Intersection Graphs of Jordan Arcs

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Abstract. A family of Jordan arcs, such that two arcs are nowhere tangent, defines a hypergraph whose vertices are the arcs and whose edges are the intersection points. We shall say that the hypergraph has a *strong intersection representation* and, if each intersection point is interior to at most one arc, we shall say that the hypergraph has a *strong contact representation*.

We first characterize those hypergraphs which have a strong contact representation and deduce some sufficient conditions for a simple planar graph to have a strong intersection representation.

Then, using the Four Color Theorem, we prove that a large class of simple planar graphs have a strong intersection representation.

1. Introduction

A family of Jordan arcs, such that two arcs are nowhere tangent, defines a hypergraph whose vertices are the arcs and whose edges are the intersection points. We shall say that the hypergraph has a *strong intersection representation* and, if each intersection point is interior to at most one arc, we shall say that the hypergraph has a *strong contact representation*.

Such a family classically defines a *string graph*, which is a simple graph, whose vertices are the arcs, two vertices being adjacent if the corresponding arcs intersect at least once [**3**]. It is not known whether deciding whether a graph is a string graph is algorithmically decidable [**11**]. However, all planar graphs are obviously string graphs.

If two arcs may only intersect once, we shall say that the intersection graph has an *intersection representation*. Deciding whether a given graph has an intersection representation is known to belong to the NP-complete class [**9**], even if strong restrictions are requested on the family of arcs [**10**]. It has been a challenge for long to prove or disprove that all planar graphs have an intersection representation. It is clear that a simple graph has an intersection representation if and only if it has a strong intersection representation.

First, we shall introduce a general framework on strong contact representation.

Then, the introduction of a 4-coloration will lead to prove that a large class of planar graphs have an intersection representation.

Most of the results proved here were first presented in [**5**] and [**7**].

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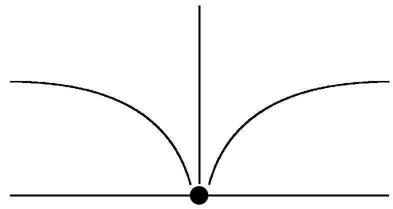

FIGURE 1. Locally one-sided contact

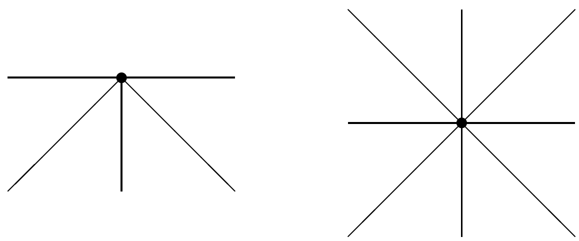

FIGURE 2. A contact and an intersection representations of K_4

General Framework

Arc intersection problems arise topological difficulties that vanish when arcs are only in contact, as the arc-point incidence then defines a plane graph (i.e. a planar graph together with an embedding).

A *contact family* $\mathcal{F} = (\mathcal{V}, \mathcal{P})$ of Jordan arcs is a finite family $\mathcal{V}$ of Jordan arcs in the plane and a finite point set $\mathcal{P}$ on these arcs, such that two arcs share at most one point (called *contact point*), and such that all the contact points belong to $\mathcal{P}$ and are interior to at most one arc. Furthermore, at each point p interior to an arc ν, the arc ν appears twice consecutively in a clockwise traversal around p (i.e. we consider only locally one-sided contacts). This family defines a particular type of intersection graph, called *contact graph* (which may be not planar) and also a colored planar bipartite *arc-point incidence graph* $\text{Incid}(\mathcal{F})$, whose vertex set is the union of the arc set $\mathcal{V}$ (colored white) and the contact point set $\mathcal{P}$ (colored black), and whose incidence is the inclusion relation of a point in an arc.

2. Graphs, Hypergraphs and Frames

2.1. Graphs. Given a graph G, we use the following notations:

- $V(G)$ is the vertex set of G,
- $E(G)$ is the edge set of G,
- $N(x)$ is the neighbor set of the vertex x,
- $N(A) = \bigcup_{x \in A} N(x)$ is the neighbor set of a subset A of vertices,
- $d_G(x)$ is the degree of the vertex x,
- G_A is the subgraph induced by the subset A of vertices,
- $T(G)$ is the set of bounded triangular faces of G, if G is a plane graph (i.e. a planar graph embedded in the plane).

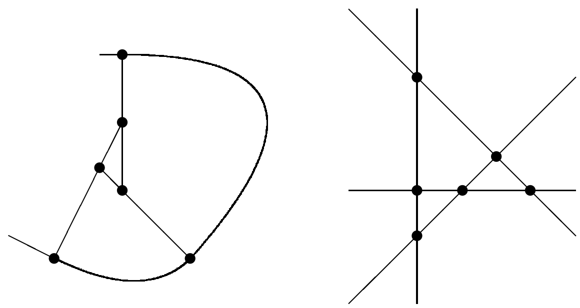

FIGURE 3. A strong contact and a strong intersection representations of K_4

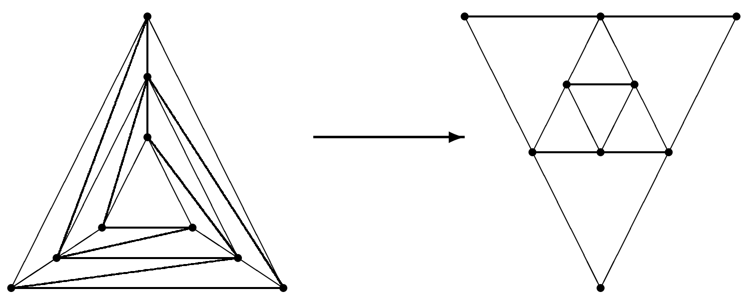

FIGURE 4. A planar graph, which has a contact representation, but no strong contact representation

DEFINITION 2.1. A graph G has a *strong contact representation* if there exists a contact family $\mathcal{F}$ such that the arc-point incidence of $\mathcal{F}$ is the vertex-edge incidence of G

REMARK 2.2. In a strong contact representation, the points represent the edges and thus belong to exactly 2 arcs, unlike the contact points of a contact representation which represent cliques.

2.2. Hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (X, \mathcal{E})$, where X is a finite set and $\mathcal{E}$ is a family $(E_i, i \in I)$ of subsets of X, such that: $E_i \neq \emptyset (\forall i \in I)$ and $\bigcup_{i \in I} E_i = X$. The elements of X and $\mathcal{E}$ are respectively the *vertices* and the *edges* of the hypergraph. Two vertices $x, y \in X$ are *adjacent* if they both belong to some edge of $\mathcal{H}$; two edges E_i, E_j are *adjacent* if their intersection is not empty. A vertex $x \in X$ is *incident* to an edge $E_i \in \mathcal{E}$ if x belongs to E_i (see [**1**]).

A hypergraph $\mathcal{H}$ is *linear* if any two edges have at most one common element:

$$\forall i \neq j, \quad \big|E_i \cap E_j\big| \leq 1$$

The *sub-hypergraph* of $\mathcal{H}$ *induced* by a subset $Y \subseteq X$ is the hypergraph $\mathcal{H}_Y = (Y, \mathcal{E}_Y)$, where

$$\mathcal{E}_Y = \{E_i \cap Y, \quad E_i \in \mathcal{E}; E_i \cap Y \neq \emptyset\}$$

DEFINITION 2.3. The *equivalent edge number* of $\mathcal{H}$ is the sum

$$\mu(\mathcal{H}) = \sum_{i \in I} \Big(\big|E_i\big| - 1\Big)$$

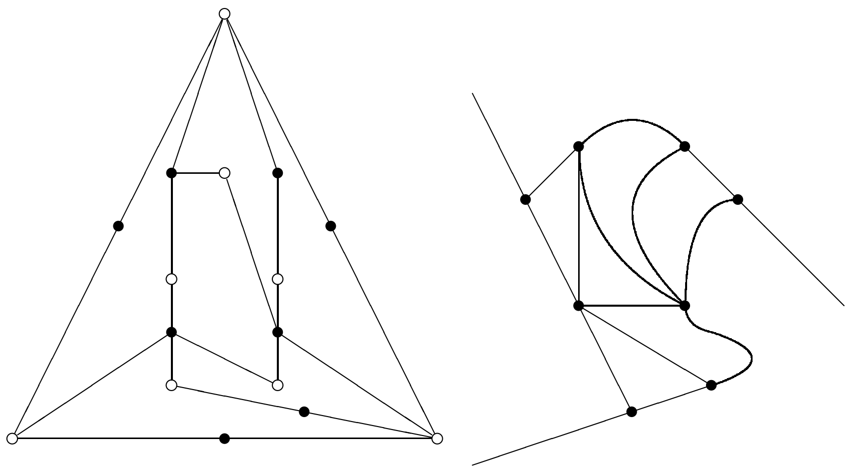

FIGURE 5. The incidence graph $\mathrm{Incid}(\mathcal{H})$ of a planar hypergraph $\mathcal{H}$ and a strong contact representation of $\mathcal{H}$

DEFINITION 2.4. The *incidence graph* $\mathrm{Incid}(\mathcal{H})$ of $\mathcal{H}$ is the colored bipartite graph on $(X, \mathcal{E})$ defined by the vertex-edge incidence, a vertex being colored white (resp. black) if it belongs to X (resp. $\mathcal{E}$).

As a special case, if G is a graph, $\mathrm{Incid}(G)$ is the bicolored vertex-edge incidence graph of G.

DEFINITION 2.5. A hypergraph $\mathcal{H}$ is *planar* if $\mathrm{Incid}(\mathcal{H})$ is planar.

The definition of strong contact representation of graphs extends naturally to hypergraphs:

DEFINITION 2.6. A hypergraph $\mathcal{H}$ has a *strong contact representation* if there exists a contact family $\mathcal{F}$, whose arc-point incidence is the vertex-edge incidence of $\mathcal{H}$, so that arcs represents vertices and points represent edges (see Fig. 5).

2.3. Frames.

DEFINITION 2.7. A *frame* is a pair $\Phi = (\Gamma, \Pi)$, where Γ is a plane graph and $\Pi = (F_1, \dots, F_p)$ is a partition of the edge set $E(\Gamma)$, such that each class is either a bounded 2-connected face of Γ or a single edge.

A vertex $x \in V(\Gamma)$ is incident to a class $F_i \in \Pi$ if it is incident to at least one edge in F_i.

A frame Φ defines a hypergraph $\mathcal{H}_\Phi = (V(\Gamma), \mathcal{E})$, where E_i is the set of the vertices incident to the class F_i of Π. We shall say that Φ is a *frame representation* of $\mathcal{H}_\Phi$ (see Fig. 6).

PROPOSITION 2.1. *A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is planar if and only if it has a frame representation.*

PROOF. The bijection between the embedded incidence graph of a planar hypergraph and frame representations is shown on Fig. 7:

- From the incidence graph of a planar hypergraph $\mathcal{H}$, each black vertex (corresponding to some E_i) is split into a C_k (if its degree is $k > 2$), a K_2 (if its degree is 2), or a loop (if its degree is 1), which edges forms a set F_i. Then, the black-white edges are contracted. The resulting graph together with the partition $(F_1, \dots, F_m)$ is a frame representation of $\mathcal{H}$.

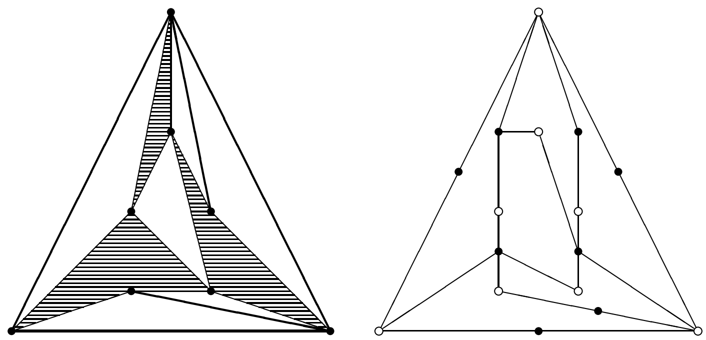

FIGURE 6. A frame representation of a planar hypergraph $\mathcal{H}$ and the incidence graph of $\mathcal{H}$ (the edges of $\mathcal{H}$ are the dashed faces and the thick line segments of the frame)

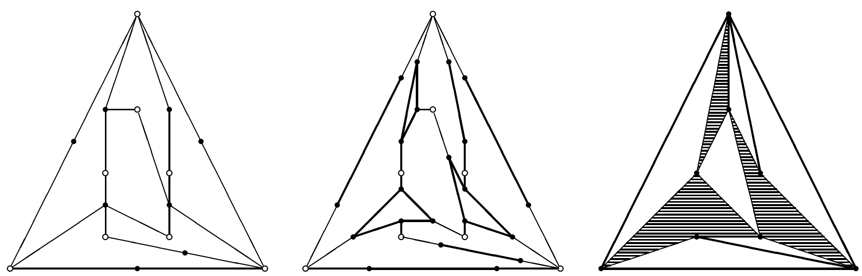

FIGURE 7. Bijection between incidence graphs of planar hypergraphs and frames

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- From a frame $\Phi = (\Gamma, \Pi)$, we add a vertex in each face in Π (including the faces defined by a loop) and join it to the vertices of the face, we bisect the non-loop edges that forms a single element class in Π and then delete all the original edges of Γ that have not been bisected.

□

REMARK 2.8. According to the definition of a linear hypergraph, the following conditions are equivalent:

- $\mathcal{H}$ is a linear planar hypergraph,
- $\mathcal{H}$ is a hypergraph which has a C_4-free frame representation,
- $\mathcal{H}$ is a planar hypergraph and all the frame representations of $\mathcal{H}$ are C_4-free.

3. Strong Contact Representation

3.1. Strong contact representation of graphs.

THEOREM 3.1. *A graph G has a strong contact representation if and only if it is planar and satisfies:*

$$\forall A \subseteq V, \quad \left|E(G_A)\right| \leq 2\left|A\right| \tag{3.1}$$

PROOF. Equation (3.1) is equivalent to the existence of a 2-orientation of G, that is an orientation of G such that each vertex has at most 2 incoming edges.

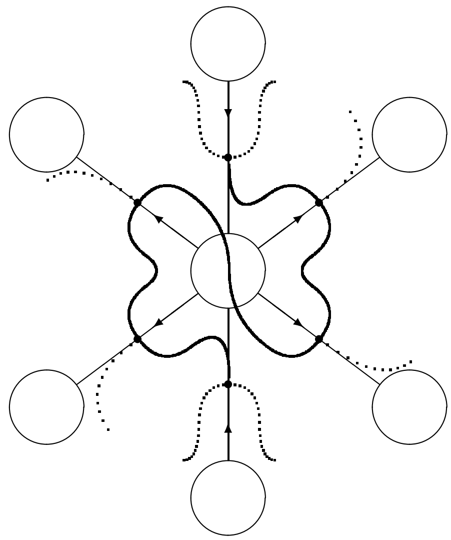

FIGURE 8. A contact representation induced by a 2-orientation

- A strong contact representation of a graph G induces a planar embedding of G and a 2-orientation of it: an edge incident to a vertex v is oriented from v if it corresponds to an internal point of the arc ν representing v; edges corresponding to point which are internal to no arc are oriented arbitrarly.
- Conversely, an embedding and a 2-orientation of G defines a strong contact representation of G (see Fig. 8).

□

REMARK 3.2. This theorem implies that outer-planar graphs and triangle free planar graphs have a strong contact representation. Actually, bipartite planar graphs have a strong contact representation using segments in two directions [**2**].

3.2. Strong contact representation of hypergraphs. We may now generalize this result to hypergraphs. That for, we need first to prove an orientation lemma:

LEMMA 3.3. *The incidence graph* Incid($\mathcal{H}$) *of an hypergraph* $\mathcal{H}$ *has a 2-1 orientation, that is an orientation such that each white (resp. black) vertex has at most 2 (resp. 1) incoming edges, if and only if it satisfies*

$$\forall A \subseteq X, \quad \mu(\mathcal{H}_A) \leq 2|A| \tag{3.2}$$

PROOF. By classical arguments, Incid($\mathcal{H}$) has a 2-1 orientation if and only if it satisfies:

$$\forall Y \subseteq X \cup \mathcal{E}, \quad |E(\mathrm{Incid}(\mathcal{H})_Y)| \leq 2|Y \cap X| + |Y \cap \mathcal{E}| \tag{3.3}$$

That is:

$$\forall A \subseteq X, \forall \mathcal{E}' \subseteq \mathcal{E}, \quad |E(\mathrm{Incid}(\mathcal{H})_{A \cup \mathcal{E}'})| \leq 2|A| + |\mathcal{E}'| \tag{3.4}$$

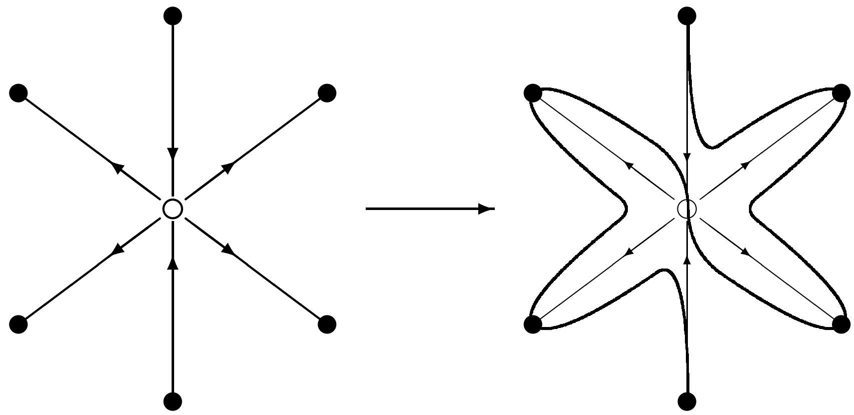

FIGURE 9. Representation of the 2-oriented planar hypergraph from a 2-1 orientation of its incidence graph

It is obviously sufficient to consider pairs $(A, \mathcal{E}')$ where $\mathcal{E}'$ is the neighbor set $N(A)$ of A. Hence, (3.4) may be rewritten:

$$\forall A \subseteq X, \quad \big|E(\mathrm{Incid}(\mathcal{H})_{A\cup N(A)})\big| \leq 2\big|A\big| + \big|N(A)\big| \tag{3.5}$$

Finally,

$$\big|E(\mathrm{Incid}(\mathcal{H})_{A\cup N(A)})\big| - \big|N(A)\big| = \sum_{E\in N(A)} d_{\mathrm{Incid}(\mathcal{H})_{A\cup N(A)}}(E) - \big|N(A)\big| \tag{3.6}$$

$$= \sum_{E\in N(A)} \Big(\big|E\cap A\big| - 1\Big) \tag{3.7}$$

$$= \mu(\mathcal{H}_A) \tag{3.8}$$

Hence, the graph $\mathrm{Incid}(\mathcal{H})$ has a 2-1 orientation if and only if (3.2) holds. □

THEOREM 3.4. *A hypergraph $\mathcal{H}$ has a strong contact representation if and only if $\mathcal{H}$ is planar and satisfies*

$$\forall A \subseteq X, \quad \mu(\mathcal{H}_A) \leq 2\big|A\big| \tag{3.9}$$

PROOF.
- A strong contact representation of a hypergraph $\mathcal{H}$ induces a planar embedding and a 2-1 orientation of $\mathrm{Incid}(\mathcal{H})$: an edge $\{v, E\}$ is oriented from v to E if the point representing E is interior to the arc representing v.
- Conversely, if $\mathcal{H}$ is a planar hypergraph, an embedding and a 2-1 orientation of $\mathrm{Incid}(\mathcal{H})$ defines a strong contact representation of $\mathcal{H}$ (see Fig. 9).

The theorem now follows from Lemma 3.3. □

3.3. Strong contact representation of frame-represented hypergraphs. We introduce some definitions and notations related to triangulation:

Let $\mathcal{H}$ be a planar hypergraph and $\Phi = (\Gamma, \Pi)$ be a frame representation of $\mathcal{H}$.

- Let C be a polygon, then $\mathrm{Triang}(C)$ is the set of the maximal outer-planar graphs which may be obtained by triangulating the inside of C.
- Let E be an edge of $\mathcal{H}$. Then $\mathrm{Triang}(E) = \{E\}$ if E as cardinality at most two and $\mathrm{Triang}(E) = \mathrm{Triang}(C)$ if C is the polygon of Γ induced by the edges of the class of Π corresponding to the edge E of $\mathcal{H}$.
- $\mathrm{Triang}(\Phi)$ is the set of the plane multigraphs which may be obtained from Γ by triangulating the faces belonging to Π.

- For any plane graph $G \in \mathrm{Triang}(\Phi)$, $\mathrm{T}_\Phi(G)$ denotes the union of the set of all the triangular faces of G which are not faces of Γ and the set of all the triangular faces of Γ which belong to Π.

  For any induced plane subgraph H of a plane graph $G \in \mathrm{Triang}(\Phi)$, $\mathrm{T}_\Phi(H)$ is the set of all the triangular faces of H which belong to $\mathrm{T}_\Phi(G)$. Notice that this set does not depend on the choice of G in $\mathrm{Triang}(\Phi)$.

LEMMA 3.5. *$\mathcal{H}$ is a linear hypergraph if and only if all the graphs in* $\mathrm{Triang}(\Phi)$ *are simple.*

PROOF.
- If $\mathcal{H}$ is not linear, there exists two edges E, E' of $\mathcal{H}$ having at least two vertices x and y in common. Hence, there exists a multigraph in $\mathrm{Triang}(\Phi)$ such that x an y are linked by an edge in $\mathrm{Triang}(E)$ and by an edge induced in $\mathrm{Triang}(E')$.
- If $\mathcal{H}$ is linear, two vertices may simultaneously belong to at most one edge of $\mathcal{H}$ and hence are linked by at most one edge in a graph belonging to $\mathrm{Triang}(\Phi)$.

□

LEMMA 3.6. *Let C be a polygon. Then, for any subset $W \subseteq V(C)$, we have:*

$$\max_{G\in \mathrm{Triang}(C)} \left|E(G_W)\right| - \left|T(G_W)\right| = \begin{cases} \left|W\right| - 1 & \text{if } W \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

PROOF. The number $\left|T(G_W)\right|$ of bounded triangular faces of G_W is one less than the number of faces of G_W. Hence, $\left|E(G_W)\right| - \left|T(G_W)\right|$ is equal to $\left|W\right| - c(G_W)$, where $c(G_W)$ is the number of connected components of G_W. If W is nonempty, this number is at most $\left|W\right| - 1$, and this value is achieved by any triangulation G of C, such that G_W is connected. □

LEMMA 3.7. *Let $\Phi = (\Gamma, \Pi)$ be a frame representation of a hypergraph $\mathcal{H}$. Let $W \subseteq V$ be a subset of the vertex set of Γ. Then,*

$$\mu(\mathcal{H}_W) = \max_{G\in \mathrm{Triang}(\Phi)} \Big(\left|E(G_W)\right| - \left|\mathrm{T}_\Phi(G_W)\right|\Big) \tag{3.11}$$

PROOF. Let Γ_i be the partial graph of Γ induced by (edges of) $F_i \in \Pi$. Let W_i be the subset of W of the vertices incident to F_i. Any graph $G \in \mathrm{Triang}(\Phi)$ is uniquely defined by the list of the graphs $G^{(i)} \in \mathrm{Triang}(\Gamma_i)$.

The number $\left|E(G_W)\right|$ of edges of the subgraph of G induced by W is the sum of the numbers $\left|E(G^{(i)}_{W_i})\right|$ of edges of the subgraphs of the $G^{(i)}$ induced by W_i.

Similarly, the number $\left|\mathrm{T}_\Phi(G_W)\right|$ is the sum of the numbers $\left|T(G^{(i)}_{W_i})\right|$ of triangles of the subgraphs of the $G^{(i)}$ induced by W_i, as a triangle belongs to $\mathrm{T}_\Phi(G_W)$ if and only if its vertices belongs to the same Γ_i.

Hence,

$$\mu(\mathcal{H}_W) = \sum_{i, W_i \neq \emptyset} \Big(\left|W_i\right| - 1\Big) \tag{3.12}$$

$$= \sum_i \left(\max_{H\in \mathrm{Triang}(\Gamma_i)} \Big(\left|E(H_{W_i})\right| - \left|T(H_{W_i})\right|\Big)\right) \tag{3.13}$$

$$= \max\left\{ \sum_i \Big(\left|E(G^{(i)}_{W_i})\right| - \left|T(G^{(i)}_{W_i})\right|\Big), \quad \forall j, G^{(j)} \in \mathrm{Triang}(\Gamma_j) \right\} \tag{3.14}$$

$$= \max_{G\in \mathrm{Triang}(\Phi)} \Big(\left|E(G_W)\right| - \left|\mathrm{T}_\Phi(G_W)\right|\Big) \tag{3.15}$$

□

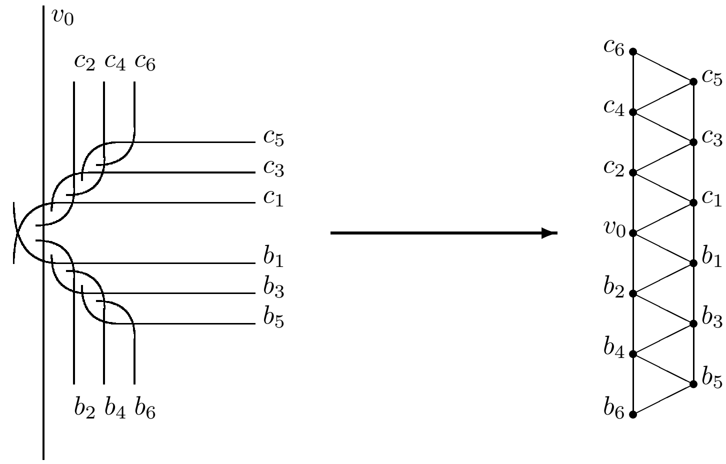

FIGURE 10. The local deformation of a contact point representing a zig-zag graph

THEOREM 3.8. *Let $\Phi = (\Gamma, \Pi)$ be a frame representation of a plane linear hypergraph $\mathcal{H}$. Then, $\mathcal{H}$ has a strong contact representation if and only if the following inequality holds:*

$$\max_{G \in \mathrm{Triang}(\Phi), W \subseteq V(\Gamma)} \left|E(G_W)\right| - \left|\mathrm{T}_\Phi(G_W)\right| - 2\left|W\right| \leq 0 \tag{3.16}$$

PROOF. This is a direct consequence of Theorem 3.4 and Lemma 3.7. □

4. Local Intersections

Let H be a maximal outer-planar graph and let $\mathcal{F}$ be a family of $\left|V(H)\right|$ arcs which share a single contact point p.

The graph H is a *local intersection graph* if, for any vertex v of H, there exists a local deformation of $\mathcal{F}$ at p which represents H, and such that

- the arc to which p is interior represents v,
- the circular order of the arcs is the same as the circular order of the vertices they represent.

A *zig-zag graph* is a maximal outer-planar graph which interior edges defines a path.

LEMMA 4.1. *A zig-zag graph is a local intersection graph.*

PROOF. The construction of a local deformation of a contact point into a zig-zag graph is shown on Fig. 10. □

REMARK 4.2. One can prove the reverse, that is: a maximal outer-planar graph is a local intersection graph if and only if it is a zig-zag graph.

Let $\mathcal{Z}(\Phi)$ denotes the set of all possible graphs obtained from Γ by triangulating the faces in Π using paths, and let $K(\Phi)$ be the graph obtained from Γ by replacing each face in Π by a clique.

THEOREM 4.3. *Let $\Phi = (\Gamma, \Pi)$ be a frame.*
If the following condition holds:

$$\max_{G \in \mathrm{Triang}(\Phi), W \subseteq V(\Gamma)} \left|E(G_W)\right| - \left|\mathrm{T}_\Phi(G_W)\right| - 2\left|W\right| \leq 0 \tag{4.1}$$

then:

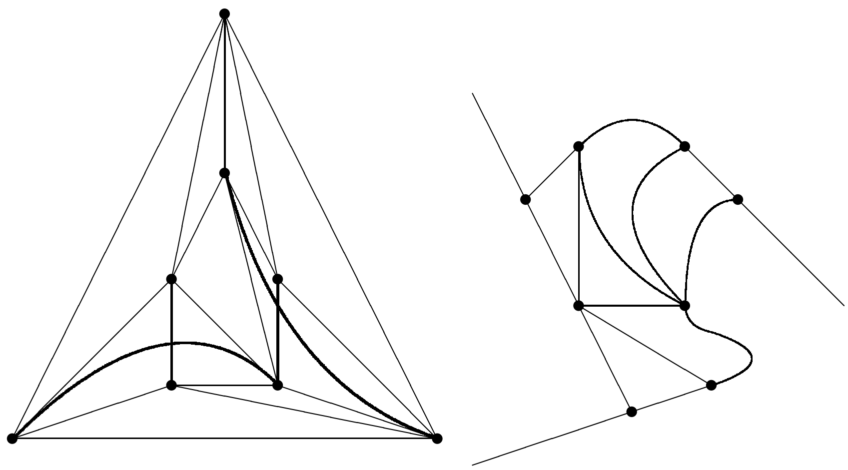

FIGURE 11. A contact representation of $K(\Phi)$

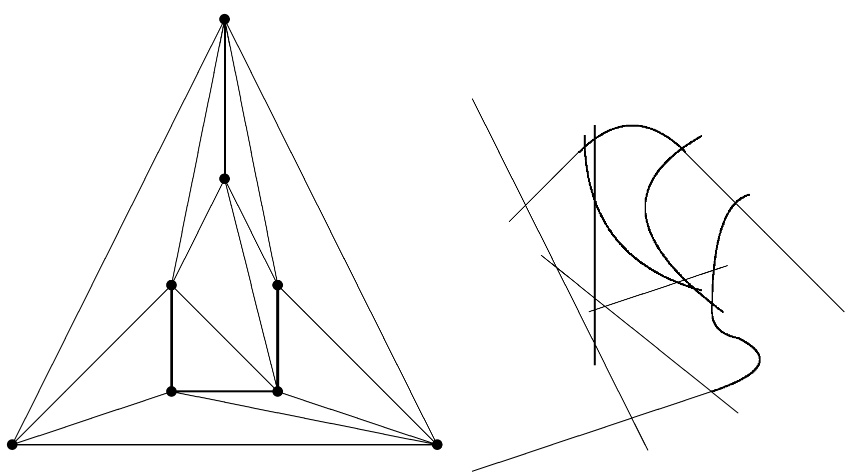

FIGURE 12. A strong intersection representation of a graph $G \in \mathcal{Z}(\Phi)$

- *$K(\Phi)$ has a contact representation,*
- *any graph in $\mathcal{Z}(\Phi)$ has an intersection representation.*

PROOF. The conclusion follows from the previous lemma and Theorem 3.8. □

COROLLARY 4.3.1. *Let $\Gamma(V, E)$ be a plane graph and let $\mathcal{I}$ be an independent (i.e. edge-disjoint) set of triangular faces (defining $\Pi = \mathcal{I} \cup \big\{\{e\}, \quad e \in E \setminus (\bigcup_{T \in \mathcal{I}} T)\big\}$). If the following condition holds:*

$$\max_{W \subseteq V} \big|E(\Gamma_W)\big| - \big|T(\Gamma_W) \cap \mathcal{I}\big| - 2\big|W\big| \leq 0 \tag{4.2}$$

then Γ has a contact representation.

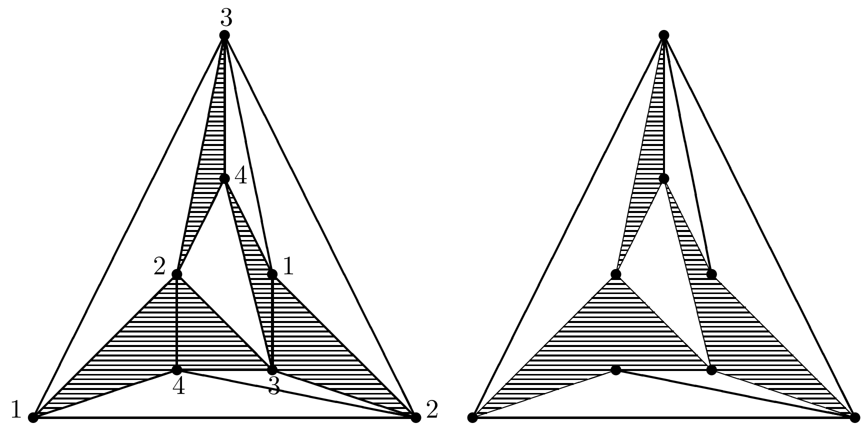

FIGURE 13. The 4-colored plane graph G (the dashed triangles are the members of $\mathrm{T}^+(G)$) and the associated frame

REPRESENTATION OF PLANAR GRAPHS

5. The Coloration Method

Let G be a 2-connected 4-colored plane graph. Let $\mathrm{T}^+(G)$ be the set of all the bounded triangular faces of G in which two consecutive colors appear clockwise.

First notice that a triangle belonging to $\mathrm{T}^+(G)$ is adjacent to at most one other triangle belonging to $\mathrm{T}^+(G)$. More precisely, two bounded adjacent triangles both belong to $\mathrm{T}^+(G)$ if and only if the vertices of their union are colored $(1, 2, 3, 4)$ in clockwise order.

A frame $\Phi = (\Gamma, \Pi)$ is obtained from G and $\mathrm{T}^+(G)$:

- The graph Γ is the graph obtained from G by erasing the edges belonging to two triangles belonging to $\mathrm{T}^+(G)$.
- The partition Π has the following classes:
  - $\{e\}$, if the edge e belongs to no triangle in $\mathrm{T}^+(G)$,
  - T, if the triangle T belongs to $\mathrm{T}^+(G)$ and no triangle adjacent to it belongs to $\mathrm{T}^+(G)$,
  - $(T_1 \cup T_2) \setminus (T_1 \cap T_2)$, if the triangles T_1 and T_2 are adjacent and both belong to $\mathrm{T}^+(G)$.

LEMMA 5.1. *If G has no induced C_4 colored $(1, 2, 3, 4)$ and no separating C_3 with two consecutive colors appearing clockwise, then all the graphs in* Triang(Φ) *have the same property*

PROOF. All the graphs in Triang(Φ) may be obtained from G by a sequence of edge-switchings performed on edges not in $E(\Gamma)$.

If such a switching creates a C_3 separator with two consecutive colors appearing clockwise (say v_3, v_1, v_2, where the edge (v_3, v_1) comes from the switching of an edge (x_2, x_4)), then the original graph has (v_3, x_4, v_1, v_2) as an induced C_4.

Similarly a C_3 separator with two consecutive colors appearing clockwise may only occur if an induced C_4 colored $(1, 2, 3, 4)$ exists before the edge-switching. $\square$

LEMMA 5.2. *For any $G \in$* Triang(Φ)*, and for any induced subgraph $H \subseteq G$, we have:*

$$\mathrm{T}^+(H) = \mathrm{T}_\Phi(H) \tag{5.1}$$

PROOF. This is a direct consequence of the fact that there exists no C_3 separator with two consecutive colors appearing clockwise in G. □

LEMMA 5.3. *If G has no induced C_4 colored $(1,2,3,4)$ in the clockwise order, we have:*

$$\big|E(G)\big| - \big|\mathrm{T}^+(G)\big| - 2\big|V(G)\big| \leq -3 \tag{5.2}$$

PROOF. Let us prove the lemma by induction on the size sequence of the faces.

A graph is minimal if each face has length 3 and hence is a maximal planar graph. Then, any triangle of G includes exactly one edge colored $(1,2)$ or $(3,4)$ and such an edge belongs to two adjacent triangles. This matching shows that $\mathrm{T}^+(G)$ includes either the half of the triangles of G or the half the triangles of G number minus 1 (depending on the coloration of the outer face which never belongs to $T^+(G)$). Hence:

$$\begin{aligned}\big|E(G)\big| - \big|\mathrm{T}^+(G)\big| - 2\big|V(G)\big| &\leq \Big(3\big|V(G)\big| - 6\Big) - \Big(\frac{1}{2}\Big(2\big|V(G)\big| - 4\Big) + 1\Big) - 2\big|V(G)\big| \\ &\leq -3\end{aligned}$$

Otherwise, we shall construct a 4-colored 2-connected plane graph G' with no face of length 4 colored $(1,2,3,4)$ in the clockwise order that satisfies (together with the associated subset $\mathrm{T}^+(G')$ of its face set):

$$\big|E(G')\big| - \big|\mathrm{T}^+(G')\big| - 2\big|V(G')\big| \geq \big|E(G)\big| - \big|\mathrm{T}^+(G)\big| - 2\big|V(G)\big| \tag{5.3}$$

Let F be a non-triangular face of G of maximal size. We shall consider several cases:

- There exists a color c, such that 4 vertices of F at least are not colored c.

  Then G' is obtained from G by adding a vertex x in F colored c, which is linked to all the $k \geq 4$ vertices of F, which are not colored c. We get:

$$\big|V(G')\big| = \big|V(G)\big| + 1 \tag{5.4}$$

$$\big|E(G')\big| = \big|E(G)\big| + k \tag{5.5}$$

$$\big|\mathrm{T}^+(G')\big| \leq \big|T(G)\big| + \left\lfloor\frac{2k}{3}\right\rfloor \tag{5.6}$$

  Thus,

$$\begin{aligned}\big|E(G')\big| - \big|\mathrm{T}^+(G')\big| - 2\big|V(G')\big| &\geq \left\lceil\frac{k}{3}\right\rceil - 2 + \big|E(G)\big| - \big|\mathrm{T}^+(G)\big| - 2\big|V(G)\big| \\ &\geq \big|E(G)\big| - \big|\mathrm{T}^+(G)\big| - 2\big|V(G)\big|\end{aligned}$$

  Remark that no face of length 4 colored $(1,2,3,4)$ in clockwise order may be created that way.

- Otherwise, no color is present twice on F. If $V_i(F)$ denotes the set of the i-colored vertices of F and l denotes the length of F, then:

$$4 \leq l = \Big|\bigcup_i V_i(F)\Big| = \frac{1}{3}\sum_i \Big|\bigcup_{j\neq i} V_j(F)\Big| \leq 4$$

  Hence, F has length 4.

  We triangulate F by adding an edge and, as F is not colored $(1,2,3,4)$ in clockwise order, at most one additional triangle is added to $\mathrm{T}^+(G')$. Hence,

$$\big|E(G')\big| - \big|\mathrm{T}^+(G')\big| - 2\big|V(G')\big| \geq \big|E(G)\big| - \big|\mathrm{T}^+(G)\big| - 2\big|V(G)\big| \tag{5.7}$$

□

LEMMA 5.4. *If G has no induced C_4 colored $(1,2,3,4)$ in the clockwise order and no separating C_3 with two consecutive colors appearing clockwise, we have:*

$$\max_{W \subseteq V(G)} \left|E(G_W)\right| - \left|\mathrm{T}^+(G_W)\right| - 2\left|V(G_W)\right| \leq -3 \tag{5.8}$$

PROOF. Let us first prove that the inequality

$$\left|E(G_W)\right| - \left|\mathrm{T}^+(G_W)\right| - 2\left|V(G_W)\right| \leq -3 \tag{5.9}$$

holds for any induced 2-connected subgraph $H \subseteq G$ in place of G_W.

Actually, let $\mathrm{T}^+(H)$ be the set of all the bounded triangles of H in which two consecutive colors appear clockwise. As G has no separating C_3 with two consecutive colors appearing clockwise, $\mathrm{T}^+(H) = T(H) \cap \mathrm{T}^+(G)$. Moreover, H has no face of length 4 colored $(1,2,3,4)$ in clockwise order as it would be an induced 4-cycle of G.

Hence, by the preceding lemma, any 2-connected induced subgraph H of G satisfies:

$$\left|E(H)\right| - \left|\mathrm{T}^+(H)\right| - 2\left|V(H)\right| \leq -3 \tag{5.10}$$

¿From this inequality, we deduce that the same holds for any induced subgraph H of G (remark that the weaker inequality $\left|E(H)\right| - \left|\mathrm{T}^+(H)\right| - 2\left|V(H)\right| \leq 0$ could not be extended to not 2-connected graphs). □

THEOREM 5.5. *Let G be a 4-colored 2-connected plane graph with no induced 4-cycle colored $(1,2,3,4)$ in the clockwise order and no separating C_3 with two consecutive colors appearing clockwise.*

Then, G has an intersection representation.

PROOF. ¿From the previous lemmas, we get

$$\max_{G \in \mathrm{Triang}(\Phi), W \subseteq V(\Gamma)} \left|E(G_W)\right| - \left|\mathrm{T}_\Phi(G_W)\right| - 2\left|W\right| \leq -3 \tag{5.11}$$

and, as G belongs to $\mathcal{Z}(\Phi) = \mathrm{Triang}(\Phi)$, the results follows. □

COROLLARY 5.5.1. *Let G be an internally 5-connected planar graph. Then, G has an intersection representation.*

THEOREM 5.6. *Let G be a 3-colored 2-connected plane graph with no separating C_3 having two consecutive colors appearing clockwise,*

Then, G has a contact representation.

PROOF. As previously, we get

$$\max_{G \in \mathrm{Triang}(\Phi), W \subseteq V(\Gamma)} \left|E(G_W)\right| - \left|\mathrm{T}_\Phi(G_W)\right| - 2\left|W\right| \leq -3 \tag{5.12}$$

and, as G is equal to $K(\Phi)$, the results follows. □

COROLLARY 5.6.1. *Let G be a 4-connected 3-colorable planar graph. Then, G has a contact representation.*

6. Cut and Paste

By cutting C_3-separated components and pasting back their representation we obtain the following extension of Theorem 5.5:

THEOREM 6.1. *Let G be a 4-colored plane graph with no induced 4-cycle colored $(1,2,3,4)$ in the clockwise order.*

Then, G has an intersection representation.

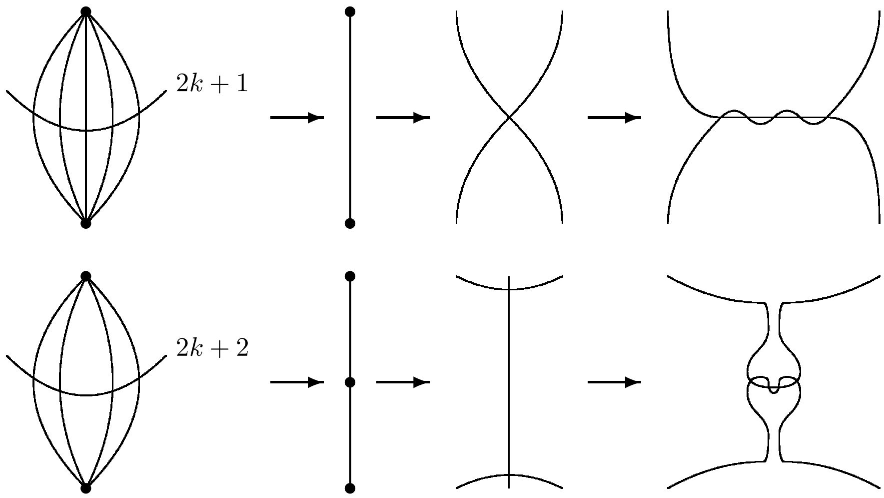

Figure 14. The existence of a strong intersection representation for all simple planar graphs is equivalent to the existence of a strong intersection representation for all planar multigraphs

7. Conclusion

Here we consider general graphs and hypergraphs (with possible loops and multiple edges).

Definition 7.1. A hypergraph $\mathcal{H}$ (or a multigraph G) has a *strong intersection representation* if there exists a family $\mathcal{F}$ of Jordan arcs, whose arc-point incidence is the vertex-edge incidence of $\mathcal{H}$ (or G): arcs represents vertices and points represent edges.

Problem 1. Has any planar simple graph an intersection representation?

Problem 2. Has any planar multigraph a strong intersection representation?

Proposition 7.1. *The problems 1 and 2 are equivalent:*
Any planar simple graph has an intersection representation if and only if any planar multigraph has a strong intersection representation.

Proof. If any planar multigraph has a strong intersection representation, any planar simple graph has indeed an intersection representation.

Conversely, if any planar simple graph has an intersection representation, any planar simple graph has also a strong intersection representation obtained by a trivial local deformation of the arcs at the intersection points. If a planar multigraph G has multiple edges, we consider the planar simple graph G' obtained from G by replacing any edge set E_i linking two vertices x and y by:

- a single edge $\{x, y\}$ if $|E_i|$ is odd,
- a bisected edge $\{x, z\}, \{z, y\}$ if $|E_i|$ is even.

Then, a strong intersection representation of G' induces a strong intersection representation of G (see Fig. 14). □

Problem 3. Has any planar hypergraph a strong intersection representation?

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