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ABSTRACT

It is shown that a d-dimensional classical $SU(N)$ Yang-Mills theory can be formulated in a $d + 2$-dimensional space, with the extra two dimensions forming a surface with non-commutative geometry.

1 Introduction

With Quantum Mechanics, W. Heisenberg introduced non-commutativity in phase space. This property was found to be crucial to solve the problem of the instability of the classical Rutherford atom due to the short-distance singularity of the Coulomb potential. So, when few years later it was realised that these short-distance singularities reappear if relativistic corrections are consistently taken into account, it is not surprising that it was Heisenberg again who first proposed a non-commutative geometry for space itself as a means to provide an effective ultraviolet cut-off [1], [2]. As it turned out, this motivation did not prove fruitful for elementary particle physics. With the development of the renormalisation programme in the framework of quantum field theories, the problem of ultraviolet divergences took a completely different turn. While a space cut-off makes all theories finite, the renormalisation programme applies to very few and very specific field theories. It is a most remarkable fact that they are precisely the ones chosen by Nature. It is not finiteness but rather lack of sensitivity to unknown physics at very short distances that turned out to be the important criterion. The geometry of physical space may still produce an ultraviolet cut-off, but its presence is not relevant for the calculation of physical processes among elementary particles.

However, almost at the same time, a new motivation for studying theories in a non-commutative space appeared, although only recently it was fully appreciated. In 1930 L.D. Landau [3] solved the problem of the motion of an electron in an external constant magnetic field and, besides computing the energy levels, the so-called "Landau levels", he showed that the components of the velocity operator of the electron do not commute. A simple way to visualise this result is to think of the classical case where the electron follows a spiral trajectory whose projection on a plane perpendicular to the field is a circle. In Landau's quantum mechanical solution its coordinates are:

$$x_c = \frac{c p_y}{eH} + x \quad ; \quad y_c = -\frac{c p_x}{eH} \tag{1}$$

which shows that the two coordinates do not commute. The magnetic field has induced a non-commutative structure on space itself. Following Heisenberg's suggestion, R. Peierls [4] showed that, at least the lowest Landau level, can be obtained by using this space non-commutativity. Since the presence of non-vanishing magnetic-type external fields is a common feature in many modern supergravity and string models, the study of field theories formulated on spaces with non-commutative geometry [5] has become quite

fashionable. A new element was added a few years ago with the work of N. Seiberg and E. Witten [6] who showed the existence of a map between gauge theories formulated in spaces with commuting and non-commuting coordinates. A very incomplete list of recent articles includes [7], [8]. Here I want to mention a different but related result which was obtained in collaboration with E.G. Floratos [9]. I shall state and, to a certain extend, prove, the following statement:

Statement: Given an $SU(N)$ Yang-Mills theory in a $d-$dimensional space with potentials

$$A_\mu(x) \;=\; A_\mu^a(x)\; t_a \tag{2}$$

where t_a are the standard $SU(N)$ matrices, there exists a reformulation in which the gauge fields and the gauge potentials become:

$$A_\mu(x) \to \mathcal{A}_\mu(x, z_1, z_2) \qquad F_{\mu\nu}(x) \to \mathcal{F}_{\mu\nu}(x, z_1, z_2) \tag{3}$$

where $\mathcal{A}$ and $\mathcal{F}$ are fields in a $(d+2)-$dimensional space, greek indices still run from 0 to $d-1$ and z_1 and z_2 are local coordinates on a two-dimensional surface endowed with non-commutative geometry. They will be shown to satisfy the commutation relation

$$[z_1, z_2] = \frac{2i}{N} \tag{4}$$

The commutators in the original $SU(N)$ Yang-Mills theory are replaced by the Moyal brackets [10], [11] with respect to the non-commuting coordinates.

$$\begin{aligned} [A_\mu(x), A_\nu(x)] &\to \{\mathcal{A}_\mu(x, z_1, z_2), \mathcal{A}_\nu(x, z_1, z_2)\}_{Moyal} \\ [A_\mu(x), \Omega(x)] &\to \{\mathcal{A}_\mu(x, z_1, z_2), \Omega(x, z_1, z_2)\}_{Moyal} \end{aligned} \tag{5}$$

where Ω is the function of the gauge transformation and $\{,\}_{Moyal}$ denotes the Moyal bracket with respect to the two operators z_1 and z_2. The trace over the group indices in the original Yang-Mills action becomes a two dimensional integral over the surface:

$$\int d^4x \; Tr\left(F_{\mu\nu}(x)F^{\mu\nu}(x)\right) \quad \to \quad \int d^4x dz_1 dz_2 \; \mathcal{F}_{\mu\nu}(x, z_1, z_2) * \mathcal{F}^{\mu\nu}(x, z_1, z_2) \tag{6}$$

The *-product will be defined later. When N goes to infinity, the two z's commute and the *-product reduces to the ordinary product.

In what follows I shall give a partial proof of this statement.

Let me start by recalling a well-known algebraic result: The Lie algebra of the group $SU(N)$, at the limit when N goes to infinity, with the generators appropriately rescaled, becomes the algebra of the area preserving diffeomorphisms of a surface. There exist explicit proofs of this theorem for the case of the sphere and the torus [12], [13], [14]. This implies a corresponding field theoretic result: A classical $SU(N)$ Yang-Mills theory on a d-dimensional space at an appropriate large N limit is equivalent to a field theory on $d + 2$ dimensions with the matrix commutators replaced by Poisson brackets with respect to the two new coordinates [15], [16], [17]. This is just the large N limit of the relations (2)-(6) we want to prove, in other words we want to establish the equivalence between Yang-Mills theories and field theories on surfaces to any order in $1/N$.

In order to be specific, let us consider the case of the sphere. One way to prove the algebraic result [13], is to start with the remark that the spherical harmonics $Y_{l,m}(\theta, \phi)$ are harmonic homogeneous polynomials of degree l in three euclidean coordinates x_1, x_2, x_3:

$$x_1 = cos\phi \; sin\theta, \quad x_2 = sin\phi \; sin\theta, \quad x_3 = cos\theta \qquad (7)$$

$$Y_{l,m}(\theta, \phi) = \sum_{\substack{i_k = 1,2,3 \\ k = 1,...,l}} \alpha^{(m)}_{i_1...i_l} \, x_{i_1}...x_{i_l} \qquad (8)$$

where $\alpha^{(m)}_{i_1...i_l}$ is a symmetric and traceless tensor. For fixed l there are $2l + 1$ linearly independent tensors $\alpha^{(m)}_{i_1...i_l}$, $m = -l, ..., l$.

Let us now choose, inside $SU(N)$, an $SU(2)$ subgroup by choosing three $N \times N$ hermitian matrices which form an $N-$dimensional irreducible representation of the Lie algebra of $SU(2)$.

$$[S_i, S_j] = i\epsilon_{ijk} S_k \qquad (9)$$

The S matrices, together with the α tensors introduced before, can be used to construct a basis of $N^2 - 1$ matrices acting on the fundamental

representation of $SU(N)$ [18].

$$S_{l,m}^{(N)} = \sum_{\substack{i_k=1,2,3 \\ k=1,\ldots,l}} \alpha_{i_1\ldots i_l}^{(m)} \, S_{i_1}\ldots S_{i_l}$$

$$[S_{l,m}^{(N)}, \, S_{l',m'}^{(N)}] = i f_{l,m; \, l',m'}^{(N)l'',m''} \, S_{l'',m''}^{(N)}$$

(10)

where the f's appearing in the r.h.s. of (10) are just the $SU(N)$ structure constants in a somehow unusual notation. The important, although trivial, observation is that the three $SU(2)$ generators S_i, rescaled by a factor proportional to $1/N$, will have well-defined limits as N goes to infinity.

$$S_i \rightarrow T_i = \frac{2}{N} S_i \tag{11}$$

Indeed, all matrix elements of T_i are bounded by $|(T_i)_{ab}| \leq 1$. They satisfy the rescaled algebra:

$$[T_i, T_j] = \frac{2i}{N} \epsilon_{ijk} T_k \tag{12}$$

and the Casimir element

$$T^2 = T_1^2 + T_2^2 + T_3^2 = 1 - \frac{1}{N^2} \tag{13}$$

in other words, under the norm $\|x\|^2 = Tr x^2$, the limits as N goes to infinity of the generators T_i are three objects x_i which commute by (12) and are constrained by (13):

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{14}$$

If we consider two polynomial functions $f(x_1, x_2, x_3)$ and $g(x_1, x_2, x_3)$ the corresponding matrix polynomials $f(T_1, T_2, T_3)$ and $g(T_1, T_2, T_3)$ have commutation relations for large N which follow from (12):

$$\frac{N}{2i} \, [f, g] \rightarrow \quad \epsilon_{ijk} \, x_i \, \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k} \tag{15}$$

If we replace now in the $SU(N)$ basis (10) the $SU(2)$ generators S_i by the rescaled ones T_i, we obtain a set of $N^2 - 1$ matrices $T_{l,m}^{(N)}$ which, according to (8), (10) and (15), satisfy:

$$\frac{N}{2i} \, [T_{l,m}^{(N)}, T_{l',m'}^{(N)}] \rightarrow \quad \{Y_{l,m}, Y_{l',m'}\} \tag{16}$$

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The relation (16) completes the algebraic part of the proof. It shows that the $SU(N)$ algebra, under the rescaling (11), does go to that of $[SDiff(S^2)]$. Since the classical fields of an $SU(N)$ Yang-Mills theory can be expanded in the basis of the matrices $T_{l,m}^{(N)}$, the relation (15) proves also the field theoretical result.

Here we want to argue that the equivalence between Yang-Mills theories and field theories on surfaces is in fact valid to any order in $1/N$. Coming back to the case of the sphere, because of the condition (13), we can parametrise the T_i's in terms of two operators, z_1 and z_2. As a first step we write:

$$T_1 = cos z_1 \ (1 - z_2^2)^{\frac{1}{2}}, \quad T_2 = sin z_1 \ (1 - z_2^2)^{\frac{1}{2}}, \quad T_3 = z_2 \qquad (17)$$

These relations should be viewed as defining the operators z_1 and z_2. A similar parametrisation has been given by T. Holstein and H. Primakoff in terms of creation and annihilation operators [19]. At the limit of $N \to \infty$, they become the coordinates ϕ and $cos\theta$ of a unit sphere. To leading order in $1/N$, the commutation relations (12) induce the commutation relation (4) between the z_i's:

$$[z_1, z_2] = \frac{2i}{N} \qquad (18)$$

i.e. the z_i's satisfy a Heisenberg commutation relation with the unity operator at the right-hand side. $1/N$ plays the role of $\hbar$. In higher orders, however, the definitions (17) must be corrected because the operators T_1 and T_2 are no more hermitian. It turns out that a convenient choice is to use T_+ and T_-. We thus write:

$$
\begin{aligned}
T_+ &= T_1 + iT_2 = e^{\frac{iz_1}{2}} \ (1 - z_2^2)^{\frac{1}{2}} \ e^{\frac{iz_1}{2}} \\
T_- &= T_1 - iT_2 = e^{-\frac{iz_1}{2}} \ (1 - z_2^2)^{\frac{1}{2}} \ e^{-\frac{iz_1}{2}} \\
T_3 &= z_2
\end{aligned}
\qquad (19)
$$

In order to avoid any misunderstanding, let me emphasise that (19) and (17) are not invertible as matrix relations and do not imply a representation of the z's in terms of finite dimensional matrices. The claim instead is that the $SU(N)$ algebra can be expressed in two equivalent ways: We can start from the non-commutative coordinates of the fuzzy sphere z_1 and z_2 which are assumed to satisfy the quantum mechanical commutation relations (4).

Through (19) we define three operators T_1, T_2 and T_3. We shall prove that they satisfy exactly, without any higher order corrections, the $SU(2)$ relations (12) and (13) and, consequently, they can be used as basis for the entire $SU(N)$ algebra. The opposite is also true. The $SU(2)$ commutation relations (12) imply the quantum mechanical relation (4). We can express the $SU(2)$ generators T_i $i = 1, 2, 3$, through (19), in terms of two operators z_i $i = 1, 2$. We can again prove that they satisfy the Heisenberg algebra (4) to all orders in $1/N$.

We start with the first part of the statement, which is straightforward calculation.

We assume (4) and we want to compute the commutator of T_+ and T_- given by (19).

$$[T_+, T_-] = e^{\frac{iz_1}{2}} \left(1 - z_2^2\right) e^{-\frac{iz_1}{2}} - e^{-\frac{iz_1}{2}} \left(1 - z_2^2\right) e^{\frac{iz_1}{2}}$$
$$= \left(e^{\frac{iz_1}{2}} z_2 e^{-\frac{iz_1}{2}}\right)^2 - \left(e^{-\frac{iz_1}{2}} z_2 e^{\frac{iz_1}{2}}\right)^2 \tag{20}$$

A useful form of the Cambell-Haussdorf relation for two operators A and B is:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + ... \tag{21}$$

Applying (21) to (20), we obtain:

$$[T_+, T_-] = \left(\frac{1}{N} + z_2\right)^2 - \left(\frac{1}{N} - z_2\right)^2 = \frac{4}{N} z_2 \tag{22}$$

Similarly, we check that the other two commutation relations of $SU(2)$ are satisfied. We can also compute the Casimir operator T^2 and we find the value $1 - 1/N^2$ of (13).

We proceed now with the proof of the opposite statement, namely the equivalence between the $SU(2)$ and the quantum mechanical commutation relations. The essence of the story is that any corrections on the r.h.s. of (4) will affect the $SU(2)$ commutation relations (12). The argument is inductive, order by order in $1/N$.

Let us start with the first term and write the general form of (4) as:

$$[z_1, z_2] = \frac{1}{N} t_1(z_1, z_2) + O(\frac{1}{N^2}) \tag{23}$$

with t_1 some function of the z's. Using (23) we compute the $1/N$ term in the commutator of two T's given by (19). If we assume that they satisfy the $SU(2)$ commutation relations we determine $t_1(z_1, z_2)$:

$$t_1(z_1, z_2) = 2i \tag{24}$$

We can now go back to (23) and determine the next term in the expansion. We write:

$$[z_1, z_2] = \frac{2i}{N} + \frac{1}{N^2} t_2(z_1, z_2) + O(\frac{1}{N^3}) \tag{25}$$

We look now at the T_i's given by (19) and compute the commutator between T_+ and T_3 to order $1/N^2$. Imposing the absence of such terms, we get:

$$t_2(z_1, z_2) = 0 \tag{26}$$

It is now clear how the induction works: We assume the commutator

$$[z_1, z_2] = \frac{2i}{N} + \frac{1}{N^k} t_k(z_1, z_2) + O(\frac{1}{N^{k+1}}) \tag{27}$$

and set the coefficient of the corresponding term in the $SU(2)$ commutation relation equal to zero. This gives again:

$$t_k(z_1, z_2) = 0 \tag{28}$$

The commutation relation (4) is the main step of the argument. Any function f of the $SU(N)$ generators, in particular any polynomial of the Yang-Mills fields and their space-time derivatives, can be rewritten, using (19), as a function of z_1 and z_2. Since they satisfy the quantum mechanics commutation relations (4), the usual proof of Moyal [10] goes through and the commutator of two such functions f and g will have an expansion in powers of $1/N$ of the form:

$$[f, g] \sim \frac{1}{N} \{f(z_1, z_2), g(z_1, z_2)\} + \frac{1}{N^2} \left(\{\frac{\partial f}{\partial z_1}, \frac{\partial g}{\partial z_2}\} + \{\frac{\partial f}{\partial z_2}, \frac{\partial g}{\partial z_1}\} \right) + \dots \tag{29}$$

with the Poisson brackets defined the usual way:

$$\{f, g\} = \left(\frac{\partial f}{\partial z_1} \frac{\partial g}{\partial z_2} - \frac{\partial f}{\partial z_2} \frac{\partial g}{\partial z_1} \right) \tag{30}$$

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The first term in this expansion is unambiguous but the coefficients of the higher orders depend on the particular ordering convention one may adopt. For example, in the symmetric ordering, only odd powers of $1/N$ appear.

For the symmetric ordering, we can introduce, formally, a *-product through:

$$f(z) * g(z) = exp(\xi \ \epsilon_{ij} \ \partial_z^i \partial_w^j) f(z) g(w)|_{w=z} \qquad (31)$$

with $z = (z_1, z_2)$ and $\xi = \frac{2i}{N}$. The $SU(N)$ commutators in the Yang-Mills Lagrangian can now be replaced by the *-products on the non-commutative surface. This equality will be exact at any given order in the $1/N$ expansion. This completes the proof of our statement.

Before closing, a few remarks:

The proof has been given only for the case of the sphere. The extension to the two-dimensional torus is straightforward, using the results of reference [16]. In principle, however, such a formulation should be possible for arbitrary genus surfaces, [20], [21], although I do not know of any explicit proof.

It is straightforward to generalize these results and include matter fields, provided they also belong to the adjoint representation of $SU(N)$. In particular, the supersymmetric Yang-Mills theories have the same property. The special case of $\mathcal{N} = 4$ supersymmetry is of obvious interest because of its conformal properties. In this theory the duality $g \rightarrow 1/g$ makes us believe that the two large N limits, namely 't Hooft's and the one used here, are related.

We believe that one could also include fields belonging to the fundamental representation of the gauge group. In 't Hooft's limit such matter multiplets are restricted to the edges of the diagram, so we expect in our case the generalization to involve open surfaces.

The equivalence between the original d-dimensional Yang-Mills theory and the new one in $d + 2$ dimensions holds at the classical level. For the new formulation however, the ordinary perturbation series, even at the large N limit, is divergent. The reason is that the quadratic part of this action does not contain derivatives with respect to z_1 or z_2. This is not surprising because these divergences represent the factors of N in the diagrams of the original theory which have not been absorbed in the redefinition of the coupling constant. However, we expect a perturbation expansion around some appropriate non-trivial classical solution to be meaningful and to contain

interesting information concerning the strong coupling limit of the original theory.

A final remark: Could one have anticipated the emergence of this action in the $1/N$ expansion? It is clear that, starting from a set of N fields $\phi^i(x)$ $i = 1, ..., N$, we can always replace $\phi^i(x)$, at the limit when N goes to infinity, with $\phi(\sigma, x)$ where $0 \leq \sigma \leq 2\pi$. In this case the sum over i will become an integral over σ. However, for a general interacting field theory, the ϕ^4 term will no more be local in σ. So, the only surprising feature is that, for a Yang-Mills theory, the resulting expression is local.

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