Exponential growth of Lie algebras of finite global dimension

Yves Félix, Steve Halperin, Jean-Claude Thomas

To cite this version:

HAL Id: hal-00008857
https://hal.archives-ouvertes.fr/hal-00008857
Submitted on 19 Sep 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
An asymptotic formula for the ranks of the homotopy groups of a finite complex

Yves Felix, Steve Halperin and Jean-Claude Thomas

September 23, 2005

Abstract

Let $X$ be a finite simply connected CW complex of dimension $n$. The loop space homology $H_*(\Omega X; \mathbb{Q})$ is the universal enveloping algebra of a graded Lie algebra $L_X$ isomorphic with $\pi_{*-1}^*(X) \otimes \mathbb{Q}$. Let $Q_X \subset L_X$ be a minimal generating subspace, and set $\alpha = \lim \sup \frac{\log \text{rk} \pi_i(X)}{i}$. Theorem:

If $\dim L_X = \infty$ and $\lim \sup (\text{dim} (Q_X)_k)^{1/k} < \lim \sup (\text{dim} (L_X)_k)^{1/k}$, then

$$\sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X) = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \to 0 \text{ as } k \to \infty.$$ 

In particular $\sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X)$ grows exponentially in $k$.

AMS Classification : 55P35, 55P62, 17B70

Key words : Homotopy Lie algebra, graded Lie algebra, exponential growth

1 Introduction

Suppose $X$ is a finite simply connected CW complex of dimension $n$. The homotopy groups of $X$ then have the form

$$\pi_i(X) = \mathbb{Z}^{\rho_i} \oplus T_i,$$

where $T_i$ is a finite abelian group and $\rho_i = \text{rk} \pi_i(X)$ is finite. It is known [6] that either $\pi_i(X) = T_i$, $i \geq 2n$ ($X$ is rationally elliptic) or else for all $k \geq 1$, $\sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X) > 0$. In this case $X$ is called rationally hyperbolic.
In [7] it is shown that in the rationally hyperbolic case \( \sum_{i=1}^{n-1} \text{rk} \pi_{k+i}(X) \) grows faster than any polynomial in \( k \). Here we show that with an additional hypothesis this sum grows exponentially in \( k \) and, in fact, setting \( \alpha = \lim \sup \frac{\log \text{rk} \pi_i}{k} \) we have

\[
\sum_{i=1}^{n-1} \text{rk} \pi_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \to 0 \text{ as } k \to \infty.
\]

In subsequent papers we will identify a large class of spaces for which the additional hypothesis holds: in fact it may well hold for all finite simply connected CW complexes.

Note that \( \text{rk} \pi_i(X) = \dim \pi_i(X) \otimes \mathbb{Q} \). Thus we work more generally with simply connected spaces \( X \) such that each \( H_i(X; \mathbb{Q}) \) is finite dimensional. In this case \( \dim \pi_i(X) \otimes \mathbb{Q} \) is also finite for each \( i \). On the other hand, a theorem of Milnor-Moore-Cartan-Serre asserts that the loop space homology \( H_*(\Omega X; \mathbb{Q}) \) is the universal enveloping algebra of a graded Lie algebra \( L_X \) and that the Hurewicz homomorphism is an isomorphism \( \pi_*(\Omega X) \otimes \mathbb{Q} \cong L_X \). Since there are natural isomorphisms \( \pi_i(X) \cong \pi_{i-1}(\Omega X) \) it follows that the results above can be phrased in terms of the integers \( \dim (L_X)_i \).

For any graded vector space \( V \) concentrated in positive degrees we define the \emph{logarithmic index} of \( V \) by

\[
\log \text{index } V = \lim \sup \frac{\log \text{dim } V_k}{k}.
\]

In ([5], [6]) it is shown that if \( X \) (simply connected) has finite Lusternik-Schnirelmann category (in particular, if \( X \) is a finite CW complex) and if \( X \) is rationally hyperbolic then

\[
\log \text{index } L_X > 0.
\]

Now let \( Q_X \) denote a minimal generating subspace for the Lie algebra \( L_X \), and let \( \alpha \) denote \( \log \text{index } L_X \).

**Theorem 1.** Let \( X \) be a simply connected topological space with finite dimensional rational homology concentrated in degrees \( \leq n \). Suppose \( \log \text{index } Q_X < \log \text{index } L_X \), then

\[
\sum_{i=1}^{n-1} \dim (L_X)_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \to 0 \text{ as } k \to \infty.
\]

In particular this sum grows exponentially in \( k \).
Theorem 2. Let $X$ be a simply connected topological space with finite dimensional rational homology in each degree, and finite Lusternik-Schnirelmann category. Suppose $\log \text{index } Q_X < \log \text{index } L_X < \infty$. Then for some $d > 0$,

$$
\sum_{i=1}^{d-1} \dim (L_X)_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where } \varepsilon_k \to 0 \text{ as } k \to \infty.
$$

In particular, $\sum_{i=1}^{d-1} \dim (L_X)_{k+i}$ grows exponentially in $k$.

Corollary: The conclusion of Theorem 1 holds for finite simply connected CW complexes $X$ for which $L_X$ is infinite, but finitely generated. The conclusion of Theorem 2 holds for simply connected spaces of finite LS category and finite rational Betti numbers provided that $L_X$ is infinite, but finitely generated, and, $\log \text{index } L_X < \infty$.

Recall that the depth of a graded Lie algebra $L$ is the least $m$ (or $\infty$) such that $\text{Ext}^m_{UL}(\mathbb{Q}, UL) \neq 0$.

The key ingredients in the proofs of Theorems 1 and 2 are

- A growth condition for $L_X$ established in ([5],[6])
- The fact that depth $L_X < \infty$, established in ([3],[6])

We shall use Lie algebra arguments in Theorem 3 below to deduce the conclusion of Theorems 1 and 2 from these ingredients, and then deduce Theorems 1 and 2.

Theorems 1 and 2 may be compared with the results in [7] and in [8] that assert for $n$-dimensional finite CW complexes (respectively for simply connected spaces with finite type rational homology and finite Lusternik-Schnirelmann category) that $\sum_{i=1}^{n-1} \dim (L_X)_{k+i}$ (resp. $\sum_{i=1}^{d-1} \dim (L_X)_{k+i}$) grows faster than any polynomial in $k$. These results use only the fact that $L_X$ has finite depth, and require no hypothesis on $Q_X$.

The hypothesis on $Q_X$ in Theorems 1 and 2 may be restated as requiring that the formal series $\sum_q \dim \text{Tor}_{1,q}^{ULX} z^q$ have a radius of convergence strictly greater than that of the formal series $\sum_q \dim (ULX)_q z^q$. Lambrechts [9] has proved a much stronger result under the hypothesis that the formal series $\sum_q \left( \sum_p (-1)^p \dim \text{Tor}_{p,q} \right) z^q$ has a radius of convergence strictly larger than that of $\sum_q \dim (ULX)_q z^q$. 

3
2 Lie algebras

In this section we work over any ground field $k$ of characteristic different from $2$; graded Lie algebras $L$ are defined as in [7] and, in particular, are assumed to satisfy $[x,[x,x]] = 0, x \in L_{odd}$ (This follows from the Jacobi identity except when $\text{char } k = 3$).

A graded Lie algebra $L$ is connected and of finite type if

$$L = \{L_i\}_{i \geq 1} \text{ and each } L_i \text{ is finite dimensional.}$$

We shall refer to these as cft Lie algebras. The minimal generating subspaces $Q$ of a cft Lie algebra $L$ are those subspaces $Q$ for which $Q \to L/[L,L]$ is a linear isomorphism.

A growth sequence for a cft Lie algebra $L$ is a sequence $(r_i)$ such that $r_i \to \infty$ and

$$\lim_{i \to \infty} \frac{\log \dim L_{r_i}}{r_i} = \log \text{index } L.$$

A quasi-geometric sequence $(\ell_i)$ is a sequence such that for some integer $m$, $\ell_i < \ell_{i+1} \leq m\ell_i$, all $i$; if additionally $(\ell_i)$ is a growth sequence then it is a quasi-geometric growth sequence.

Of particular interest here are the growth conditions

$$0 < \log \text{index } L < \infty; \quad \text{(A.1)}$$

and

$$\log \text{index } Q < \log \text{index } L. \quad \text{(A.2)}$$

**Proposition.** Let $L$ be a cft Lie algebra satisfying (A.1) and (A.2), and assume $L$ has a quasi-geometric growth sequence $(r_j)$. Then any sequence $(s_i)$ such that $s_i \to \infty$ has a subsequence $(s_i)$ for which there are growth sequences $(t_j)$ and $(p_j)$ such that

$$t_j \leq s_i < p_j \quad \text{and} \quad p_j/t_j \to 1.$$

**Proof.** First note that because of (A.2),

$$\log \text{index } L/Q = \log \text{index } L$$
Now adopt the following notation, for \( i \geq 1 \):

\[
\begin{align*}
\log \text{index } L/Q &= \alpha \\
\dim L_i e^{(\alpha + \sigma_i)i} &= \\
\dim Q_i &= e^{(\alpha + \sigma_i)i} \\
\dim L_i/Q_i &= e^{(\alpha + \tau_i)i} \\
\dim (UL)_i &= e^{(\alpha + \delta_i)i}
\end{align*}
\]

(1)

Then because of (A.1) and (A.2), \( 0 < \alpha < \infty \), and \( \lim \sup \epsilon_i = 0 \). Moreover, by a result of Babenko ([2], [6]), \( \log \text{index } UL = \log \text{index } L \), and so \( \lim \sup (\delta_i) = 0 \). Finally (A.2) implies that \( \lim \sup (\sigma_i) = \sigma < 0 \), and that \( \tau_{q_i} \to 0 \) as \( i \to \infty \).

Next, since \( r_j \) is a quasi-geometric sequence, for some fixed \( m \) we have \( r_j < r_{j+1} \leq mr_j \), all \( j \). It follows that for each \( s_i \) in our sequence we may choose \( q_i \) in the sequence \( (r_j) \) so that

\[
s_i < q_i \leq ms_i.
\]

(2)

Thus \( q_i \to \infty \) as \( i \to \infty \). The adjoint representation of \( UL \) in \( L \) defines surjections

\[
\bigoplus_{(\ell,k,t) \in J_i} (UL)_\ell \otimes Q_k \otimes L_t \to \bigoplus_{(\ell,k,t) \in J_i} (UL)_\ell \otimes [Q_k, L_t] \to L_{q_i}/Q_{q_i},
\]

(3)

where \( J_i \) consists of those triples for which \( \ell + k + t = q_i \) and \( t \leq s_i < t + k \). Thus

\[
3(q_i + 1) \max_{(\ell,k,t) \in J_i} \dim (UL)_\ell \dim [Q_k, L_t] \geq \dim L_{q_i}/Q_{q_i}.
\]

For each \( s_i \) we may therefore choose \( (\ell_i, k_i, t_i) \in J_i \) so that

\[
3(q_i + 1) \dim (UL)_{\ell_i} \dim [Q_{k_i}, L_{t_i}] \geq \dim L_{q_i}/Q_{q_i}.
\]

(4)

**Lemma 1.** \( (k_i + t_i) \geq 1/m \) \( q_i \), and \( (k_i + t_i) \geq 1/m - 1 \ell_i \). In particular, \( k_i + t_i \to \infty \) as \( i \to \infty \).

**Proof.** It follows from (2) that \( k_i + t_i \geq s_i \geq 1/m \) \( q_i \). Since \( q_i = k_i + t_i + \ell_i \), \( (k_i + t_i) \geq 1/m - 1 \ell_i \). Since \( s_i \to \infty \) so does \( k_i + t_i \). \( \square \)

Next, define \( \varepsilon(k_i, t_i) \) by

\[
\dim [Q_{k_i}, L_{t_i}] = e^{[\alpha + \varepsilon(k_i, t_i)](k_i + t_i)}.
\]
Lemma 2. $\varepsilon(k_i, t_i) \to 0$ and $\varepsilon_{k_i+t_i} \to 0$ as $i \to \infty$.

Proof. Define $\varepsilon(q_i)$ by $3(q_i + 1) = e^{\varepsilon(q_i)}q_i$. Then (4) reduces to

$$\varepsilon(q_i)q_i + (\alpha + \delta_{\ell_i})\ell_i + [\alpha + \varepsilon(k_i, t_i)](k_i + t_i) \geq (\alpha + \tau_{q_i})q_i.$$  

Since $\ell_i + k_i + t_i = q_i$,

$$\varepsilon(q_i)q_i \frac{q_i}{k_i + t_i} + \delta_{\ell_i} \frac{\ell_i}{k_i + t_i} + \varepsilon(k_i, t_i) \geq \tau_{q_i} \frac{q_i}{k_i + t_i}.$$  

Now as $i \to \infty$, $q_i \geq s_i \to \infty$. Thus $\varepsilon(q_i) \to 0$. Moreover, since $q_i$ belongs to a growth sequence, $\varepsilon_{q_i} \to 0$ and $\tau_{q_i} \to 0$. Use Lemma 1 to conclude that

$$\varepsilon(q_i)q_i \frac{q_i}{k_i + t_i} \to 0 \quad \text{and} \quad \tau_{q_i} \frac{q_i}{k_i + t_i} \to 0,$$

and hence

$$\liminf \varepsilon(k_i, t_i) \geq -\limsup \delta_{\ell_i} \frac{\ell_i}{k_i + t_i}.$$  

Next, since $\limsup \delta_{\ell_i} = 0$ and $\frac{\ell_i}{k_i + t_i} \leq m - 1$, it follows that

$$-\limsup \delta_{\ell_i} \frac{\ell_i}{k_i + t_i} = 0.$$  

Finally, $[Q_{k_i}, L_{t_i}]$ embeds in $L_{k_i+t_i}/Q_{k_i+t_i}$, and it follows that

$$\varepsilon_{k_i+t_i} \geq \varepsilon(k_i, t_i).$$  

But $k_i+t_i \to \infty$ and so $\limsup \varepsilon_{k_i+t_i} \leq 0$. This, together with $\liminf \varepsilon(k_i, t_i) \geq 0$, completes the proof of the lemma. \hfill \Box

Next, since $Q_{k_i} \otimes L_{t_i} \to [Q_{k_i}, L_{t_i}]$ is surjective we have

$$\sigma_{k_i, k_i} + \varepsilon_{t_i, t_i} \geq \varepsilon(k_i, t_i)(k_i + t_i) \quad (5)$$

Lemma 3. $t_i/k_i \to \infty$ and $t_i \to \infty$ as $i \to \infty$.

Proof. If $t_i/k_i$ does not converge to $\infty$ then we would have $t_{i,\nu}/k_{i,\nu} \leq T$ for some subsequence $(s_{i,\nu})$. But

$$\sigma_{k_{i,\nu}} \geq -\varepsilon_{t_{i,\nu}} \frac{t_{i,\nu}}{k_{i,\nu}} + \varepsilon(k_{i,\nu}, t_{i,\nu})(1 + t_{i,\nu}/k_{i,\nu}).$$
Since (Lemma 1) \( k_{i_{\nu}} + t_{i_{\nu}} \to \infty \) and (Lemma 2) \( \varepsilon(k_{i_{\nu}}, t_{i_{\nu}}) \to 0 \), the lim inf of the right hand side of this equation would be \( \geq 0 \). Hence \( \limsup \sigma_{k_{i_{\nu}}} \geq 0 \), which would contradict \( \limsup \sigma_i < 0 \). Finally, since \( t_{i_{\nu}} + k_{i_{\nu}} \to \infty \) it follows that \( t_{i_{\nu}} \to \infty \). \( \square \)

**Lemma 4.** Write \( k_i \lambda_i t_i \). Then,
\[
\lambda_i \to 0 \quad \text{and} \quad \varepsilon_t \to 0 \quad \text{as} \quad i \to \infty.
\]

**Proof.** Lemma 3 asserts that \( \lambda_i \to 0 \). Rewrite equation (5) as
\[
\varepsilon_t \geq (\varepsilon(k_i, t_i) - \sigma_i) \lambda_i + \varepsilon(k_i, t_i).
\]
Since \( \limsup \sigma_j \) is finite, the \( \sigma_j \) are bounded above. Thus \( -\sigma_j \geq A \), some constant \( A \). Since \( \lambda_i \to 0 \), \( \liminf(-\sigma_i \lambda_i) \geq 0 \). Since (Lemma 2) \( \varepsilon(k_i, t_i) \to 0 \) it follows from (6) that \( \liminf \varepsilon_t \to 0 \). But \( \limsup \varepsilon_t \leq \limsup \varepsilon_i = 0 \) and so \( \varepsilon_t \to 0 \). \( \square \).

The lemmas above establish the Proposition. Simply set \( p_i = t_i + k_i \) and note that \( t_i \to \infty \) (Lemma 3), \( p_i \to \infty \) (Lemma 1), \( p_i/t_i = 1 + \lambda_i \to 1 \) (Lemma 4). Furthermore \( \varepsilon_t \to 0 \) (Lemma 4) and \( \varepsilon_p \to 0 \) (Lemma 2). Thus \( t_i \) and \( p_i \) are growth sequences. \( \square \)

**Theorem 3.** Let \( L \) be a cft Lie algebra of finite depth and satisfying the growth conditions (A.1) and (A.2). Set \( \alpha = \log \text{index} \ L \). Then for some \( d \),
\[
\sum_{i=1}^{d-1} \dim L_{k+i} = e^{(\alpha + \varepsilon_k)k}, \quad \text{where} \ \varepsilon_k \to 0 \ \text{as} \ k \to \infty.
\]
In particular, this sum grows exponentially in \( k \).

**Proof.** According to [4] there is a finitely generated sub Lie algebra \( E \subset L \) such that \( \text{Ext}^r_{U_1}(k, UL) \to \text{Ext}^r_{U_1 E}(k, UL) \) is non-zero.

**Lemma 5.** The centralizer, \( Z \), of \( E \) in \( L \) is finite dimensional.

**Proof.** Since \( E \) has finite depth, \( Z \cap E \) is finite dimensional [3]. Choose \( k \) so \( Z \cap E \) is concentrated in degrees \( < k \). Suppose \( x \in Z_{\geq k} \) has even degree, and put \( F = lx \oplus E \). Then \( \text{Ext}^r_{UF}(k, UF) \to \text{Ext}^r_{UE}(k, UF) \) is zero, contradicting the hypothesis that the composite
\[
\text{Ext}^r_{UL}(k, UL) \to \text{Ext}^r_{UF}(k, UL) \to \text{Ext}^r_{UE}(k, UL)
\]
is non-zero.

It follows that $Z_{\geq k}$ is concentrated in odd degrees, hence an abelian ideal in $Z + E$. Again

$$\text{Ext}^r_{UL}(\mathbb{k}, UL) \to \text{Ext}^r_{U(Z+E)}(\mathbb{k}, UL) \to \text{Ext}^r_{UE}(\mathbb{k}, UL)$$

is non-zero. Thus $\text{Ext}^r_{U(Z+E)}(\mathbb{k}, UL) \neq 0$, $Z + E$ has finite depth and every abelian ideal in $Z + E$ is finite dimensional. \(\square\)

Choose $d$ so that $E$ is generated in degrees $\leq d - 1$. As in the Proposition, set log index $L = \log index L/Q = \alpha$. If the theorem fails we can find a sequence $s_i \to \infty$ such that

$$\sum_{j=1}^{d-1} \dim L_{s_i+j} \leq e^{(\alpha-\beta)s_i}, \quad (7)$$

some $\beta > 0$. Apply the Proposition to find growth sequences $t_i$ and $p_i$ such that $t_i \leq s_i < p_i$ and $p_i/t_i \to 1$.

We now use (7) to prove that

$$\dim (UE)_{s_i-t_i,s_i-t_i+d} \dim L_{s_i,s_i+d} < \dim L_{t_i}, \quad i \text{ large}. \quad (8)$$

In fact since $\lim \sup (\dim (UL)_i)^{1/i} = e^\alpha$ it follows that for some $\gamma > 0$,

$$\sum_{j=1}^{d-1} \dim (UE)_{j+k} \leq e^{\gamma(k+1)}, \quad \text{all } k.$$

Thus it is sufficient to show that

$$\gamma(s_i-t_i+1) + (\alpha - \beta)s_i < (\alpha + \varepsilon_{t_i})t_i, \quad \text{large } i,$$

where $\varepsilon_{t_i} \to 0$ as $i \to \infty$. Write $s_i = \mu_i t_i$; then $\mu_i \to 1$ and the inequality reduces to the obvious

$$\gamma/t_i + (\mu_i - 1) + (\alpha - \beta)\mu_i < (\alpha + \varepsilon_{t_i}), \quad \text{large } i.$$

Thus (8) is established.

Choose $s = s_i$, $t = t_i$ so that (8) holds and so that $Z_j = 0$, $j \geq t$. Write $s - t = k$. The adjoint action of $UL$ in $L$ restricts to a linear map

$$\left[\bigoplus_{j=1}^{d-1}(UE)_{k+j}\right] \otimes L_t \to \bigoplus_{j=1}^{d-1}L_{s+j}$$
and it follows from (8) that for some non-zero \( x \in L_t \),
\[
(ad a)x = 0, \quad a \in UE_{(k,k+d)}.
\]

On the other hand, since \( E \) is generated in degrees \( \leq d - 1 \), \( (UE)_{>k} = UE \cdot (UE)_{(k,k+d)} \). Thus \( (UE)_{>k} \cdot x = 0 \) and so \( (UE) \cdot x \) is finite dimensional.

A non-zero element \( y \) of maximal degree in \( UE \cdot x \) satisfies
\[
[a, y] = (ad a)(y) = 0, \quad a \in E,
\]
i.e. \( y \in Z \) in contradiction to \( Z_{>t} = 0 \). This completes the proof of the Theorem. \( \square \)

### 3 Proof of Theorems 1 and 2

**Proof of Theorem 2:** We show that \( L_X \) satisfies the hypothesis of Theorem 3. Since depth \( L_X < \infty \) ([3]) and (A.2) holds by hypothesis we have only to construct a quasi-geometric growth sequence \( (r_i) \).

Let \( \alpha = \log \text{index} L_X \). Then \( \alpha > 0 \) by [5]. Choose a sequence
\[
u_1 < \nu_2 < \cdots
\]
such that \( (\dim (L_X)_{\nu_i})^{1/\nu_i} \to e^\alpha \).

Next, suppose \( \text{cat} X = m \) and put \( a \left( \frac{1}{2(m + 1)} \right)^{(m+1)} \). By starting the sequence at some \( \nu_j \) we may assume \( \dim (L_X)_{\nu_i} > \frac{1}{a} \), all \( i \). Thus the formula in ([5], top of page 189) gives a sequence
\[
u_i = \nu_0 < \nu_1 \cdots < \nu_k \nu_{i+1}
\]
such that \( \nu_{i+1} \leq 2(m + 1)\nu_i \) and
\[
\left( \dim (L_X)_{\nu_j} \right)^{1/\nu_j} \geq [a \dim (L_X)_{\nu_0}]^{1/\nu_0}, \quad j < k.
\]
Since \( \nu_0 = \nu_i \) and \( \nu_i \to \infty \) it follows that \( a \frac{1}{\nu_0 + 1} \to 1 \) as \( i \to \infty \). Hence interpolating the sequences \( \nu_i \) with the sequences \( \nu_j \) gives a quasi-geometric growth sequence \( (r_j) \). \( \square \)

**Proof of Theorem 1:** A theorem of Adams-Hilton [1] shows that
\[
UL_X = H_*(\Omega X; \mathbb{Q}) = H(TV,d)
\]
where $TV$ is the tensor algebra on $V$ and $V_i \cong H_{i+1}(X; \mathbb{Q})$. Thus $V$ is finite dimensional. Since $TV$ has a strictly positive radius of convergence so do $H(TV,d)$ and $L_X$:

$$\log \text{index } L_X < \infty.$$

Thus $X$ satisfies the hypotheses of Theorem 2. The fact that $d$ can be replaced by $n$ is proved by Lambrechts in [10].

□

References


Université Catholique de Louvain, 1348, Louvain-La-Neuve, Belgium
University of Maryland, College Park, MD 20742-3281, USA
Université d’Angers, 49045 Bd Lavoisier, Angers, France