a.e. \((r, t) \in Q_T\).

The case \(-M \leq u_0(r) \) and \(-M \leq \bar{u}_0 \) can be dealt with, in the same manner as above, by considering \( z = u + M \) and \( z^- = \frac{1}{2} (|z| - z) \), we also obtain \( z^- = 0 \) and hence \( u(r, t) \geq -M \) for a.e. \((r, t) \in Q_T\).

From all above, one obtains \( |u(r, t)| \leq M \) a.e. \((r, t) \in Q_T\) and this ends the proof of Theorem 2.

V. ASYMPTOTIC BEHAVIOR OF THE SOLUTION AS \( t \to +\infty \).

In this part, let \( T > 0, (H_1) - (H_3) \), and \((F_1) - (F_3) \) hold. Then, there exists a unique solution \( u \) of problem (3.1) - (3.2) such that
\[ u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0), \quad r^{\gamma_p} u \in L^p(Q_T), \]
\[ tu \in L^\infty(0, T; V_1), \quad tu \in L^2(0, T; V_0). \]

We shall study asymptotic behavior of the solution \( u(t) \) as \( t \to +\infty \).

We make the following supplementary assumptions on the functions \( a, h, f \).

\[ (H_3''') \quad f \in L^\infty(0, \infty; V_0); \]
\[ (H_4) \quad \text{There exist the positive constants } C_a, C_h, C_f, \gamma_a, \gamma_h, \gamma_f, a_\infty, h_\infty \]

and a function \( f_\infty \in V_0 \) such that
First, we consider the following stationary problem

\begin{equation}
-a_x\left( u''_x(r) + \frac{r}{x} u'_x(r) \right) + F(r, u_x(r)) = f_x(r), \quad 0 < r < 1,
\end{equation}

\begin{equation}
\lim_{r \to 0^+} r^{\gamma/2} u'_x(r) < +\infty, \quad u'_x(1) + h_x u_x(1) = h_x \overline{u}_0.
\end{equation}

The weak solution of problem (5.1)-(5.2) is obtained from the following variational problem.

Find \( u_x \in V_1 \) such that

\begin{equation}
a_x \langle u''_x, v \rangle + a_x h_x u_x(1) v(1) + \langle F(r, u_x), v \rangle = \langle f_x, v \rangle + \overline{u}_0 a_x h_x v(1), \quad \text{for all } v \in V_1.
\end{equation}

We then have the following theorem.

**Theorem 3.** Let \((F_1), (F_2), (H_4)\) hold. Then there exists a solution \( u_x \) of the variational problem (5.3) such that

\[ u_x \in V_1 \text{ and } r^{\gamma/\beta} u_x \in L^p(0, 1). \]

Furthermore, if \( F \) satisfies the following condition, in addition,

\[ F(r, u) + \varepsilon u \text{ is nondecreasing with respect to variable } u, \]

\[ \text{with } 0 < \varepsilon < \frac{\gamma a_x}{\gamma + 1} \min \{1, h_x\}. \]

Then the solution is unique.

**Proof.** Denote by \( \{w_j\}, j = 1, 2, \ldots \) an orthonormal basis in the separable Hilbert space \( V_1 \). Put

\begin{equation}
y_m = \sum_{j=1}^{m} d_{mj} w_j,
\end{equation}

where \( d_{mj} \) satisfy the following nonlinear equation system:
\[ a_x \langle y'_m, w_j \rangle + a_x h_x y_m(1)w_j(1) + \langle F(r, y_m), w_j \rangle \]
\[ = \langle f_x, w_j \rangle + \overline{a}_0 a_x h_x w_j(1), \quad 1 \leq j \leq m. \]

By the Brouwer's lemma (see Lions [2], Lemma 4.3, p.53), it follows from the hypotheses \((F_1), (F_2), (H_4)\) that system (5.4), (5.5) has a solution \(y_m\).

Multiplying the \(j\)th equation of system (5.5) by \(d_m\), then summing up with respect to \(j\), we have

\[ a_x \| y'_m \|_2^2 + a_x h_x y_m^2(1) + \langle F(r, y_m), y_m \rangle \]
\[ = \langle f_x, y_m \rangle + \overline{a}_0 a_x h_x y_m(1). \]

By using the inequalities (2.5), (2.9) and by the hypotheses \((F_1), (H_4)\), we obtain

\[ C_0 \| y_m \|_1^2 + C_1 \int_0^1 r^\gamma |y_m(r)|^p dr \]
\[ \leq (\| f_x \|_0 + |\overline{a}_0 a_x h_x K_1|) \| y_m \|_1 + \frac{c'}{\gamma+1}, \]

where \( C_0 = \frac{\overline{a}_0}{\gamma+1} \min \{1, h_x\}. \)

Hence, we deduce from (5.7) that

\[ \| y_m \|_1 \leq C. \]

\[ \int_0^1 r^\gamma |y_m(r)|^p dr \leq C, \]

\( C \) is a constant independent of \(m\).

By means of (5.8), (5.9) and Lemma 2, the sequence \( \{y_m\} \) has a subsequence still denoted by \( \{y_m\} \) such that

\[ y_m \to u_\infty \text{ in } V_1 \text{ weakly}, \]

\[ y_m \to u_\infty \text{ in } V_0 \text{ strongly and } a.e. \text{ in } (0,1), \]

\[ r^{\gamma/p} y_m \to r^{\gamma/p} u_\infty \text{ in } L^p(0,1) \text{ weakly}. \]

On the other hand, by (5.11) and the hypothesis \((F_1), (F_2)\) we have
We also deduce from the hypothesis \((F_2)\) and from (5.9) that
\[
\int_0^1 \left| r^{p/\ell} F(r, y_m(r)) \right|^p dr \leq 2^{p-1} C_2^p \left[ 1 + \int_0^1 r^{p} |y_m(r)|^p dr \right] \leq C,
\]
where \(C\) is a constant independning of \(m\).

Applying Lemma 3 with \(N = 1, q = p/\ell, G_m = r^{p/\ell} F(r, y_m), G = r^{p/\ell} F(r, u_\infty)\), we deduce from (5.13), (5.14) that
\[
r^{p/\ell} F(r, y_m) \to r^{p/\ell} F(r, u_\infty) \text{ in } L^{p/\ell}(0, 1) \text{ weakly.}
\]
Passing to the limit in Eq.(5.5), we find without difficulty from (5.10), (5.15) that \(u_\infty\) satisfies the equation
\[
a_x(u_\infty, w_j) + a_x h_x u_\infty(1) w_j(1) + \langle F(r, u_\infty), w_j \rangle = \langle f, w_j \rangle + \bar{u}_0 a_x h_x w_j(1).
\]
Equation (5.16) holds for every \(j = 1, 2, \ldots\), i.e., (5.3) holds. The solution of the problem (5.3) is unique; that can be showed using the same arguments as in the proof of Theorem 1.

**Remark 4.** The result of Theorem 3 is similar to one in [7].

Now we consider asymptotic behavior of the solution \(u(t)\) as \(t \to +\infty\).

We then have the following theorem.

**Theorem 4.** Let \((F_1), (F_2), (F_4), (H_1), (H_2^d), (H_3^d), (H_4)\) hold. Then we have
\[
\|u(t) - u_\infty\|_0^2 \leq \left( \|u_0 - u_\infty\|_0^2 + \frac{\bar{C}_2}{2(\gamma_1 - \gamma_0)} \right) e^{-2\gamma_0 t}, \forall t \geq 0,
\]
where
\[
\bar{C}_2 = \frac{1}{\delta} (C^2 C_d^2 + C_f^2 + K_0^2 (|\bar{u}_0| + CK_1)^2 (C_a \|h\|_x + C_h a_x)^2),
\]
\[
\delta = \frac{1}{4} \left( \frac{\gamma_0}{\gamma + 1} \min \{1, h_0 \} - \varepsilon \right),
\]
\(\gamma_0\) is a constant depending only on the constants \(\gamma_1 = \min \{\gamma_a, \gamma_h, \gamma_f\}\) and \(\bar{C}_1 = \frac{\gamma_0}{\gamma + 1} \min \{1, h_0 \} - \varepsilon\).

**Proof.** Put \(Z_m(t) = u_m(t) - y_m\). Let us subtract (3.5) with (5.5) to obtain
By multiplying (5.17) by \(c_{mj}(t) - d_{mj}\) and summing up in \(j\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|Z_m(t)\|_0^2 + a(t) \|Z_{mr}(t)\|_0^2 + (a(t) - a_x)\langle y_{mr}, Z_{mr}(t)\rangle \\
+ a(t)h(t)Z_m(1,t) + (a(t)h(t) - a_x h_x) y_m(1)Z_m(1,t) \\
+ \langle F(r, u_m(t)) - F(r, y_m), Z_m \rangle \\
= \langle f(t) - f_x, w_j \rangle + \bar{a}_0(a(t)h(t) - a_x h_x) w_j(1), 1 \leq j \leq m,
\]

(5.17)

From the assumption \((H'_2)\) and the inequality (2.9), it follows that

\[
a(t) \|Z_{mr}(t)\|_0^2 + a(t)h(t)Z_m^2(1,t) \geq \bar{C}_0 \|Z_m(t)\|_1^2,
\]

(5.19)

where \(\bar{C}_0 = \frac{\gamma a_0}{\gamma + 1} \min\{1, h_0\}\).

By \((F_4)\), we get

\[
\langle F(r, u_m(t)) - F(r, y_m), Z_m \rangle \geq -\mathcal{E} \|Z_m(t)\|_0^2.
\]

(5.20)

It follows from (5.18)-(5.20), and (2.5), that

\[
\frac{d}{dt} \|Z_m(t)\|_0^2 + 2\bar{C}_0 \|Z_m(t)\|_0^2 \leq 2|a(t) - a_x| \|y_{mr}\|_0 \|Z_{mr}(t)\|_0 \\
+ 2|a(t)h(t) - a_x h_x| K_1 \|y_m\|_1 \|Z_m(t)\|_1 + 2\mathcal{E} \|Z_m(t)\|_0^2 \\
+ 2 \|f(t) - f_x\|_0 \|Z_m(t)\|_0 + 2|\bar{a}_0||a(t)h(t) - a_x h_x| K_1 \|Z_m(t)\|_1.
\]

(5.21)

Note that \(\|y_m\|_1 \leq C\), we obtain from (5.21) that
\[
\frac{d}{dt} \| Z_m(t) \|_0^2 + 2\tilde{C}_0 \| Z_m(t) \|_1^2
\leq 2C|a(t) - a_\infty|\| Z_m(t) \|_1 + 2\varepsilon \| Z_m(t) \|_1^2 + 2\| f(t) - f_\infty \|_0 \| Z_m(t) \|_0
+ 2K_1(|\tilde{u}_0| + CK_1)|a(t)h(t) - a_\infty h_\infty|\| Z_m(t) \|_1.
\]

Choose \( \delta > 0 \) such that \( 3\delta < \tilde{C}_0 - \varepsilon = \tilde{C}_1 \), then we have from (5.22)

\[
\frac{d}{dt} \| Z_m(t) \|_0^2 + \tilde{C}_1 \| Z_m(t) \|_1^2
\leq \frac{1}{\delta} (C_1^2 a_\infty + C_f^2 + K_1^2 (|\bar{u}_0| + CK_1)^2 (C_h \| h \|_\infty + C_h a_\infty)^2)e^{-2\gamma_1 t}
= \tilde{C}_2 e^{-2\gamma_1 t}, \quad \text{for all} \ t \geq 0.
\]

Put \( \gamma_1 = \min\{\gamma_a, \gamma_h, \gamma_f\} \), we deduce from (5.23) and \( (H_4) \) that

\[
\frac{d}{dt} \| Z_m(t) \|_0^2 + \tilde{C}_1 \| Z_m(t) \|_1^2
\leq \frac{1}{\delta} (C_1^2 a_\infty + C_f^2 + K_1^2 (|\bar{u}_0| + CK_1)^2 (C_h \| h \|_\infty + C_h a_\infty)^2)e^{-2\gamma_1 t}
\]

Put \( \gamma_0 = \frac{1}{2} \min\{\gamma_1, \tilde{C}_1\} \). Hence, we obtain from (5.24) that

\[
\| Z_m(t) \|_0^2 \leq e^{-2\gamma_0 t} \| Z_0m \|_1^2 + \tilde{C}_2 e^{-2\gamma_0 t} \int_0^t e^{-2(\gamma_1 - \gamma_0)s} ds
\leq \left( \| Z_0m \|_1^2 + \frac{\tilde{C}_2}{2(\gamma_1 - \gamma_0)} \right) e^{-2\gamma_0 t}.
\]

Letting \( m \to +\infty \) in (5.25) we obtain

\[
\| u(t) - u_\infty \|_0^2 \leq \liminf_{m \to +\infty} \| u_m(t) - y_m \|_0^2
\leq \left( \| u_0 - u_\infty \|_0^2 + \frac{\tilde{C}_2}{2(\gamma_1 - \gamma_0)} \right) e^{-2\gamma_0 t}, \quad \text{for all} \ t \geq 0.
\]

This completes the proof of Theorem 4. 

VI. NUMERICAL RESULTS

First, we present some results of numerical comparison of the approximated representation of the solution of a nonlinear problem of the type (1.1)-(1.3) and the corresponding exact
solution of this problem.
Let the problem
\begin{eqnarray}
(6.1) & u_t - \left( u_{rr} + \frac{2}{r} u_r \right) + F(u) = 0, \\
(6.2) & u_r(1, t) + u(1, t) = 0, \ u_r(0, t) = 0, \\
(6.3) & u(r, 0) = 0,
\end{eqnarray}
where
\[ f(r, t) = e^{-ar}(1 + ar) \cos t + \alpha^2 e^{-ar} \sin t(3 - ar) + e^{-\frac{3}{2}ar}(1 + ar)^{3/2} \sgn(\sin t), \]
\[ F(u) = |u|^{3/2} \sgn(u); \]
\[ \alpha = \frac{1 + \sqrt{5}}{2}, \] and the domain \( D = \{(r, t) : 0 \leq r \leq 1, 0 \leq t \leq 1\}. \]

The exact solution of the problem (6.1)-(6.3) is \( v(r, t) = e^{-ar}(1 + ar) \sin t. \)
To solve numerically the problem (6.1)-(6.3), we consider the nonlinear differential system for the unknowns \( u_k(t) = u(r_k, t), \ r_k = kh, \ h = 1/N. \)
\begin{equation}
\begin{cases}
\frac{du_k}{dt} = \frac{1}{h^2} \left( 1 - \frac{2}{k} \right) u_{k-1} + \frac{2}{h^2} \left( \frac{1}{k} - 1 \right) u_k + \frac{u_{k+1}}{h^2} - F(u_k) + f(r_k, t), \\
u_1 = u_0, \ u_N = \frac{u_{N-1}}{h+1}, \\
u_k(0) = 0, \ k = 1, 2, \ldots, N-1.
\end{cases}
\end{equation}

To solve the nonlinear differential (6.4) at the time \( t, \) we use the following linear recursive scheme generated by the nonlinear term \( F(u_k): \)
\begin{equation}
\begin{cases}
\frac{du_{k,n}}{dt} = \frac{1}{h^2} \left( 1 - \frac{2}{k} \right) u_{k-1,n} + \frac{2}{h^2} \left( \frac{1}{k} - 1 \right) u_{k,n} + \frac{u_{k+1,n}}{h^2} - F(u_{k,n}) + f(r_k, t), \\
u_{k,n}(0) = 0, \ k = 1, 2, \ldots, N-1.
\end{cases}
\end{equation}
The linear differential system (6.5) is solved by searching the associated eigenvalues and eigenfunctions. With a spatial step \( h = \frac{1}{10} \) on the interval \([0, 1]\) and for \( t \in [0, 2], \) we have drawn the corresponding approximate surface solution \((t, t) \rightarrow u(r, t)\) in figure 1, obtained by successive re-initializations in \( t \) with a time step \( \Delta t = \frac{1}{50}. \) For comparison in figure 2, we have also drawn the exact surface solution \((t, t) \rightarrow v(r, t). \)

Now consider the following problem
\begin{eqnarray}
(6.6) & u_t - \left( u_{rr} + \frac{2}{r} u_r \right) + |u|^{3/2} \sgn(u) = 0, \\
(6.7) & u_r(1, t) + u(1, t) = 0, \ u_r(0, t) = 0, \\
(6.8) & u(r, 0) = \frac{1}{4}.
\end{eqnarray}
Using the same method as previously we have drawn in figure 3 the approximate surface
solution \((t, t) \rightarrow u(r, t)\) which decreases exponentially to 0 as \(t\) tends to infinity, 0 being the unique solution of the corresponding steady state problem

\[
(6.9) \quad u_{rr} + \frac{2}{r} u_r - |u|^{3/2} \text{sgn}(u) = 0,
\]

\[
(6.10) \quad u_r(1) + u(1) = 0, \; u_r(0) = 0.
\]

Notice, since the function \(F(u) = |u|^{3/2} \text{sgn}(u)\) has a derivative positive the solution of the problem (6.6)-(6.8) is bounded and unique according section IV.

Figure 1. Approximate solution
Figure 2. Exact solution

Figure 3. Asymptotic behavior
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