Zipf's law in Multifragmentation

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Abstract

We discuss the meaning of Zipf's law in nuclear multifragmentation. We remark that Zipf's law is a consequence of a power law fragment size distribution with exponent $\tau \simeq 2$. We also recall why the presence of such distribution is not a reliable signal of a liquid-gas phase transition.

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The search for reliable signatures of the liquid-gas phase transition in nuclear multifragmentation is, both theoretically and experimentally, one of the major issues of this field of physics. The empirical observation that the size distribution of heavier clusters generated in various processes satisfies the so called Zipf's law [1], has raised interest and curiosity. This was first pointed out by Y.G. Ma [2] in the framework of the isospin dependent lattice gas model and more recently seen in nuclear fragmentation data [3, 4].

In the present context, Zipf's law [1] states that the mean size (mass or charge) $\bar{s}(r)$ of the largest, second-largest...r-largest clusters, decreases according to their rank $r = 1, 2, \cdots, n$ as

$$\bar{s}(r) \sim 1/r^{\lambda}, \tag{1}$$

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[1] In its original formulation, Zipf's law concerns the rank of the frequency of words in a text.

with $\lambda \simeq 1$.

The examination of the above mentioned numerical simulations [2] and experimental data [3, 4] , shows that a fairly good agreement with Eq. (1) is indeed obtained when the exponent is $\lambda \simeq 1$. This happens when other observables reach extreme values (maximum value of the moments of the cluster size distribution (*c.s.d.*), minimum of the effective τ parameter fit of the *c.s.d.*, maximum fluctuation of the largest fragment...). This seems to be the origin of the suggestion [2] that the fulfillment of the Zipf's law is a good signal of the liquid gas phase transition.

The aim of this note is to point out that the finding of a Zipf's law is nothing but a consequence of the power law shape of the *c.s.d.* with exponent $\tau \simeq 2$. More precisely, both exponents are connected through the formula $\lambda = 1/(\tau - 1)$.

The proof of this statement is straightforward. Let be s the size (mass or charge) of the clusters and $s(r)$ the size of the cluster of rank r. If the *c.s.d.* is a power low

$$Pr[s \in (s, s + ds)] \sim ds/s^{\tau}, \tag{2}$$

integrating over s, one gets the probability to find a cluster of size larger than S

$$Pr[s > S] \sim 1/S^{(\tau-1)}. \tag{3}$$

We now take $S = \bar{s}(r)$, where $\bar{s}(r)$ is the average size of clusters of rank r. This is a strictly decreasing real function of r, hence without degeneracy. On an infinite sampling, the event E, "one cluster randomly chosen has a size larger than $\bar{s}(r)$" is identical to the event "his rank is larger than r". Arranging in ascending order the ranks from 1 to n, the probability of E is

$$Pr(E) = Pr[s > \bar{s}(r)] = \frac{r-1}{n-1} \sim r. \tag{4}$$

On the other hand, Eq.(3) gives now

$$Pr(E) \sim 1/\bar{s}(r)^{\tau-1}.$$

Therefore

$$r \sim 1/\bar{s}(r)^{\tau-1},$$

and from Eq. (1),

$$\lambda = 1/(\tau - 1). \tag{5}$$

Similar proofs can be found in the literature (see for example ref. [5], where the same arguments are used to proof that if the ranking follows Eq. (1), then the *c.s.d.* is necessarily a power law).

The above formulae are strictly valid for infinite samplings. We have checked numerically that these remain accurate for finite samplings. We proceeded as follows. We generate partitions of an integer number N, with the condition that the mean *c.s.d.* is a power law of given exponent τ (i.e. each part of N is taken as a cluster size s). From these random numbers s, we construct the function $\bar{s}(r)$. As expected, for large N ($N > 1000$), the function $\bar{s}(r)$ is very close to a perfect power law in a large domain of r and Eq.(5) is satisfied within numerical uncertainties in the fits of the power laws. For $N \simeq 100$ and $2 \leq \tau \leq 3$, Eq.(5) is fulfilled within a few percent. In general, $\bar{s}(1)$ lies above the best fit curve. This is due to the finite size. Indeed, in the domain of s contributing to $\bar{s}(1)$, the *c.s.d.* deviates from a power law. For the same reasons, for larger values of τ the agreement is less good. Fig.1 shows an example of $\bar{s}(r)$ calculated from a power law *c.s.d.* with $\tau = 2.2$. A best fit with Eq. (1) gives $\lambda \simeq 0.86$.

It is interesting to see what happens if the *c.s.d.* is not a power law. For example, for an exponential distribution $\sim \exp(-\alpha s)$, following the same reasoning, one finds $\bar{s}(r) \sim -\log r$. Various theories of cluster formation [6, 8] offer other interesting examples. Close to their critical point, the *c.s.d.* behaves as

$$n(s) \sim s^{-\tau} f(s^{\sigma}\epsilon), \tag{6}$$

where f is a scaling function, ϵ the distance to the critical point and τ, σ two critical exponents. Right at the critical point, the scaling function $f(0) = 1$. In three dimensions $\tau \simeq 2.21$ for the lattice-gas model with clusters defined according to the Coniglio-Klein prescription [6, 7] and $\tau \simeq 2.18$ for percolation theory [8]. As expected, (approximate) Zipf's laws have been observed [2, 9] in the vicinity of the corresponding critical points.

We present below some results for a bond-percolation calculation [8] with $N = 100$ occupied sites. In Fig. (2) we show as a function of the distance to the critical bond parameter $\epsilon = p_c - p$, the evolution of λ (upper panel) and the χ^2 deviation of the $\bar{s}(r)$ with respect to two power law fits (lower panel). The continuous line (squares) is the best fit of $\bar{s}(r)$ with two parameters c/r^{λ}. The minimum of χ^2 occurs when $\lambda \simeq 0.90$ for a slope $\tau \simeq 2.16$ of the corresponding *c.s.d.*, in agreement with Eq.(5) within 4%. The dashed-dotted line (crosses) corresponds to the fit with fixed $\lambda = 1$ (i.e. a strict

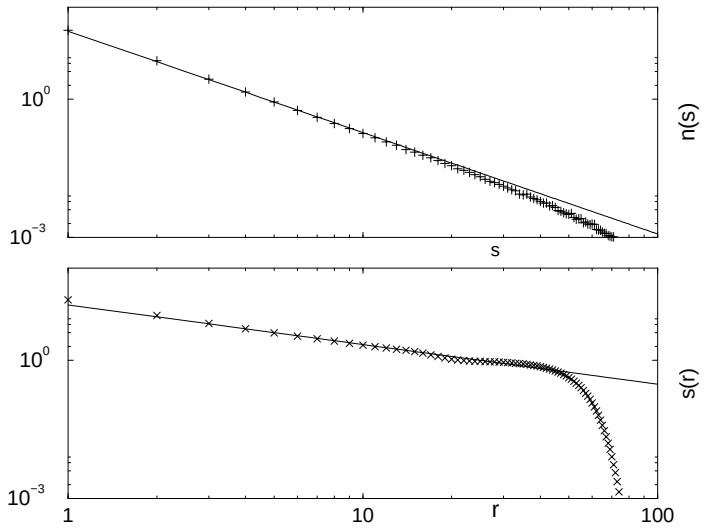

Figure 1: *Cluster size distribution $n(s)$ (upper panel) and mean size $\bar{s}(r)$ distribution of clusters of rank r (lower panel), corresponding to a power law* c.s.d. *with $\tau = 2.2$. The size s of the clusters is generated from random partitions of N=100 with the constraint that the mean* c.s.d. *is a power law with exponent $\tau = 2.2$ (see text). The straight lines are best fits, with slopes $\tau = 2.2$ and $\lambda = 0.86$, respectively.*

Zipf's law, as done in ref. [2]). This fit, that violates Eq.(5) by 26%, giving an incorrect localization of the critical point, shows as expected a larger χ^2. Similarly, for the lattice-gas model with Coniglio-Klein clusters one expects, according to Eq.(1), $\lambda \simeq 0.83$ [2].

Before to summarize, we would like to add a comment on the observation of power law *c.s.d.* and Zipf's laws. As mentioned before, various theories of cluster formation (Fischer droplets [10], lattice-gas [6], Lennard-Jones fluids with Hill's clusters [11], percolation [8]), predict power laws *c.s.d.* with $\tau \simeq 2$ at the corresponding critical points. However, such *c.s.d.* also appear elsewhere (for example, in the super-critical region of the lattice-gas and realistic Lennard-Jones fluids).

In summary, (approximate) Zipf's law is just a mathematical consequence of a power law *c.s.d.* with exponent $\tau \simeq 2$. Such distributions appear at the critical point of many theories, but also elsewhere. In consequence, we conclude that the observation of a Zipf's law is neither a new and independent

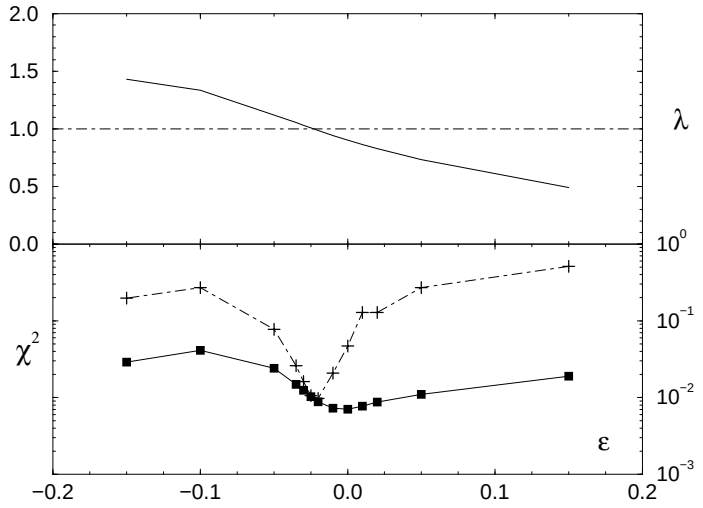

Figure 2: *The slope parameter λ (Eq. (1))(upper panel) and the χ^2 deviation from a power law of $\bar{s}(r)$ (lower panel), as a function of the distance ϵ to the percolation critical point (see text). The continuous line (squares) corresponds to the best fit with λ free and the dashed-dot line (crosses) with $\lambda = 1$ [2].*

signal of a critical behavior, nor an unambiguous signal of a thermodynamical phase transition.

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