Generalized Integral Operators and Applications

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Abstract

We extend the theory of distributional kernel operators to a framework of generalized functions, in which they are replaced by integral kernel operators. Moreover, in contrast to the distributional case, we show that these generalized integral operators can be composed unrestrictedly. This leads to the definition of the exponential, and more generally entire functions, of a subclass of such operators.

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1 Introduction

The theory of nonlinear generalized functions [2, 3, 4, 5, 9, 12, 16], which appears as a natural extension of the theory of distributions, seems to be a suitable framework to overcome the limitations of the classical theory of unbounded operators.

Following a first approach done by D. Scarpalezos in [18], we introduce a natural concept of generalized integral kernel operators in this setting. In addition, we show that these operators are characterized by their kernel. Our approach has some relationship with the one of [10, 11] but is less restrictive and uses other technics of proofs. Let us quote that classical operators with smooth or distributional kernel are represented by generalized integral kernel operators in the spaces of generalized functions, through the sheaf embeddings of $\mathcal{C}^\infty$ or $\mathcal{D}'$ into $\mathcal{G}$, the sheaf of spaces of generalized functions. This shows that our theory is a natural extension of the classical one.

Contrary to the classical case [13], we show that such operators can be composed unrestrictedly. This is done for generalized operators with kernel properly supported belonging to the classical space of generalized functions $\mathcal{G}$ and for operators with kernel in a less usual space $\mathcal{G}_{L^2}$, constructed from the algebra $\mathcal{D}_{L^2} = H^\infty$. This allows to consider their iterate composition and the question of summation of series of such operators naturally arises. In view of applications to theoretical physics, this question has been solved for the exponential, with additional assumptions on the growth of the kernel with respect to the scaling parameter. Two cases have been considered: The case of operators with compactly supported kernel for which the results have been announced and partially proved in [1]; The case of operators with kernels in the above mentioned space $\mathcal{G}_{L^2}$ for which we give an application to symmetrical operators.

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2 The mathematical framework

In order to render the paper almost self contained, we recall some elements of the theory of generalized numbers and functions without any proofs. We refer the reader to [2, 3, 4, 5, 9, 12, 14, 15, 16] for more details (except for subsection 2.4).

2.1 The sheaf of algebras of generalized functions

Let E be a sheaf of topological $\mathbb{K}$-algebras on a topological space X ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). As in [14], we assume that E satisfies the two following properties:

(i) For each open subset Ω of X, the algebra $E(\Omega)$ is endowed with a family of semi-norms $\mathcal{P}(\Omega) = (p_i)_{i \in I(\Omega)}$, which gives to $E(\Omega)$ a structure of topological vector space and satisfies

$$\forall i \in I(\Omega),\ \exists (j,k,C) \in I(\Omega)^2 \times \mathbb{R}_+^* \ / \ \forall f,g \in E(\Omega)\ ,\ p_i(fg) \leq C p_j(f) p_k(g); \qquad (1)$$

(ii) For two open subsets Ω_1 and Ω_2 of X such that $\Omega_1 \subset \Omega_2$, one has

$$\forall i \in I(\Omega_1)\ ,\ \exists j \in I(\Omega_2)\ /\ \forall u \in E(\Omega_2)\ ,\ p_i(u \mid_{\Omega_1}) \leq p_j(u);$$

(iii) Let $\mathcal{F} = (\Omega_\lambda)_{\lambda \in \Lambda}$ be any family of open subsets of X with $\Omega = \cup_{\lambda \in \Lambda} \Omega_\lambda$. Then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily of $\mathcal{F}$: $\Omega_1, \Omega_2, ..., \Omega_{s(i)}$ and corresponding semi-norms $p_1 \in \mathcal{P}(\Omega_1)$, $p_2 \in \mathcal{P}(\Omega_2)$,..., $p_{s(i)} \in \mathcal{P}(\Omega_{s(i)})$, such that, for any $u \in E(\Omega)$,

$$p_i(u) \leq \max_{1 \leq i \leq s(i)} \left(p_i(u \mid_{\Omega_i}) \right).$$

Set

$$\mathcal{H}(E,\mathcal{P})(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]} \ / \ \forall i \in I(\Omega),\ \exists\ n \in \mathbb{N}\ :\ p_i(u_\varepsilon) = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0 \right\}$$
$$\mathcal{I}(E,\mathcal{P})(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in E(\Omega)^{(0,1]} \ / \ \forall i \in I(\Omega),\ \forall\ n \in \mathbb{N}\ :\ p_i(u_\varepsilon) = O(\varepsilon^{n}) \text{ as } \varepsilon \to 0 \right\}.$$

As proved in [14], the functor $\mathcal{H}(E,\mathcal{P}) : \Omega \mapsto \mathcal{H}(E,\mathcal{P})(\Omega)$ is a sheaf of subalgebras of the sheaf $E^{(0,1]}$, the functor $\mathcal{I}(E,\mathcal{P}) : \Omega \mapsto \mathcal{I}(E,\mathcal{P})(\Omega)$ is a sheaf of ideals of $\mathcal{H}(E,\mathcal{P})$ and the constant factor sheaf $\mathcal{H}(\mathbb{K}, |\cdot|)/\mathcal{I}(\mathbb{K}, |\cdot|)$ is exactly the factor ring $\overline{\mathbb{K}} = A/I_A$, with

$$A = \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \ / \ \exists\ n \in \mathbb{N}\ :\ |r_\varepsilon| = O(\varepsilon^{-n}) \text{ as } \varepsilon \to 0\}$$
$$I_A = \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \ / \ \forall\ n \in \mathbb{N}\ :\ |r_\varepsilon| = O(\varepsilon^{n}) \text{ as } \varepsilon \to 0\}.$$

The sheaf of factor algebras $\mathcal{A}(E,\mathcal{P}) = \mathcal{H}(E,\mathcal{P})/\mathcal{I}(E,\mathcal{P})$, is called a *sheaf of nonlinear generalized functions.*

Remark 1 *If E is a sheaf of differential algebras then the same holds for $\mathcal{A}(E,\mathcal{P})$.*

In the following paragraphs, we will use this definition in the following particular cases.

Example 2 *We define $\overline{\mathbb{C}}$ (resp. $\overline{\mathbb{R}}$) to be the factor ring A/I_A of generalized complex (resp. real) numbers.*

Example 3 *Take the sheaf $E=\mathcal{C}^{\infty}$ on $X=\mathbb{R}^{d}$ ($d\in\mathbb{N}$), endowed with its usual topology. This topology can be described, for Ω an open subset of $\mathbb{R}^{d}$, by the family $\mathcal{P}(\Omega)=\{p_{K,l}\,;K\Subset\Omega,\ l\in\mathbb{N}\}$, where the notation $K\Subset\Omega$ means that K is a compact subset included in Ω and*

$$p_{K,l}(f)=\sup_{x\in K,|\alpha|\leq l}|\partial^{\alpha}f(x)|,\ \text{for all } f\in\mathcal{C}^{\infty}(\Omega)\ (\text{with } \partial^{\alpha}=\frac{\partial^{|\alpha|}f}{\partial x^{\alpha}}).$$

$\mathcal{A}(\mathcal{C}^{\infty},\mathcal{P})\,(\Omega)$ is the algebra of simplified generalized functions, introduced by the second author [2, 3, 4, 5, 12].

Notation 4 *We set $\mathcal{E}_{M}(\Omega)=\mathcal{H}(\mathcal{C}^{\infty},\mathcal{P})(\Omega)$, $\mathcal{I}(\Omega)=\mathcal{I}(\mathcal{C}^{\infty},\mathcal{P})(\Omega)$ and $\mathcal{G}(\Omega)=\mathcal{A}(\mathcal{C}^{\infty},\mathcal{P})(\Omega)$. We shall also write P_{K} instead of $P_{K,0}$ for every compact subset K of Ω.*

Since $\mathcal{G}$ is a sheaf, the support of a section $u\in\mathcal{G}(\Omega)$ is well defined. Let us recall that, for Ω' an open subset of Ω and $u\in\mathcal{G}(\Omega)$, the restriction of u to Ω' is the class in $\mathcal{G}(\Omega')$ of $\left(u_{\varepsilon|\Omega'}\right)_{\varepsilon}$ where $(u_{\varepsilon})_{\varepsilon}$ is any representative of u. We say that u is null on Ω' if its restriction to Ω' is null in $\mathcal{G}(\Omega')$. The *support* of a generalized function $u\in\mathcal{G}(\Omega)$ is the complement in Ω of the largest open subset of Ω where u is null.

Notation 5 *For Ω an open subset of $\mathbb{R}^{d}$, we will denote by $\mathcal{G}_{C}(\Omega)$ the set of generalized functions of $\mathcal{G}(\Omega)$ with compact support.*

Remark 6 *Every $f\in\mathcal{G}_{C}\,(\Omega)$ has a representative $(f_{\varepsilon})_{\varepsilon}\in\mathcal{E}_{M}\,(\Omega)$, such that each f_{ε} is supported in the same compact set. We say that such a representative has a* global compact support.

The two following examples will be used in section 6 for the definition of the exponential of some generalized integral operator. In them, we apply the Colombeau construction to presheaves of algebras in example 7 (resp. vector spaces in example 9). In these cases, property (iii) may not be satisfied but the general construction is still valid, giving presheaves of generalized algebras (resp. of vector spaces).

Example 7 *For Ω an open subset of $\mathbb{R}^{d}$, we consider $E(\Omega)=H^{\infty}(\Omega)$ with its usual topology, defined by the family $\mathcal{P}(\Omega)=\{\|\cdot\|_{m}^{\prime}\,;\ m\geq0\}$, with*

$$\|f\|_{m}^{\prime}=\|f\|_{H^{m}(\Omega)}=\sum_{|\alpha|\leq m}\|\partial^{\alpha}f\|_{L^{2}(\Omega)},\ \textit{for all } f\in H^{\infty}(\Omega).$$

Notation 8 *We set $\mathcal{E}_{L^{2}}(\Omega)=\mathcal{H}(H^{\infty},\mathcal{P})(\Omega)$, $\mathcal{I}_{L^{2}}(\Omega)=\mathcal{I}(H^{\infty},\mathcal{P}),\mathcal{P})(\Omega)$ and $\mathcal{G}_{L^{2}}(\Omega)=\mathcal{A}(H^{\infty},\mathcal{P})(\Omega)$.*

When $E\,(\Omega)$ is only a topological vector space on $\mathbb{K}$ (that is (1) is not necessarly satisfied), $\mathcal{G}(\Omega)$ is defined analogously and is still a module on $\overline{\mathbb{K}}$.

Example 9 *For Ω an open subset of $\mathbb{R}^{d}$, we consider $E^{i}(\Omega)=L^{i}(\Omega)\cap\mathcal{C}^{\infty}(\Omega)$ ($i=1$ and $i=2$) with the topology given by the norm $\|\cdot\|_{i}=\|\cdot\|_{L^{i}(\Omega)}$.*

Notation 10 *We set $\mathcal{E}_{l^{i}}(\Omega)=\mathcal{H}(L^{i}\cap\mathcal{C}^{\infty},\|f\|_{L^{i}(\Omega)})(\Omega)$, $\mathcal{I}_{l^{2}}(\Omega)=\mathcal{I}(L^{i}\cap\mathcal{C}^{\infty},\|f\|_{L^{i}(\Omega)})$ and $\mathcal{G}_{l^{i}}(\Omega)=\mathcal{E}_{l^{i}}(\Omega)/\mathcal{I}_{l^{i}}(\Omega)$.*

2.2 Embeddings of spaces of distributions into spaces of generalized functions

Let Ω be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$). The embedding of $\mathcal{C}^\infty(\Omega)$ into $\mathcal{G}(\Omega)$ is given by the canonical map

$$\sigma \ : \ \mathcal{C}^\infty(\Omega) \to \mathcal{G}(\Omega) \qquad f \mapsto Cl(f_\varepsilon)_\varepsilon, \text{ with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1]\,,$$

which is an injective homomorphism of algebras.

An embedding i_S of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ such that $i_S\mid_{\mathcal{C}^\infty(\Omega)} = \sigma$ can be constructed by the two following methods. For the first one [12], one starts from a net $(\rho_\varepsilon)_\varepsilon$ defined by $\rho_\varepsilon(\cdot) = \varepsilon^{-d}\rho(\cdot/\varepsilon)$, where $\rho \in \mathcal{S}(\mathbb{R}^d)$ satisfies

$$\textstyle\int \rho(x)\,\mathrm{d}x = 1\ ; \quad \forall m \in \mathbb{N}^d \setminus \{0\} \quad \int x^m \rho(x)\,\mathrm{d}x = 0.$$

An embedding i_0 of $\mathcal{E}'(\mathbb{R}^d)$ in $\mathcal{G}(\mathbb{R}^d)$ is defined by

$$i_0 \ : \ \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega) \qquad T \mapsto Cl((T * \rho_\varepsilon)_{|\Omega})_\varepsilon.$$

From this, for every open subset $\Omega \subset \mathbb{R}^d$, an open covering $(\Omega_\lambda)_\lambda$ of Ω with relatively compact open subsets is considered, and $\mathcal{D}'(\Omega_\lambda)$ is embedded into $\mathcal{G}(\Omega_\lambda)$ with the help of cutoff functions and i_0. Using a partition of unity subordinate to $(\Omega_\lambda)_\lambda$, the embedding i_S of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ is constructed by gluing the bits obtained before together. Finally, it is shown that the embedding i_S does not depend on the choice of $(\Omega_\lambda)_\lambda$ and other material of the construction, excepted the net $(\rho_\varepsilon)_\varepsilon$. The second method [16] starts from the same $(\rho_\varepsilon)_\varepsilon$ which is slightly modified by ad hoc cutoff functions. Consider $\chi \in \mathcal{D}(\mathbb{R})$ even such that

$$0 \le \chi \le 1\,, \qquad \chi \equiv 1 \text{ on } \bar{B}(0,1)\,, \qquad \chi \equiv 0 \text{ on } \mathbb{R}^d \setminus B(0,2)$$

and set

$$\forall x \in \mathbb{R}^d, \quad \forall \varepsilon \in (0,1]\,, \quad \Theta_\varepsilon(x) = \rho_\varepsilon(x)\,\chi(|\ln \varepsilon| x).$$

One shows that

$$\left(\textstyle\int \Theta_\varepsilon(x)\,\mathrm{d}x - 1\right)_\varepsilon \in \mathcal{I}(\mathbb{R})\ ; \quad \forall m \in \mathbb{N}^d \setminus \{0\}, \quad \left(\textstyle\int x^m \Theta_\varepsilon(x)\,\mathrm{d}x\right)_\varepsilon \in \mathcal{I}(\mathbb{R}). \tag{2}$$

Set $\Gamma_\varepsilon = \left\{x \in \Omega \ / \ d(x, \mathbb{R}^d \setminus \Omega) \ge \varepsilon\ ,\ d(x,0) \le 1/\varepsilon\right\}$ and consider $(\gamma_\varepsilon)_\varepsilon \in \mathcal{D}(\mathbb{R}^d)^{(0,1]}$ such that

$$\forall \varepsilon \in (0,1]\ , \quad 0 \le \gamma_\varepsilon \le 1, \quad \gamma_\varepsilon \equiv 1 \text{ on } \Gamma_\varepsilon.$$

Then the map

$$\mathcal{D}'(\Omega) \to \mathcal{G}(\Omega) \qquad T \mapsto Cl(\gamma_\varepsilon T * \Theta_\varepsilon)_\varepsilon$$

is equal to i_S [6]. (This last proof uses mainly (2); The additional cutoff $(\gamma_\varepsilon)_\varepsilon$, which is such that $\gamma_\varepsilon T \mapsto T$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$, is needed to obtain a well defined net $(\gamma_\varepsilon T * \Theta_\varepsilon)_\varepsilon$.)

2.3 Integration of generalized functions

We shall use integration of generalized functions on compact sets or integration of generalized functions having compact support.

Let K be a given compact subset of Ω and u an element of $\mathcal{G}(\Omega)$. The integral of u on K, denoted by $\int_K u(x)\,\mathrm{d}x$, is the class, in $\overline{\mathbb{C}}$ of the integral on K of any representative of u. (This class does not depend on the choice of the representative of u.)

The integral of a generalized function having a compact support is an immediate extension of the previous case. Indeed, if $u \in \mathcal{G}(\Omega)$ has a compact support K, let $K_1 \subset K_2$ be two compact subsets of Ω such that K is contained in the interior of K_1. Then, it can be shown that

$$\int_{K_2 \setminus \overset{\circ}{K_1}} u(x)\,\mathrm{d}x = 0 \text{ in } \overline{\mathbb{C}}.$$

Therefore, $\int_{K_1} u(x)\,\mathrm{d}x = \int_{K_2} u(x)\,\mathrm{d}x$ and this value is denoted by $\int_{\Omega} u(x)\,\mathrm{d}x$. ($\Omega$ is omitted in the sequel if no confusion may arise.)

We shall also consider integration on the space $\mathcal{G}_{l^1}(\Omega)$. The integral of $u \in \mathcal{G}_{l^1}(\Omega)$, denoted by $\int_{\Omega} u(x)\,\mathrm{d}x$, is the class, in $\overline{\mathbb{C}}$ of the integral on Ω of any representative of u. (This class does not depend on the choice of the representative of u.)

It follows immediately from the definitions that the integral of a generalized function on a set of measure zero is equal to zero, the integral of a null generalized function is equal to zero and that the classical formulas of integration by parts, change of variables, change in order of integration (Fubini's theorem), ... are valid for the integration of generalized functions.

2.4 Generalized parameter integrals

Let X (resp. Y) be an open subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$). We denote by $\mathcal{G}_{ps}(X \times Y)$ the set of generalized functions g of $\mathcal{G}(X \times Y)$ *properly supported* in the following sense:

$$\forall\ O_1 \text{ relatively compact open subset of } X,\ \exists K_2 \Subset Y \ /\ supp\ g \cap (O_1 \times Y)\ \subset O_1 \times K_2. \quad (3)$$

Clearly, $\mathcal{G}_{ps}(X \times Y)$ is a subalgebra of $\mathcal{G}(X \times Y)$.

Lemma 11 *Let g be in $\mathcal{G}_{ps}(X \times Y)$. For V relatively compact open subset of X, there exists W relatively compact open subset of Y such that $supp\ g \cap (V \times Y)\ \subset V \times W$.*
For all $\varepsilon \in (0,1]$ and $x \in V$, we set $G_\varepsilon(x) = \int_W g_\varepsilon(x,y)\,\mathrm{d}y$, where $(g_\varepsilon)_\varepsilon$ is a representative of g. The net $(G_\varepsilon)_\varepsilon$ belongs to $\mathcal{E}_M(V)$ and its class, denoted by G, is an element of $\mathcal{G}(V)$ which does not depend on the choice of the representative of g and of W.

Proof. First, the existence of W is due to the hypothesis (3). Then, for all $\varepsilon \in (0,1]$, G_ε is well defined and of class $\mathcal{C}^\infty$ by the usual regularity theorems. (Note that G_ε is the integral of a $\mathcal{C}^\infty$-function on a relatively compact open subset.)

We first show that $(G_\varepsilon)_\varepsilon$ belongs to $\mathcal{E}_M(V)$. Let K_1 be a compact subset of V and α be in $\mathbb{N}^m$. For $x \in K_1$, one has

$$\begin{aligned} |\partial^\alpha G_\varepsilon(x)| \leq \left| \int_W \partial^\alpha g_\varepsilon(x,y)\,\mathrm{d}y \right| &\leq Vol(W) \sup_{x \in K_1, y \in \bar{W}} |\partial^\alpha g_\varepsilon(x,y)| \\ &\leq Vol(W)\, C\, \varepsilon^{-q}, \text{ for } \varepsilon \text{ small enough,} \end{aligned}$$

for some $C > 0$ and $q \in \mathbb{N}$, where $Vol(W)$ denotes the volume of W.

Let us verify that G does not depend on the choice of the representative of g and on the one of W. According to M. Grosser *et al.* ([12], theorem 1.2.3), it is enough to consider estimates of order zero, what we will do in the following. Let $(g^1_\varepsilon)_\varepsilon$ and $(g^2_\varepsilon)_\varepsilon$ be two representatives of g. As previously, we can define $(G^1_\varepsilon)_\varepsilon$ and $(G^2_\varepsilon)_\varepsilon$ in $\mathcal{E}_M(V)$. We have to show that $(G^1_\varepsilon - G^2_\varepsilon)_\varepsilon$ is in $\mathcal{I}(V)$. Let K_1 be a compact subset of V. For $x \in K_1$, one has

$$\left|G^1_\varepsilon(x) - G^2_\varepsilon(x)\right| = \left| \int_W (g^1_\varepsilon - g^2_\varepsilon)(x,y)\,\mathrm{d}y \right| \leq Vol(W) \sup_{x \in K_1, y \in \bar{W}} |(g^1_\varepsilon - g^2_\varepsilon)(x,y)|.$$

As $(g^1_\varepsilon - g^2_\varepsilon)_\varepsilon$ belongs to $\mathcal{I}(X \times Y)$, we get from the previous estimate that $P_{K_1}\left(G^1_\varepsilon - G^2_\varepsilon\right) = O\left(\varepsilon^n\right)$ as $\varepsilon \to 0$ for all $n \in \mathbb{N}$.

Consider W_1 and W_2 relatively compact open subsets of Y such that, $supp\ g \cap (V \times Y) \subset V \times W_i$ for $i = 1, 2$ with, for example $W_1 \subset W_2$. For all $\varepsilon \in (0, 1]$, $x \in V$ and $i = 1, 2$, we set $G_\varepsilon^i(x) = \int_{W_i} g_\varepsilon(x, y)\,\mathrm{d}y$ where $(g_\varepsilon)_\varepsilon$ is a representative of g. Then $(G_\varepsilon^1)_\varepsilon$ and $(G_\varepsilon^2)_\varepsilon$ belong to $\mathcal{E}_M(V)$. Let K_1 be a compact subset of V. For $x \in K_1$, one has

$$\left|(G_\varepsilon^1 - G_\varepsilon^2)(x)\right| = \left|\int_{W_2 \setminus W_1} g_\varepsilon(x, y)\,\mathrm{d}y\right| \leq Vol(W_2 \setminus W_1) \sup_{x \in K_1, y \in \overline{W_2 \setminus W_1}} |g_\varepsilon(x, y)|.$$

As $supp\ g \cap (V \times Y) \subset V \times W$, the restriction of g to $V \times (W_2 \setminus W_1)$ is null. Therefore, the previous estimate shows that $P_{K_1}\left(G_\varepsilon^1 - G_\varepsilon^2\right) = O\left(\varepsilon^n\right)$ as $\varepsilon \to 0$ for all $n \in \mathbb{N}$.

Lemma 12 *Let g be in $\mathcal{G}_{ps}(X \times Y)$, $(V_i)_{i \in I}$ be a family of relatively compact open subsets of X such that $\cup_{i \in I} V_i = X$ and define $G_i \in \mathcal{G}(V_i)$ as in lemma 11. Then, there exists $G \in \mathcal{G}(X)$ such that the restriction of G to V_i is equal to G_i, for all $i \in I$. Moreover, G only depends on g, but not on $(V_i)_{i \in I}$.*

Proof. For $i \neq j$ such that $V_i \cap V_j \neq \varnothing$, we remark that $V_i \cup V_j$ is a relatively compact open subset of X. There exists W a relatively compact open subset of Y such that $supp\ g \cap ((V_i \cup V_j) \times Y) \subset (V_i \cup V_j) \times W$. According to lemma 11, we can define $\Phi = Cl\left(\Phi_\varepsilon\right)_\varepsilon \in \mathcal{G}(V_i \cup V_j)$, with

$$\forall x \in V_i \cup V_j, \quad \Phi_\varepsilon\left(x\right) = \int_W g_\varepsilon(x, y)\,\mathrm{d}y.$$

Then, $\left(\Phi_{\varepsilon|V_i}\right)_\varepsilon$ (resp. $\left(\Phi_{\varepsilon|V_j}\right)_\varepsilon$) is a representative of G_i (resp. G_j) since those representatives depend neither on the representative of g nor on the choice of appropriate W. Then $G_{i|V_i \cap V_j} = G_{j|V_i \cap V_j}$. Thus, $(G_i)_{i \in I}$ is a coherent family, which implies the existence of G since $\mathcal{G}(X)$ is a sheaf. The proof of the independence of G with respect to $(V_i)_{i \in I}$ follows the same lines.

Lemmas 11 and 12 give immediately the following:

Proposition 13 *For g in $\mathcal{G}_{ps}(X \times Y)$, there exists $G \in \mathcal{G}(X)$ such that, for all relatively compact open subset O_1 of X,*

$$G_{|O_1} = Cl\left(\left(x \mapsto \int_{K_2} g_\varepsilon(x, y)\,\mathrm{d}y\right)_{|O_1}\right)_\varepsilon,$$

where (g_ε) is a representative of g and $K_2 \Subset Y$ is such that $supp\, g \cap (O_1 \times Y) \subset O_1 \times K_2$.

Notation 14 *By a slight abuse of notation, we shall set $G = \int_Y g(\cdot, y)\,\mathrm{d}y$ or $G\left(\cdot_1\right) = \int_Y g(\cdot_1, y)\,\mathrm{d}y$. We shall omit the set on which the integration is performed when no confusion may arise.*

Example 15 *Proposition 13 can be used to define the Fourier's transform of a compactly supported generalized function. Indeed, if u is in $\mathcal{G}_C(Y)$ then the generalized function $\{(x, y) \mapsto e^{-ixy} u(y)\}$ is in $\mathcal{G}_{ps}(X \times Y)$, then $\hat{u}(x) = \int e^{-ixy} u(y)\,\mathrm{d}y$ is well defined.*

3 Generalized integral operators

In this section, we introduce the notion of generalized integral operator and study their basic properties. The reader can find another approach in [10, 11]. Let X (resp. Y) be an open subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$).

Definition 16 *Let H be in $\mathcal{G}_{ps}(X \times Y)$. We call* generalized integral operator *the map*

$$\begin{array}{rcll} \widehat{H} & : & \mathcal{G}(Y) & \rightarrow & \mathcal{G}(X) \\ & & f & \mapsto & \widehat{H}(f) = \int H(\cdot, y) f(y) \,\mathrm{d}y, \end{array}$$

with the meaning introduced in proposition 13 and notation 14. We say that H is the kernel *of the generalized integral operator $\widehat{H}$.*

This map is well defined due to proposition 13 since the application $H(\cdot_1, \cdot_2) f(\cdot_2)$ is clearly is in $\mathcal{G}_{ps}(X \times Y)$.

Remark 17 *If $H \in \mathcal{G}(X \times Y)$ has a compact support then H satisfies (3) and $\widehat{H}$ is well defined. Furthermore, the definition of $\widehat{H}$ does not need to refer to proposition 13 in this case. Indeed, if H is in $\mathcal{G}_C(X \times Y)$ with supp $H \subset \mathring{K}_1 \times \mathring{K}_2$ ($K_1 \Subset X$, $K_2 \Subset Y$) and f in $\mathcal{G}(Y)$, we have*

$$\widehat{H}(f) = Cl \left(x \mapsto \int_{K_2} H_\varepsilon(x, y) f_\varepsilon(y) \,\mathrm{d}y \right)_\varepsilon$$

where $(H_\varepsilon)_\varepsilon$ (resp. $(f_\varepsilon)_\varepsilon$) is any representative of H (resp. f). Furthermore, as supp $H \subset K_1 \times K_2$, we have $H(\cdot_1, \cdot_2) f(\cdot_2)_{|(X \backslash K_1) \times Y} = 0$, hence $\widehat{H}(f)_{|X \backslash K_1} = 0$ and supp $\widehat{H}(f) \subset K_1$. (The proof uses arguments similar to the one of lemma 11.) Finally, the image of $\widehat{H}$ is included in $\mathcal{G}_C(X)$ and, more precisely, in $\{g \in \mathcal{G}_C(X) \; / \; \text{supp } g \subset K_1\}$.

Remark 18 *If H is in $\mathcal{G}(X \times Y)$ without any other hypothesis, we can define a map $\widehat{H}$: $\mathcal{G}_C(Y) \rightarrow \mathcal{G}(X)$ in the same way. Indeed, for all f in $\mathcal{G}_C(Y)$ with supp $f = K_2$ and for all O_1 relatively compact open subset of X, supp $H(\cdot_1, \cdot_2) f(\cdot_2) \cap (\; O_1 \times Y) \;\subset O_1 \times K_2$, that is $H(\cdot_1, \cdot_2) f(\cdot_2)$ is in $\mathcal{G}_{ps}(X \times Y)$. In this case, the generalized integral operator can be defined globally since f has a representative with global compact support.*

This remark leads us to make the link between the classical theory of integral operators acting on $\mathcal{D}(Y)$ and the generalized one. This is detailed in section 4.

Remark 19 *In all previous cases, $\widehat{H}$ is a linear map of $\overline{\mathbb{C}}$-modules. This holds also for the map $\widehat{\ }$: $\mathcal{G}_{ps}(X \times Y) \rightarrow \mathcal{L}(\mathcal{G}(Y), \mathcal{G}(X))$, which associates $\widehat{H}$ to H. Moreover, $\widehat{H}$ is continuous for the sharp topologies [18]. Conversely, the third author showed in [7] that any continuous linear map from $\mathcal{G}_C(Y)$ to $\mathcal{G}(X)$, satisfying appropriate growth hypotheses with respect to the regularizing parameter ε, can be written as a generalized integral operator, giving a Schwartz kernel type theorem in the framework of integral generalized operators.*

Example 20 *The identity map of subspaces of compactly generalized functions with limited growth [7] admits as kernel*

$$\Phi = Cl \left((x, y) \mapsto \Theta_\varepsilon(x - y) \right)_\varepsilon,$$

where $(\Theta_\varepsilon)_\varepsilon$ is defined in paragraph 2.2.

Theorem 21 (*Characterization of generalized integral operators by their kernel*) *One has $\widehat{H} = 0$ if and only if $H = 0$.*

A first proof, due to the second author, is based on embeddings of Sobolev's spaces in spaces of smooth functions. The proof given below is due to V. Valmorin (personal communication) and uses the following:

Lemma 22 *Let Ω be an open subset of $\mathbb{R}^d$ and K be a compact of Ω, of diameter D. If Φ belongs to $\mathcal{D}_K(\Omega)$ then*

$$\sup_{y\in K} |\Phi(y)| \leq \left(D^d \int_\Omega |\frac{\partial^d}{\partial y_1 \partial y_2 ... \partial y_d} \Phi(y)|^2 \ \mathrm{d}y \right)^{1/2}.$$

Proof. First, let us assume that $H = 0$ in $\mathcal{G}(X \times Y)$ and fix $f \in \mathcal{G}(Y)$. For any O_1 relatively compact open subset of X, there exists $K_2 \Subset Y$ such that $supp\, g \cap (O_1 \times Y) \ \subset O_1 \times K_2$ and

$$\widehat{H}\,(f)_{|O_1} = Cl(\,(x \mapsto \int_{K_2} H_\varepsilon(x,y) f_\varepsilon(y) \ \mathrm{d}y)_{|O_1}\ )_\varepsilon \,,$$

where $(H_\varepsilon)_\varepsilon$ (resp. $(f_\varepsilon)_\varepsilon$) is any representative of H (resp. f). As H is null, the map $H_\varepsilon(\cdot_1, \cdot_2) f(\cdot_2)$ is null on $O_1 \times Y$. Thus $\widehat{H}\,(f)_{|O_1}$ is null and, by sheaf properties, $\widehat{H}\,(f)$ is null.

Conversely, suppose that $\widehat{H} = 0$. In order to prove that $H = 0$ in $\mathcal{G}(X \times Y)$, we shall prove that $H_{|O_1 \times Y} = 0$ in $\mathcal{G}(O_1 \times Y)$, for any O_1 relatively compact open subset of X and conclude by using the sheaf properties of $\mathcal{G}(\cdot)$. Let K_1 and K_2 two compacts subsets of O_1 and Y respectively. From (3), we can find W a relatively compact open subset of Y such that $K_2 \subset W$ and *supp* $H \cap (O_1 \times Y) \ \subset O_1 \times W$. Let $(H_\varepsilon)_\varepsilon$ be a representative of H and set $\varphi_{\varepsilon,x}(y) = H_\varepsilon(x,y)\rho(y)$, for all $y \in Y$ and $x \in O_1$, where ρ is a $\mathcal{C}^\infty$-function on Y such that $\rho = 1$ on W and *supp* $\rho \subset O_2$, with O_2 a relatively compact open subset of Y. Thus, for all x in O_1, $\varphi_{\varepsilon,x} \in \mathcal{D}_{\bar{O}_2}(Y)$. This implies

$$\sup_{y\in \bar{O}_2} |\varphi_{\varepsilon,x}(y)| \leq \left(diam(\bar{O}_2)^n \int_{O_2} |\frac{\partial^n}{\partial y^n} \varphi_{\varepsilon,x}(y)|^2 \, \mathrm{d}y \right)^{1/2},$$

where $\frac{\partial^n}{\partial y^n}$ is the derivative $\frac{\partial^n}{\partial y_1 \partial y_2 ... \partial y_n}$ and $diam(\cdot)$ denotes the diameter. As $K_2 \subset W \subset \bar{O}_2$, we have

$$\sup_{y\in K_2} |\varphi_{\varepsilon,x}(y)| \leq \left(diam(\bar{O}_2)^n \int_{O_2} |\frac{\partial^n}{\partial y^n} \varphi_{\varepsilon,x}(y)|^2 \, \mathrm{d}y. \right)^{1/2}.$$

Set

$$\psi_\varepsilon(x) = \int_{O_2} |\frac{\partial^n}{\partial y^n} \varphi_{\varepsilon,x}(y)|^2 \, \mathrm{d}y \ , \ \forall x \in O_1.$$

Since H_ε and ρ are $\mathcal{C}^\infty$-functions, ψ_ε is continuous on K_1. Therefore, ψ_ε has its maximum at a point $x(\varepsilon) \in K_1$. Consequently,

$$\sup_{x\in K_1, y\in K_2} |H_\varepsilon(x,y)| = \sup_{x\in K_1, y\in K_2} |\varphi_{\varepsilon,x}(y)| \leq \sqrt{diam(\bar{O}_2)^n \psi_\varepsilon(x(\varepsilon))}.$$

By choosing $f_\varepsilon = \overline{\frac{\partial^n}{\partial y^n} \varphi_{\varepsilon,x(\varepsilon)}(\cdot)}$, one has $(f_\varepsilon)_\varepsilon$ in $\mathcal{E}_M(Y)$, since $x(\varepsilon) \in K_1$, so its class f is an element of $\mathcal{G}(Y)$, as well as $\frac{\partial^n}{\partial y^n} f$. Since $W \subset O_2$, one has

$$\begin{aligned}
(\widehat{H}(\frac{\partial^n}{\partial y^n} f))_\varepsilon(x(\varepsilon)) &= \int_{O_2} H_\varepsilon(x(\varepsilon), y) \frac{\partial^n}{\partial y^n} f_\varepsilon(y) \, \mathrm{d}y \\
&= \int_{O_2} \varphi_{\varepsilon,x(\varepsilon)}(y) \frac{\partial^n}{\partial y^n} \overline{\frac{\partial^n}{\partial y^n} \varphi_{\varepsilon,x(\varepsilon)}(y)} \, \mathrm{d}y \\
&= (-1)^n \int_{O_2} |\frac{\partial^n}{\partial y^n} \varphi_{\varepsilon,x(\varepsilon)}(y)|^2 \, \mathrm{d}y = (-1)^n \psi_\varepsilon(x(\varepsilon)),
\end{aligned}$$

since $(x,y) \mapsto \varphi_{\varepsilon,x}(y)$ is another representative of H on $O_1 \times Y$. As $\frac{\partial^n f}{\partial y^n}$ belongs to $\mathcal{G}(Y)$, we have $\widehat{H}(\frac{\partial^n f}{\partial y^n}) = 0$ in $\mathcal{G}(X)$, thus its representative is in $\mathcal{I}(O_1)$. Since $(x(\varepsilon))_\varepsilon$ is bounded in K_1, $(\widehat{H}(\frac{\partial^n f}{\partial y^n}))_\varepsilon(x(\varepsilon))$ is in I_A and the previous equality implies that $(\psi_\varepsilon(x(\varepsilon)))_\varepsilon \in I_A$, that is $\psi_\varepsilon(x(\varepsilon)) = O(\varepsilon^n)$ as $\varepsilon \to 0$, with $n \in \mathbb{N}$. Furthermore

$$\sup_{(x,y)\in K} |H_\varepsilon(x,y)| \leq \sup_{(x,y)\in K_1\times K_2} |H_\varepsilon(x,y)| \leq \sqrt{diam(\bar{O}_2)^n \psi_\varepsilon(x(\varepsilon))}.$$

Hence $\sup_{(x,y)\in K} |H_\varepsilon(x,y)| = O(\varepsilon^p)$ as $\varepsilon \to 0$, for all $p \in \mathbb{N}$. Thus $(H_\varepsilon)_\varepsilon$ is in $\mathcal{I}(O_1 \times Y)$, which ends the proof.

Corollary 23 *The linear map* $\widehat{\ }$*, defined in remark 19, is injective.*

Remark 24 *If H is in $\mathcal{G}_{L^2}(X \times Y)$, then H can be embedded in $\mathcal{G}(X \times Y)$ using Sobolev's embeddings, but H may not be properly supported. Nevertheless, we can define a generalized integral operator acting on $\mathcal{G}_{L^2}$ type spaces as follows:*

$$\begin{array}{rccl} \widehat{H}: & \mathcal{G}_{L^2}(Y) & \to & \mathcal{G}_{L^2}(X) \\ & f & \mapsto & Cl\left(x \mapsto \int_Y H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y\right)_\varepsilon, \end{array}$$

independently of the representative $(H_\varepsilon)_\varepsilon$ (resp. $(f_\varepsilon)_\varepsilon$) of H (resp. f).

Indeed, set $\Phi_\varepsilon(x) = \int_Y H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y$ for ε in $(0,1]$ and $x \in X$. The function Φ_ε is well defined, since $(H_\varepsilon)_\varepsilon \in \mathcal{E}_{L^2}(X \times Y)$ and $(f_\varepsilon)_\varepsilon \in \mathcal{E}_{L^2}(Y)$. Moreover, we have, for all $x \in X$ and ε in $(0,1]$

$$|\Phi_\varepsilon(x)| \leq \left| \int_Y H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y \right| \leq \|H_\varepsilon(x,\cdot)\|_2 \, \|f_\varepsilon\|_2 \, .$$

Thus Φ_ε is in $L^2(X)$, with $\|\Phi_\varepsilon\|_2 \leq \|H_\varepsilon\|_2 \, \|f_\varepsilon\|_2$. As for all $\alpha \in \mathbb{N}^n$, $\partial^\alpha \Phi_\varepsilon$ exists and satisfies $\|\partial^\alpha \Phi_\varepsilon\|_2 \leq \|\partial_x^\alpha H_\varepsilon\|_2 \, \|f_\varepsilon\|_2$: $(\Phi_\varepsilon)_\varepsilon$ is in $\mathcal{E}_{L^2}(X)$. A straightforward computation shows that $Cl\,(\Phi_\varepsilon)_\varepsilon$ does not depend on the representative of H and f. Thus, the operator $\widehat{H}$ is well defined.

4 Link with the classical theory and regularity properties

In this section, we compare our definition to the classical one, that is when H is a $\mathcal{C}^\infty$-function or a distribution. Let X (resp. Y) be an open subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$).

Theorem 25 *If $h \in \mathcal{C}^\infty(X \times Y)$ then the diagram*

$$\begin{array}{ccc} \mathcal{E}'(Y) & \overset{\widehat{h}}{\to} & \mathcal{C}^\infty(X) \\ \downarrow i_S & & \downarrow \sigma \\ \mathcal{G}_C(Y) & \overset{\widehat{\sigma(h)}}{\to} & \mathcal{G}(X) \end{array}$$

is commutative.

<u>Proof</u>. We have to prove that $\sigma \circ \widehat{h} = \widehat{\sigma(h)} \circ i_S$. Let T be in $\mathcal{E}'(Y)$. Due to the local structure of distributions, there exist $r \in \mathbb{N}$, a finite family $(f_\alpha)_{0\leq|\alpha|\leq r}$ $(\alpha \in \mathbb{N}^n)$ of continuous on $\mathbb{R}^n$ having their support contained in the same arbitrary neighborhood of the support of T, such that $T = \sum_{0\leq|\alpha|\leq r} \partial^\alpha f_\alpha$. By the linearity of the operators under consideration, we can assume that $T = \partial^\alpha f$, where f is a continuous function on $\mathbb{R}^n$ whose support is contained in a neighborhood of the support of T. In this proof and the one of the following theorem, the exponents of the mollifiers are 1 for the space X, 2 for Y and none for $X \times Y$. In this case, a representative of $\sigma \circ \widehat{h}(T)$ is defined, for all $x \in X$, by

$$\widehat{h}(T)(x) = \langle T, h(x,\cdot)\rangle = \langle \partial^\alpha f, h(x,\cdot)\rangle = (-1)^{|\alpha|}\langle f, \partial_y^\alpha h(x,\cdot)\rangle = (-1)^{|\alpha|}\int f(y)\partial_y^\alpha h(x,y)\,\mathrm{d}y.$$

A representative of $\widehat{\sigma(h)} \circ i_S(T)$ is

$$\int h(x,y)(T * \Theta_\varepsilon^2)(y)\,\mathrm{d}y = \int h(x,y)(f * \partial^\alpha \Theta_\varepsilon^2)(y)\,\mathrm{d}y = \int\int h(x,y)f(\lambda)\partial^\alpha\Theta_\varepsilon^2(y-\lambda)\,\mathrm{d}\lambda\mathrm{d}y.$$

As the functions f and $\partial^\alpha\Theta_\varepsilon^2$ have compact supports, the two previous integrals are integrals on compacts sets. Thus, we can apply Fubini's theorem and obtain

$$\int\int h(x,y)f(\lambda)\partial^\alpha\Theta_\varepsilon^2(y-\lambda)d\lambda dy = (-1)^{|\alpha|}\int\left(\partial_y^\alpha h(x,\cdot) * \Theta_\varepsilon^2\right)(\lambda)f(\lambda)\,\mathrm{d}\lambda.$$

As the function $y \mapsto \left(\partial_y^\alpha h(x,\cdot) * \Theta_\varepsilon^2\right)(y)$ is a representative of $i_S(\partial_y^\alpha h(x,\cdot))$ in $\mathcal{G}(Y)$ and since $\partial_y^\alpha h(x,\cdot)$ is a $\mathcal{C}^\infty$-function and $i_s\mid_{\mathcal{C}^\infty} = \sigma$, one has

$$\left(\partial_y^\alpha h(x,\cdot) * \Theta_\varepsilon^2 - \partial_y^\alpha h(x,\cdot)\right)_\varepsilon \in \mathcal{I}(Y).$$

Moreover, f is compactly supported so the difference of the representatives of $\widehat{\sigma(h)} \circ i_S(T)$ and $\sigma \circ \widehat{h}(T)$ is in $\mathcal{I}(X)$. Thus $\widehat{\sigma(h)} \circ i_S(T) = \sigma \circ \widehat{h}(T)$ in $\mathcal{G}(X)$, which implies the required result.

Definition 26 *The kernel $H \in \mathcal{G}(X \times Y)$ of a generalized integral operator is called regular when $\widehat{H}(\mathcal{G}_C(Y) \subset \mathcal{G}^\infty(X)$, where, for all Ω open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$),*

$$\mathcal{G}^\infty(\Omega) = \mathcal{E}^\infty(\Omega)/\mathcal{I}(\Omega).$$

with

$$\mathcal{E}^\infty(\Omega) = \left\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} \;/\; \forall K \Subset \Omega\ ,\ \exists n \in \mathbb{N},\ \forall l \in \mathbb{N}\ ,\ p_{K,l}(u_\varepsilon) = O(\varepsilon^{-n}),\ \textit{as}\ \varepsilon \to 0\right\}.$$

The reader can find more details about this algebra in [17].

Proposition 27 *If h is in $\mathcal{C}^\infty(X \times Y)$, then $\sigma(h)$ is regular in the above sense.*

<u>Proof</u>. Let f be in $\mathcal{G}_C(Y)$. Then there exists K_2 compact of Y and $(f_\varepsilon)_\varepsilon$ a representative of f such that *supp* $f_\varepsilon \subset K_2$. Consequently, a representative of $\widehat{\sigma(h)}(f)$ is $(\psi_\varepsilon)_\varepsilon$ with

$$\psi_\varepsilon : x \mapsto \int_{K_2} h(x,y)f_\varepsilon(y)\,\mathrm{d}y$$

and, for all K_1 compact of X, α in $\mathbb{N}^m$, x in K_1,

$$\left|\partial_x^\alpha\left(\int_{K_2} h(x,y)f_\varepsilon(y)\,\mathrm{d}y\right)\right| \leq Vol(K_2)\sup_{(x,y)\in K_1\times K_2}|\partial_x^\alpha h(x,y)|\sup_{y\in K_2}|f_\varepsilon(y)| \leq C(\alpha)\varepsilon^{-q},$$

as ε tends to 0, where q does not depend on α. That shows that $\widehat{\sigma(h)}(f)$ is in $\mathcal{G}^\infty(X)$.

Theorem 28 *If $h \in \mathcal{D}'(X \times Y)$ then the diagram*

$$\begin{array}{ccc} \mathcal{D}(Y) & \overset{\widehat{h}}{\to} & \mathcal{D}'(X) \\ \downarrow \sigma & & \downarrow i_S \\ \mathcal{G}_C(Y) & \overset{\widehat{i_S(h)}}{\to} & \mathcal{G}(X) \end{array}$$

is commutative.

Proof. We have to prove that $i_S \circ \widehat{h} = \widehat{i_S(h)} \circ \sigma$. Let f be in $\mathcal{D}(Y)$. A representative of $i_S \circ \widehat{h}(f)$ is defined, for all $x \in X$, by

$$\begin{aligned}\left(\gamma_\varepsilon^1 \widehat{h}(f) * \Theta_\varepsilon^1\right)(x) &= \langle \gamma_\varepsilon^1 \widehat{h}(f), \{\xi \mapsto \Theta_\varepsilon^1(x-\xi)\}\rangle \\ &= \langle \widehat{h}(f), \{\xi \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\}\rangle \\ &= \langle h, \{\xi \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\} \otimes f\rangle.\end{aligned}$$

A representative of $\widehat{i_S(h)} \circ \sigma(f)$ is, for all $x \in X$,

$$\begin{aligned}&\int \langle \gamma_\varepsilon^1 \otimes \gamma_\varepsilon^2 h, \{(\xi,\eta) \mapsto \Theta_\varepsilon^1(x-\xi)\Theta_\varepsilon^2(y-\eta)\}\rangle f(y)\,\mathrm{d}y \\ &= \langle \gamma_\varepsilon^1 \otimes \gamma_\varepsilon^2 h, \{(\xi,\eta) \mapsto \Theta_\varepsilon^1(x-\xi) \int \Theta_\varepsilon^2(y-\eta) f(y)\,\mathrm{d}y\}\rangle \\ &= \langle h, \{(\xi,\eta) \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\gamma_\varepsilon^2(\eta) \int \Theta_\varepsilon^2(y-\eta) f(y)\,\mathrm{d}y\}\rangle \\ &= \langle h, \{(\xi,\eta) \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\gamma_\varepsilon^2(\eta)\left(\Theta_\varepsilon^2 * f\right)(\eta)\}\rangle,\end{aligned}$$

since Θ_ε^2 is even. Consequently, the difference of these representatives of $i_S \circ \widehat{h}(f)$ and $\widehat{i_S(h)} \circ \sigma(f)$ is equal to

$$\langle h, \{\xi \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\} \otimes \{f - \gamma_\varepsilon^2\left(\Theta_\varepsilon^2 * f\right)\}\rangle.$$

As $f - \gamma_\varepsilon^2\left(\Theta_\varepsilon^2 * f\right)$ is a representative of $\sigma(f) - i_S(f)$, which is equal to zero in $\mathcal{G}(Y)$ since $i_s\,|_{\mathcal{C}^\infty(Y)} = \sigma$, this representative is in $\mathcal{I}(Y)$. Furthermore, $(\gamma_\varepsilon^1\Theta_\varepsilon^1(x-\cdot))_\varepsilon$ is in $\mathcal{E}_M(X)$. Thus, $(\{\xi \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\} \otimes \{f - \gamma_\varepsilon^2\left(\Theta_\varepsilon^2 * f\right)\})_\varepsilon$ is in $\mathcal{I}(X \times Y)$. Let K_1 be a compact of X, then there are Ω_1 relatively compact open subset of X such that $K_1 \subset \Omega_1$ and $\varepsilon_1 > 0$ such that, for all x in K_1, for all $\varepsilon < \varepsilon_1$, $\gamma_\varepsilon^1\Theta_\varepsilon^1(x-\cdot)$ is in $\mathcal{D}(\Omega_1)$. Furthermore, by setting *supp* $f = K_2$, there are Ω_2 relatively compact open subset of Y such that $K_2 \subset \Omega_2$ and $\varepsilon_2 > 0$ such that, for all $\varepsilon < \varepsilon_2$, $f - \gamma_\varepsilon^2\left(\Theta_\varepsilon^2 * f\right)$ is in $\mathcal{D}(\Omega_2)$. Consequently, $\{\xi \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\} \otimes \{f - \gamma_\varepsilon^2\left(\Theta_\varepsilon^2 * f\right)\}$ is in $\mathcal{D}(\Omega_1 \times \Omega_2)$. By using the local structure of distributions, one can write h as a derivative of a continuous function on $\mathbb{R}^m \times \mathbb{R}^n$, whose support is contained in an arbitrary neighborhood of $\Omega_1 \times \Omega_2$ and one shows that

$$\left(\langle h, \{\xi \mapsto \gamma_\varepsilon^1(\xi)\Theta_\varepsilon^1(x-\xi)\} \otimes \{f - \gamma_\varepsilon^2\left(\Theta_\varepsilon^2 * f\right)\}\rangle\right)_\varepsilon$$

is in $\mathcal{I}(X)$, which implies the required result.

Proposition 29 *If H is in $\mathcal{G}^\infty(X \times Y)$ then H is regular in the sense given by definition 26.*

Proof. Let H be in $\mathcal{G}^\infty(X \times Y)$ and f be in $\mathcal{G}_C(Y)$. There exists K_2 compact of Y and $(f_\varepsilon)_\varepsilon$ a representative of f such that *supp* $f_\varepsilon \subset K_2$. Let us denote by $(H_\varepsilon)_\varepsilon$ a representative of H. A representative of $\widehat{H}(f)$ is

$$(x,y) \mapsto \int_{K_2} H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y$$

and, for all K_1 compact of X, α in $\mathbb{N}^m$, x in K_1,

$$\left|\partial_x^\alpha \left(\int_{K_2} H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{d}y \right)\right| \leq Vol(K_2) \sup_{(x,y)\in K_1\times K_2} |\partial_x^\alpha H_\varepsilon(x,y)| \sup_{y\in K_2} |f_\varepsilon(y)| \leq C(\alpha)\varepsilon^{-q},$$

as ε tends to 0, where q does not depend on α. That shows that $\widehat{H}(f)$ is in $\mathcal{G}^\infty(X)$.

This result has also been proved in [10].

5 Composition of generalized integral operators

5.1 Operators with kernel in $\mathcal{G}_{ps}(.)$

Theorem 30 *Let X, Y and Ξ be three open subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively and $H_1 \in \mathcal{G}_{ps}(X\times\Xi)$, $H_2 \in \mathcal{G}_{ps}(\Xi\times Y)$. The operators $\widehat{H}_1 : \mathcal{G}(\Xi)\to\mathcal{G}(X)$ and $\widehat{H}_2 : \mathcal{G}(Y)\to\mathcal{G}(\Xi)$ can be composed. Moreover, $\widehat{H}_1\circ\widehat{H}_2$ is a generalized integral operator, whose kernel is L, defined by $L(\cdot_1,\cdot_2) = \int_\Xi H_1(\cdot_1,\xi)H_2(\xi,\cdot_2)\,\mathrm{d}\xi$ (with the meaning of notation 14) and L belongs to $\mathcal{G}_{ps}(X\times Y)$.*

Proof. For all f in $\mathcal{G}(Y)$, $\widehat{H}_2(f)$ is well defined in $\mathcal{G}(\Xi)$, then we can define $\widehat{H}_1(\widehat{H}_2(f))$ in $\mathcal{G}(X)$ and $\widehat{H}_1\circ\widehat{H}_2$ is well defined. We have to show that L is well defined, properly supported and that $\widehat{H}_1\circ\widehat{H}_2 = \widehat{L}$. We set $\Phi(\cdot_1,\cdot_2,\cdot_3) = H_1(\cdot_1,\cdot_3)H_2(\cdot_3,\cdot_2)$ ($\cdot_3$ refers to the ξ variable). Choose O_1 (resp. O_2) a relatively compact open subset of X (resp. Y).

Since $H_1 \in \mathcal{G}_{ps}(X\times\Xi)$, there exists a compact subset $K_3\subset\Xi$ such that

$$supp\ H_1 \cap (O_1\times\Xi) \subset O_1\times K_3.$$

Therefore, $supp\ \Phi\cap(O_1\times O_2\times\Xi)\subset O_1\times O_2\times K_3$ and Φ is in $\mathcal{G}_{ps}(X\times Y\times\Xi)$. Proposition 13 implies the existence of L in $\mathcal{G}(X\times Y)$, denoted by $\int H_1(\cdot_1,\xi)H_2(\xi,\cdot_2)\,\mathrm{d}\xi$.

With the same notations as above, since $H_2\in\mathcal{G}_{ps}(X\times\Xi)$, there exists for $O_3 = \overset{\circ}{K}_3$, a compact subset $K_2\subset Y$ such that

$$supp\ H_2\cap(O_3\times Y)\subset O_3\times K_2.$$

We have $\Phi_{|O_1\times(Y\backslash K_2)\times\Xi} = 0$ since $H_{1|O_1\times(\Xi\backslash K_3)} = 0$ and $H_{2|K_3\times(Y\backslash K_2)} = 0$. Therefore, L is null on $O_1\times(Y\backslash K_2)$, so $supp\ L\cap(O_1\times Y)$ is included in $O_1\times K_2$ which shows that L is in $\mathcal{G}_{ps}(X\times Y)$.

Moreover, for any $f\in\mathcal{G}(Y)$, we have (the compact sets on which the integration are performed are indicated contrary to the notations above),

$$\begin{aligned}
\widehat{L}(f)_{|O_1} &= \int_{K_2} L(\cdot_1,y)\,f(y)\,\mathrm{d}y \\
&= \int_{K_2}\left(\int_{K_3} H_1(\cdot_1,\xi)H_2(\xi,y)\,\mathrm{d}\xi\right) f(y)\,\mathrm{d}y \\
&= \int_{K_2\times K_3} H_1(\cdot_1,\xi)H_2(\xi,y)f(y)\,\mathrm{d}\xi\mathrm{d}y \text{ (by fubini's theorem)} \\
&= \int_{K_3} H_1(\cdot_1,\xi)\left(\int_{K_2} H_2(\xi,y)f(y)\,\mathrm{d}\xi\right)\mathrm{d}y = \widehat{H}_1(\widehat{H}_2(f))_{|O_1}.
\end{aligned}$$

Using the sheaf structure of $\mathcal{G}(X)$, it follows that $\widehat{L}(f) = \left(\widehat{H}_1\circ\widehat{H}_2\right)(f)$.

Remark 31 *It is straightforward to verify that the composition of generalized integral operators is associative.*

Example 32 *Take* $\delta \in \mathcal{D}'(\mathbb{R})$ *the classical delta function and* χ *an integrable function on* $\mathbb{R}$. *Set* $H = \delta_x \otimes \mathbf{1}_y$ *and* $K = \chi \otimes \delta_y$. *Such* H *and* K *are distributions on* $\mathbb{R}^2$. *The kernel operators associated with* H *and* K *are respectively*

$$\begin{array}{rcccc} \widehat{H} & : & \mathcal{D}(\mathbb{R}) & \to & \mathcal{D}'(\mathbb{R}) \\ & & f & \mapsto & \delta_x \int f(y)\,\mathrm{d}y \end{array} \qquad \text{and} \qquad \begin{array}{rcccc} \widehat{K} & : & \mathcal{D}(\mathbb{R}) & \to & \mathcal{D}'(\mathbb{R}) \\ & & f & \mapsto & f(0)\chi. \end{array}$$

By noticing that $\widehat{H} : L^1(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ *and* $\widehat{K} : \mathcal{D}(\mathbb{R}) \to L^1(\mathbb{R})$, *one can define* $\widehat{H} \circ \widehat{K}$ *by the following*

$$\begin{array}{rccc} \widehat{H} \circ \widehat{K} : & \mathcal{D}(\mathbb{R}) & \to & \mathcal{D}'(\mathbb{R}) \\ & f & \mapsto & \delta_x f(0) \int \chi(\xi)\,\mathrm{d}\xi, \end{array}$$

which admits as kernel

$$\delta_x \delta_y \int \chi(\xi)\,\mathrm{d}\xi. \tag{4}$$

Conversely, $\widehat{K} \circ \widehat{H}$ *cannot be defined classically. We are going to define it in the context of generalized functions. By using the notations of paragraph 2.2, one has*

$$i_S(H) = Cl\left((x,y) \mapsto (\Theta_\varepsilon \otimes \mathbf{1}_y)(x,y)\right)_\varepsilon$$

and

$$i_S(K) = Cl\left((x,y) \mapsto ((\chi * \Theta_\varepsilon) \otimes \Theta_\varepsilon)(x,y)\right)_\varepsilon,$$

so $\widehat{i_S(K)} \circ \widehat{i_S(H)} = \widehat{L}$ *is well defined from* $\mathcal{G}_C(\mathbb{R})$ *to* $\mathcal{G}(\mathbb{R})$, *with*

$$\begin{aligned} L &= Cl\left((x,y) \mapsto \int ((\chi * \Theta_\varepsilon) \otimes \rho_\varepsilon)(x,\xi)(\Theta_\varepsilon \otimes \mathbf{1}_y)(\xi,y)\,\mathrm{d}\xi\right)_\varepsilon \\ &= Cl\left((x,y) \mapsto (\chi * \Theta_\varepsilon)(x) \int (\Theta_\varepsilon(\xi))^2\,\mathrm{d}\xi\right)_\varepsilon, \end{aligned}$$

that is $L = \int i_S(\delta)^2(\xi)\,\mathrm{d}\xi \cdot i_S(\chi) \otimes \mathbf{1}_y = \int \delta^2(\xi)\,\mathrm{d}\xi \cdot \chi \otimes \mathbf{1}_y$ *with a slight abuse of notation. For the case of* $\widehat{H} \circ \widehat{K}$, *one can easily verify that the image by* i_S *of the classical distributional kernel given by (4) is equal to the kernel obtained by theorem 30.*

Corollary 33 *For* H *in* $\mathcal{G}_{ps}(X^2)$ *(*X *open subset of* $\mathbb{R}^d$*) and* $n \geq 2$, $\widehat{H}^n = \underbrace{\widehat{H} \circ \cdots \circ \widehat{H}}_{n \text{ times}} : \mathcal{G}(X) \to \mathcal{G}(X)$ *is a well defined generalized integral operator, whose kernel* $L_n \in \mathcal{G}_{ps}(X^2)$ *is defined by*

$$L_n(\cdot_1, \cdot_2) = \int H(\cdot_1, \xi_1) H(\xi_1, \xi_2) \cdots H(\xi_{n-1}, \cdot_2)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-1},$$

with the meaning of notation 14.

Proof. We prove this proposition by induction. Theorem 30 gives the result for $n = 2$, by considering $X = \Xi = Y$ and $H_1 = H_2$. Suppose now that $\widehat{H}^{n-1}$ is well defined with its kernel L_{n-1} defined by

$$L_{n-1}(\cdot_1, \cdot_2) = \int_{X^{n-2}} H(\cdot_1, \xi_1) H(\xi_1, \xi_2) \cdots H(\xi_{n-2}, \cdot_2)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-2}$$

in $\mathcal{G}_{ps}(X^2)$.

We apply theorem 30 with $H_1 = H$ and $H_2 = L_{n-1}$. It follows that $\widehat{H}^n = \widehat{H}^{n-1} \circ \widehat{H}$ is a well defined operator which admits as kernel

$$\begin{aligned} L(\cdot_1, \cdot_2) &= \int_X \left(\int_{X^{n-2}} H(\cdot_1, \xi_1) H(\xi_1, \xi_2) \cdots H(\xi_{n-2}, \xi_{n-1}) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-2} \right) H(\xi_{n-1}, \cdot_2) \, \mathrm{d}\xi_{n-1} \\ &= \int_{X^{n-1}} H(\cdot_1, \xi_1) H(\xi_1, \xi_2) \cdots H(\xi_{n-2}, \xi_{n-1}) H(\xi_{n-1}, \cdot_2) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-2} \mathrm{d}\xi_{n-1}, \end{aligned}$$

by applying Fubini's theorem. This is possible since the integrals are always performed on compact sets, using the local definition (proposition 13). Thus $L(\cdot_1, \cdot_2)$, which is properly supported, is equal to $L_n(\cdot_1, \cdot_2)$ which satisfies the required properties.

For operators with compactly supported kernels, we can give a more precise result.

Proposition 34 *With the notations of theorem 30, for H_1 in $\mathcal{G}_C(X \times \Xi)$ and H_2 in $\mathcal{G}_C(\Xi \times Y)$, $\widehat{H}_1 \circ \widehat{H}_2 : \mathcal{G}(Y) \to \mathcal{G}_C(X)$ is a generalized integral operator whose kernel L is an element of $\mathcal{G}_C(X \times Y)$. Moreover, if K_1 (resp. K_2 ; K_3) is a compact subset of X, (resp. Ξ ; Y) such that the support of H_1 (resp. H_2) is contained in the interior of $K_1 \times K_2$ (resp. $K_2 \times K_3$) then L can be defined globally by $L(\cdot_1, \cdot_2) = \int_{K_2} H_1(\cdot_1, \xi) H_2(\xi, \cdot_2) \, \mathrm{d}\xi$ and the support of L is contained in $K_1 \times K_3$.*

Proof. We only have to verify the assertions related to L. Denote by $(H_{1,\varepsilon})_\varepsilon$ (resp. $(H_{2,\varepsilon}))_\varepsilon$ a representative of H_1 (resp. H_2) and set $O_1 = X \setminus K_1$, $O_3 = Y \setminus K_3$. The net

$$\left((x, y) \mapsto \int_{K_2} H_{1,\varepsilon}(x, \xi) H_{2,\varepsilon}(\xi, y) \, \mathrm{d}\xi \right)_\varepsilon$$

is a representative of L, which justifies the global definition of L. For $U \Subset X$ and $V \Subset Y$ such that $U \times V \subset X \times Y \backslash K_1 \times K_3$, we have either $U \subset O_1$ or $V \subset O_3$. We shall suppose, for example, that $U \subset O_1$. For $(x, y) \in U \times V$, we have

$$|L_\varepsilon(x, y)| = \left| \int_{K_2} H_{1,\varepsilon}(x, \xi) H_{2,\varepsilon}(\xi, y) \, \mathrm{d}\xi \right| \leq Vol(K_2) p_{U \times K_2}(H_{1,\varepsilon}) p_{K_2 \times V}(H_{2,\varepsilon}).$$

Therefore

$$p_{U \times V}(L_\varepsilon) \leq Vol(K_2) p_{U \times K_2}(H_{1,\varepsilon}) p_{K_2 \times V}(H_{2,\varepsilon}). \tag{5}$$

As $(H_{1,\varepsilon|O_1 \times \Xi})_\varepsilon$ is in $\mathcal{I}(O_1 \times \Xi)$ and $U \cap K_2 \subset O_1 \times \Xi$, it follows that $p_{U \times K_2}(H_{1,\varepsilon}) = O(\varepsilon^m)$ as $\varepsilon \to 0$, for all $m \in \mathbb{N}$. Since $(H_{2,\varepsilon})_\varepsilon$ is in $\mathcal{E}_M(\Xi \times Y)$, relation (5) implies that $p_{U \times V}(L_\varepsilon) = O(\varepsilon^m)$ as $\varepsilon \to 0$, for all $m \in \mathbb{N}$. Finally, $(L_\varepsilon)_\varepsilon$ satisfies the null estimate of order 0 for all compact subsets included in $X \times Y \backslash K_1 \times K_3$. Therefore, $L_{|X \times Y \backslash K_1 \times K_3} = 0$ and the support of L is contained in $K_1 \times K_3$.

From corollary 33 and proposition 34, we immediately deduce the following:

Corollary 35 *If H belongs to $\mathcal{G}_C(X^2)$, with supp $H \subset (\mathring{K}_2)^2$ ($K_2 \Subset X$), then the image of $\widehat{H}^n$ is included in $\mathcal{G}_C(X)$ and the support of L_n is contained in $(\mathring{K}_2)^2$. Moreover, $L_n \in \mathcal{G}_C(X^2)$ can be defined globally by*

$$L_n(\cdot_1, \cdot_2) = \int_{K_2} H(\cdot_1, \xi_1) H(\xi_1, \xi_2) \cdots H(\xi_{n-1}, \cdot_2) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-1}.$$

5.2 Operators with kernel in $\mathcal{G}_{L^2}(\cdot)$

Proposition 36 *For H_1 in $\mathcal{G}_{L^2}(X\times\Xi)$ and H_2 in $\mathcal{G}_{L^2}(\Xi\times Y)$, $\widehat{H}_1\circ\widehat{H}_2:\mathcal{G}_{L^2}(Y)\to\mathcal{G}_{L^2}(X)$ is a generalized integral operator whose kernel is $L\in\mathcal{G}_{L^2}(X\times Y)$ defined by*

$$L = Cl\left(\left((x,y)\mapsto\int_{\Xi}H_{1,\varepsilon}(x,\xi)H_{2,\varepsilon}(\xi,y)\,\mathrm{d}\xi\right)_{\varepsilon}\right),$$

where $(H_{1,\varepsilon})_\varepsilon$ (resp. $(H_{2,\varepsilon})_\varepsilon$) is a representative of H_1 (resp. H_2).

Proof. With the notations given in the proposition, set

$$L_\varepsilon(x,y)=\int_{\Xi}H_{1,\varepsilon}(x,\xi)H_{2,\varepsilon}(\xi,y)\,\mathrm{d}\xi,\ \text{for all } (x,y) \text{ in } X\times Y,$$

Then

$$\begin{aligned}\|L_\varepsilon\|_2^2 &= \int\int\left(\int H_{1,\varepsilon}(x,\xi)H_{2,\varepsilon}(\xi,y)\,\mathrm{d}\xi\right)^2\mathrm{d}x\mathrm{d}y\\ &\le \int\int\|H_{1,\varepsilon}(x,\cdot)\|_2^2\|H_{2,\varepsilon}(\cdot,y)\|_2^2\,\mathrm{d}x\mathrm{d}y\\ &\le \int\|H_{1,\varepsilon}(x,\cdot)\|_2^2\,\mathrm{d}x\int\|H_{2,\varepsilon}(\cdot,y)\|_2^2\,\mathrm{d}y=\|H_{1,\varepsilon}\|_2^2\|H_{2,\varepsilon}\|_2^2.\end{aligned}$$

Furthermore, for all $\alpha,\beta\in\mathbb{N}^d\setminus\{(0,0)\}$ and $(x,y)\in X\times Y$, by derivation in the sense of distributions, we have

$$\partial_{x\,y}^{(\alpha,\beta)}L_\varepsilon(x,y)=\int_{\Xi}\partial_x^\alpha H_{1,\varepsilon}(x,\xi)\,\partial_y^\beta H_{2,\varepsilon}(\xi,y)\,\mathrm{d}\xi\quad(\text{with } \partial_{x\,y}^{(\alpha,\beta)}L_\varepsilon=\frac{\partial^{|\alpha|+|\beta|}L_\varepsilon}{\partial x^\alpha\partial y^\beta}).$$

Thus

$$\left\|\partial_{x\,y}^{(\alpha,\beta)}L_\varepsilon\right\|_2^2\le\|\partial_x^\alpha H_{1,\varepsilon}\|_2^2\left\|\partial_y^\beta H_{2,\varepsilon}\right\|_2^2.$$

As $(H_{1,\varepsilon})_\varepsilon\in\mathcal{E}_{L^2}(X\times\Xi)$ and $(H_{2,\varepsilon})_\varepsilon\in\mathcal{E}_{L^2}(\Xi\times Y)$, we get that $(L_\varepsilon)_\varepsilon$ is in $\mathcal{E}_{L^2}(X\times Y)$. We set $L=Cl(L_\varepsilon)_\varepsilon$ in $\mathcal{G}_{L^2}(X\times Y)$. Moreover, for any $f\in\mathcal{G}_{L^2}(X)$ and any of its representative $(f_\varepsilon)_\varepsilon\in\mathcal{E}_{L^2}(\Omega)$, a representative of $\left(\widehat{H}_1\circ\widehat{H}_2\right)(f)$ is given by the net $(\Psi_\varepsilon)_\varepsilon$ with

$$\Psi_\varepsilon(x)=\int_{\Xi}H_{1,\varepsilon}(x,\xi)\left(\int_Y H_{2,\varepsilon}(\xi,y)f_\varepsilon(y)\,\mathrm{d}y\right)\mathrm{d}\xi,\quad\text{for all } x\in X.$$

Then, for all $x\in X$,

$$\Psi_\varepsilon(x)=\int_{\Xi}\int_Y H_{1,\varepsilon}(x,\xi)H_{2,\varepsilon}(\xi,y)f_\varepsilon(y)\,\mathrm{d}y\mathrm{d}\xi=\int_Y L_\varepsilon(x,y)f_\varepsilon(y)\,\mathrm{d}y,$$

by applying Fubini's theorem. Thus, $(\Psi_\varepsilon)_\varepsilon$ is a representative of $\widehat{L}(f)$ and $\widehat{H}_1\circ\widehat{H}_2=\widehat{L}$.

Corollary 37 *For H in $\mathcal{G}_{L^2}(X^2)$ (X open subset of $\mathbb{R}^d$), for all $n\ge 2$, $\widehat{H}^n:\mathcal{G}_{L^2}(X)\to\mathcal{G}_{L^2}(X)$ is a generalized integral operator whose kernel is $L_n\in\mathcal{G}_{L^2}(X^2)$ defined by $L_n=Cl\,(L_{n,\varepsilon})_\varepsilon$, with*

$$L_{n,\varepsilon}:(x,y)\mapsto\int_{X^{n-1}}H_\varepsilon(x,\xi_1)H_\varepsilon(\xi_1,\xi_2)\cdots H_\varepsilon(\xi_{n-1},y)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_{n-1},$$

where $(H_\varepsilon)_\varepsilon$ is a representative of H.

Proof. The proof of this result is an adaptation of the one of corollary 33 to the L^2-case.

6 Application: Exponential of generalized integral operators

In this section, we define the exponential of generalized integral operators in two particular cases and study some of their properties. We first define some convenient spaces. Let us set, for Ω open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$),

$$\mathcal{H}_{ln}(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^{(0,1]} \ / \ \forall K \Subset \Omega \ , \ \forall l \in \mathbb{N} \ , \ p_{K,l}(u_\varepsilon) = O(|\ln \varepsilon|) \text{ as } \varepsilon \to 0 \right\}$$

and

$$\mathcal{H}_{L^2\,ln}(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in H^\infty(\Omega)^{(0,1]} \ / \ \forall m \geq 0 \ , \ \|u_\varepsilon\|_m = O(|\ln \varepsilon|) \text{ as } \varepsilon \to 0 \right\}.$$

Then $\mathcal{H}_{ln}(\Omega)$ (resp. $\mathcal{H}_{L^2\,ln}(\Omega)$) is a linear subspace of $\mathcal{E}_M(\Omega)$ (resp. $\mathcal{E}_{L^2}(\Omega)$), but is not a subalgebra. Then we set

$$\mathcal{G}_{ln}(\Omega) = \mathcal{H}_{ln}(\Omega)/\mathcal{I}(\Omega)\,, \quad \mathcal{G}_{C\,ln}(\Omega) = \mathcal{G}_{ln}(\Omega) \cap \mathcal{G}_C(\Omega) \quad \text{and} \quad \mathcal{G}_{L^2\,ln}(\Omega) = \mathcal{H}_{L^2\,ln}(\Omega))/\mathcal{I}_{L^2}(\Omega).$$

The space $\mathcal{G}_{ln}(\Omega)$ (resp. $\mathcal{G}_{L^2\,ln}(\Omega)$) is a subvector space of $\mathcal{G}(\Omega)$ (resp. $\mathcal{G}_{L^2}(\Omega)$).

6.1 Exponential of generalized integral operators whose kernel is in $\mathcal{G}_{C\,ln}(X^2)$

Theorem-definition 38 *Let H be in $\mathcal{G}_{C\,ln}(X^2)$ (X an open subset of $\mathbb{R}^d$). For $n \geq 1$, denote by L_n the kernel of $\widehat{H}^n : \mathcal{G}(X) \to \mathcal{G}_C(X)$ defined as in corollary 35 (with $L_1 = H$) and by $(L_{n,\varepsilon})_\varepsilon$ a representative of L_n. For all $\varepsilon \in (0,1]$, the series $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$ converges for the usual topology of $\mathcal{C}^\infty(X^2)$. Its sum, denoted by S_ε, defines an element $(S_\varepsilon)_\varepsilon$ of $\mathcal{E}_M(X^2)$. Furthermore, $S = Cl\,(S_\varepsilon)_\varepsilon$ defines a compactly supported element of $\mathcal{G}(X^2)$ only depending on H.*
The well defined operator $e^{\widehat{H}} = \widehat{S} + Id$ (where Id is the operator identity) will be called the exponential *of $\widehat{H}$.*

Proof. We divide it in three parts. The first part contains the estimates of $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$ for a particular representative of L_n, constructed from a representative $(H_\varepsilon)_\varepsilon$. The second part deals with the independence of $Cl\,(S_\varepsilon)_\varepsilon$ of the chosen representative of H. The third part shows that S is compactly supported.

• Let H be in $\mathcal{G}_{C\,ln}(X^2)$ and $(H_\varepsilon)_\varepsilon$ one of its representative. According to corollary 35, we have $\widehat{H}^n = \widehat{L_n} : \mathcal{G}(X) \to \mathcal{G}_C(X)$ and $L_n \in \mathcal{G}_C(X^2)$ admits as representative $(L_{n,\varepsilon})_\varepsilon$ with

$$L_{n,\varepsilon}(\cdot_1, \cdot_2) = \int_{K^{n-1}} H_\varepsilon(\cdot_1, \xi_1) H_\varepsilon(\xi_1, \xi_2) \cdots H_\varepsilon(\xi_{n-1}, \cdot_2)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-1},$$

where K is a compact subset of X such that *supp* $H \subset \mathring{K}^2$.

For all compact subset of X^2 of the form $K_1 \times K_2$, $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$ and $(x,y) \in K_1 \times K_2$, one has

$$\left| \partial_{x\,y}^{(\alpha,\beta)} L_{2,\varepsilon}(x,y) \right| = \left| \int_K \partial_x^\alpha H_\varepsilon(x,\xi) \partial_y^\beta H_\varepsilon(\xi, y)\, \mathrm{d}\xi \right| \leq \int_K p_{K_1\times K, |\alpha|}(H_\varepsilon) p_{K\times K_2, |\beta|}(H_\varepsilon)\, \mathrm{d}\xi.$$

It follows that

$$p_{K_1\times K_2, |(\alpha,\beta)|}(L_{2,\varepsilon}) \leq Vol(K) p^2_{V^2, |(\alpha,\beta)|}(H_\varepsilon),$$

where V is a compact subset of X containing K,K_1 and K_2. By induction, we show that, for all $n \geq 2$,

$$p_{K_1\times K_2, |(\alpha,\beta)|}(L_{n,\varepsilon}) \leq Vol(K)^{n-1} p^n_{V^2, |(\alpha,\beta)|}(H_\varepsilon).$$

This last inequality implies that the series $\sum_{n\geq 1}\frac{L_{n,\varepsilon}}{n!}$ converges for the usual topology of $\mathcal{C}^\infty\left(X^2\right)$. Set, for $\varepsilon\in(0,1]$,

$$S_\varepsilon=\sum_{n=1}^{+\infty}\frac{L_{n,\varepsilon}}{n!}.$$

As L_n is in $\mathcal{G}_C(X^2)$ and since the convergence is uniform on compact sets, S_ε belongs to $\mathcal{C}^\infty(X^2)$ for all $\varepsilon\in(0,1]$. Furthermore, for all compact subset of X^2 of the form $K_1\times K_2$ and $(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d$, one has

$$\begin{aligned}p_{K_1\times K_2,|(\alpha,\beta)|}(S_\varepsilon)&\leq\sum_{n=1}^{+\infty}\frac{1}{n!}p_{K_1\times K_2,|(\alpha,\beta)|}(L_{n,\varepsilon})\\&\leq\sum_{n=1}^{+\infty}\frac{1}{n!}Vol(K)^{n-1}p^n_{V^2,|(\alpha,\beta)|}(H_\varepsilon)\leq\frac{1}{Vol(K)}e^{Vol(K)p_{V^2,|(\alpha,\beta)|}(H_\varepsilon)}.\end{aligned}$$

Since H is in $\mathcal{G}_{C\,ln}(X^2)$, $p_{V^2,|(\alpha,\beta)|}(H_\varepsilon)=O(|\ln\varepsilon|)$ as $\varepsilon\to 0$, that is there exists $k\in\mathbb{N}$ such that $p_{V^2,|(\alpha,\beta)|}(H_\varepsilon)\leq k\ln(1/\varepsilon)$, so

$$p_{K_1\times K_2,|(\alpha,\beta)|}(S_\varepsilon)\leq C_K\varepsilon^{-kVol(K)},$$

where C_K is a constant depending only on K. Consequently, $(S_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(X^2)$ and we denote by S its class in $\mathcal{G}(X^2)$.

• Let us show now that S does not depend on the choice of the representative of H. Let $(H^1_\varepsilon)_\varepsilon$ and $(H^2_\varepsilon)_\varepsilon$ be two representatives of H in $\mathcal{G}_{C\,ln}(X^2)$. From $(H^1_\varepsilon)_\varepsilon$, we define $(L^1_{n,\varepsilon})_\varepsilon$ and $(S^1_\varepsilon)_\varepsilon$, and from $(H^2_\varepsilon)_\varepsilon$, $(L^2_{n,\varepsilon})_\varepsilon$ and $(S^2_\varepsilon)_\varepsilon$. Let K be a compact subset of X such that the support of H is contained in the interior of K^2. For all $n\geq 2$, $K_1\times K_2$ compact subset of X^2 and $(x,y)\in K_1\times K_2$, one has

$$\begin{aligned}\left(L^1_{n,\varepsilon}-L^2_{n,\varepsilon}\right)(x,y)&=\int_{K^{n-1}}H^1_\varepsilon(x,\xi_1)H^1_\varepsilon(\xi_1,\xi_2)\cdots H^1_\varepsilon(\xi_{n-1},y)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_{n-1}\\&\quad-\int_{K^{n-1}}H^2_\varepsilon(x,\xi_1)H^2_\varepsilon(\xi_1,\xi_2)\cdots H^2_\varepsilon(\xi_{n-1},y)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\cdots\mathrm{d}\xi_{n-1}.\end{aligned}$$

Thus, $L^1_{n,\varepsilon}-L^2_{n,\varepsilon}$ can be written as a sum of n integrals. The term under each integral sign is itself formed by the product of n functions, one of them being equal to $H^1_\varepsilon-H^2_\varepsilon$ and the $(n-1)$ others being equal to H^1_ε or H^2_ε. Consequently,

$$p_{K_1\times K_2}\left(L^1_{n,\varepsilon}-L^2_{n,\varepsilon}\right)\leq nVol(K)^{n-1}p_{V^2}\left(H^1_\varepsilon-H^2_\varepsilon\right)\left(\max_{1\leq i\leq 2}(p_{V^2}(H^i_\varepsilon))\right)^{n-1}.$$

Since $(H^1_\varepsilon)_\varepsilon$ and $(H^2_\varepsilon)_\varepsilon$ are in $\mathcal{G}_{C\,ln}(X^2)$, there exists $k\in\mathbb{N}$ such that

$$\max_{1\leq i\leq 2}(p_{V^2}(H^i_\varepsilon))\leq k|\ln\varepsilon|.$$

As $(H^1_\varepsilon-H^2_\varepsilon)_\varepsilon$ is in $\mathcal{I}(X^2)$, for all $m\in\mathbb{N}$, there exists $C>0$ such that $p_{V^2}\left(H^1_\varepsilon-H^2_\varepsilon\right)\leq C\varepsilon^m$ for ε small enough. Then

$$p_{K_1\times K_2}\left(L^1_{n,\varepsilon}-L^2_{n,\varepsilon}\right)\leq C\varepsilon^m nVol(K)^{n-1}k^{n-1}|\ln\varepsilon|^{n-1},\ \text{for } \varepsilon \text{ small enough.}$$

Thus

$$p_{K_1\times K_2}\left(S^1_\varepsilon-S^2_\varepsilon\right)\leq C\varepsilon^m\sum_{n=1}^{+\infty}\frac{1}{(n-1)!}Vol(K)^{n-1}k^{n-1}|\ln\varepsilon|^{n-1}\leq C\varepsilon^m\varepsilon^{-kVol(K)},$$

for ε small enough. Therefore, $p_{K_1\times K_2}\left(S^1_\varepsilon - S^2_\varepsilon\right) = O\left(\varepsilon^{m'}\right)$ as $\varepsilon \to 0$, for all $m' \in \mathbb{N}$, that is $(S^1_\varepsilon - S^2_\varepsilon)_\varepsilon$ belongs to $\mathcal{I}(X^2)$. Consequently, S does not depend on the choice of the representative of H in $\mathcal{G}_{C\,ln}(X^2)$.

• It remains to prove that S is in $\mathcal{G}_C(X^2)$, in order to define $\widehat{S}$. Set $O = X^2 \setminus K^2$. For all $n \geq 2$, $K_1 \times K_2$ compact subset of O, one has

$$p_{K_1\times K_2}(L_{n,\varepsilon}) \leq Vol(K)^{n-1} p_{K_1\times K}(H_\varepsilon) p^{n-2}_{K^2}(H_\varepsilon) p_{K\times K_2}(H_\varepsilon).$$

Therefore

$$p_{K_1\times K_2}(S_\varepsilon) \leq p_{K_1\times K_2}(H_\varepsilon) + \sum_{n=2}^{+\infty} \frac{1}{n!} Vol(K)^{n-1} p_{K_1\times K}(H_\varepsilon) p^{n-2}_{K^2}(H_\varepsilon) p_{K\times K_2}(H_\varepsilon).$$

We have either $K_1 \cap K = \varnothing$ or $K_2 \cap K = \varnothing$ since $K_1 \times K_2 \subset O$. Suppose, for example, that $K_1 \cap K = \varnothing$. Then $K_1 \times K \subset O$. As H is in $\mathcal{I}(O)$ there exists, for all $m \in \mathbb{N}$, a constant $C > 0$ such that

$$p_{K_1\times K_2}(H_\varepsilon) \leq C\varepsilon^m, \quad p_{K_1\times K}(H_\varepsilon) \leq C\varepsilon^m.$$

As H is in $\mathcal{H}_{ln}(X^2)$, there exists $k > 0$ such that

$$p_{K\times K_2}(H_\varepsilon) \leq k|\ln\varepsilon|, \quad p_{K^2}(H_\varepsilon) \leq k|\ln\varepsilon|.$$

Thus

$$\begin{aligned} p_{K_1\times K_2}(S_\varepsilon) &\leq C\varepsilon^m + C\varepsilon^m \sum_{n=2}^{+\infty} \frac{1}{(n-1)!} Vol(K)^{n-1} k^{n-1} |\ln\varepsilon|^{n-1} \\ &\leq C\varepsilon^m + C\varepsilon^m e^{Vol(K)k|\ln\varepsilon|} = O\left(\varepsilon^{m-kVol(K)}\right) \text{ as } \varepsilon \to 0, \end{aligned}$$

which implies that $(S_\varepsilon)_\varepsilon$ is in $\mathcal{I}(O)$. Consequently, S is null on O and has a compact support included in K^2.

6.2 Exponential of generalized integral operators whose kernel is in $\mathcal{G}_{L^2\,ln}(X^2)$

Theorem-definition 39 *Let H be in $\mathcal{G}_{L^2\,ln}(X^2)$ (X open subset of $\mathbb{R}^d$). Denote by L_n the kernel of $\widehat{H}^n : \mathcal{G}_{L^2}(X) \to \mathcal{G}_{L^2}(X)$ defined as in corollary 37 (with $L_1 = H$) and by $(L_{n,\varepsilon})_\varepsilon$ a representative of L_n. For all $\varepsilon \in (0,1]$, the series $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$, where $(L_{n,\varepsilon})_\varepsilon \in \mathcal{H}_{L^2\,ln}(X^2)$ is a representative of L_n, converges in L^2-norm. Its sum, denoted by S_ε, defines an element $(S_\varepsilon)_\varepsilon$ of $\mathcal{E}_{L^2}(X^2)$.*
By setting $S = Cl\,(S_\varepsilon)_\varepsilon$ in $\mathcal{G}_{L^2}(X^2)$, we define the exponential *of $\widehat{H}$ as $e^{\widehat{H}} = \widehat{S} + Id$ where Id is the operator identity.*

Proof. We divide it in two parts. The first part contains the estimates of $\sum_{n\geq 1} \frac{L_{n,\varepsilon}}{n!}$ for a particular representative of L_n, constructed from a representative $(H_\varepsilon)_\varepsilon$ of H, and shows the existence of $(S_\varepsilon)_\varepsilon$. The second part deals with the independence of $Cl\,(S_\varepsilon)_\varepsilon$ with respect to the chosen representative of H.

• Let H be in $\mathcal{G}_{L^2\,ln}(X^2)$. Applying corollary 37, we have $\widehat{H}^n = \widehat{L_n} : \mathcal{G}_{L^2}(X) \to \mathcal{G}_{L^2}(X)$, for all $n \geq 2$, with $L_n \in \mathcal{G}_{L^2}(X^2)$ defined by $L_n = Cl\,(L_{n,\varepsilon})$ where, for all $\varepsilon \in (0,1]$ and $(x,y) \in X^2$,

$$L_{n,\varepsilon}(x,y) = \int H_\varepsilon(x,\xi_1) H_\varepsilon(\xi_1,\xi_2) \cdots H_\varepsilon(\xi_{n-1},y)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdots \mathrm{d}\xi_{n-1}$$

and $(H_\varepsilon)_\varepsilon$ denote a representative of H.
We are going to show by induction that $\|L_{n,\varepsilon}\|_2^n \le \|H_\varepsilon\|_2^n$, for all n greater than 2. First, for all $\varepsilon \in (0,1]$ and $(x,y) \in X^2$, one has

$$|L_{2,\varepsilon}(x,y)| = \left| \int H_\varepsilon(x,\xi) H_\varepsilon(\xi,y)\,\mathrm{d}\xi \right| \le \|H_\varepsilon(x,\cdot)\|_2 \,\|H_\varepsilon(\cdot,y)\|_2 \,.$$

Thus $\|L_{2,\varepsilon}\|_2 \le \|H_\varepsilon\|_2^2$ and the first step is done. Assume that, for all $(x,y) \in X^2$,

$$|L_{n-1,\varepsilon}(x,y)| \le \|H_\varepsilon(x,\cdot)\|_2 \,\|H_\varepsilon\|_2^{n-3}\,\|H_\varepsilon(\cdot,y)\|_2 \ \text{ and } \ \|L_{n-1,\varepsilon}\|_2 \le \|H_\varepsilon\|_2^{n-1}\,.$$

Then, for all $(x,y) \in X^2$,

$$|L_{n,\varepsilon}(x,y)| = \left| \int L_{n-1,\varepsilon}(x,\xi) H_\varepsilon(\xi,y)\,\mathrm{d}\xi \right| \le \|L_{n-1,\varepsilon}(x,\cdot)\|_2 \,\|H_\varepsilon(\cdot,y)\|_2 \,.$$

As

$$\|L_{n-1,\varepsilon}(x,\cdot)\|_2^2 \le \int \|H_\varepsilon(x,\cdot)\|_2^2 \,\|H_\varepsilon\|_2^{2n-6}\,\|H_\varepsilon(\cdot,y)\|_2^2 \,\mathrm{d}y \le \|H_\varepsilon(x,\cdot)\|_2^2 \,\|H_\varepsilon\|_2^{2n-4},$$

we get $\|L_{n-1,\varepsilon}(x,\cdot)\|_2 \le \|H_\varepsilon(x,\cdot)\|_2 \,\|H_\varepsilon\|_2^{n-2}$ and $|L_{n,\varepsilon}(x,y)| \le \|H_\varepsilon(x,\cdot)\|_2 \,\|H_\varepsilon\|_2^{n-2}\,\|H_\varepsilon(\cdot,y)\|_2\,.$
Consequently,

$$\|L_{n,\varepsilon}\|_2^2 \le \int\int \|H_\varepsilon(x,\cdot)\|_2^2 \,\|H_\varepsilon\|_2^{2n-4}\,\|H_\varepsilon(\cdot,y)\|_2^2 \,\mathrm{d}x\mathrm{d}y = \|H_\varepsilon\|_2^{2n},$$

that is

$$\|L_{n,\varepsilon}\|_2 \le \|H_\varepsilon\|_2^n\,. \tag{6}$$

With a similar method, we can prove that, for all $n \ge 2$, $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^d \setminus \{(0,0)\}$ and $(x,y) \in X^2$,

$$\left|\partial_{x\,y}^{(\alpha,\beta)} L_{n,\varepsilon}(x,y)\right| \le \|\partial_x^\alpha H_\varepsilon(x,\cdot)\|_2 \,\|H_\varepsilon\|_2^{n-2}\left\|\partial_y^\beta H_\varepsilon(\cdot,y)\right\|_2,$$

and

$$\left\|\partial_{x\,y}^{(\alpha,\beta)} L_{n,\varepsilon}\right\|_2 \le \|\partial_x^\alpha H_\varepsilon\|_2 \,\|H_\varepsilon\|_2^{n-2}\left\|\partial_y^\beta H_\varepsilon\right\|_2. \tag{7}$$

From the inequalities (6) and (7), we deduce that the series $\sum_{n\ge1} \frac{L_{n,\varepsilon}}{n!}$ and $\sum_{n\ge1} \frac{1}{n!}\partial_{x\,y}^{(\alpha,\beta)} L_{n,\varepsilon}$ converge, in L^2-norm. We set

$$S_\varepsilon(x,y) = \sum_{n=1}^{+\infty} \frac{L_{n,\varepsilon}}{n!}(x,y), \ \text{ for all } (x,y) \in X^2,$$

and

$$D_\varepsilon^{\alpha,\beta}(x,y) = \sum_{n=1}^{+\infty} \frac{1}{n!}\partial_{x\,y}^{(\alpha,\beta)} L_{n,\varepsilon}(x,y), \ \text{ for all } \ (\alpha,\beta) \in \left(\mathbb{N}^d \times \mathbb{N}^d\right) \setminus \{(0,0)\}, (x,y) \in X^2.$$

We turn to the study of these series. One has

$$\|S_\varepsilon\|_2 \le \sum_{n=1}^{+\infty} \frac{1}{n!}\,\|L_{n,\varepsilon}\|_2 \le \sum_{n=1}^{+\infty} \frac{1}{n!}\,\|H_\varepsilon\|_2^n \le e^{\|H_\varepsilon\|_2}.$$

As H is in $\mathcal{G}_{L^2\,ln}(X^2)$, H_ε is in $L^2(X^2)$ and S_ε also, for all $\varepsilon \in (0,1]$. Furthermore, $\|H_\varepsilon\|_2 = O(|\ln\varepsilon|)$ as ε tends to 0, that is there exists $k \in \mathbb{N}$ such that $\|H_\varepsilon\|_2 \le k \ln|\varepsilon|$. It follows that

$$\|S_\varepsilon\|_2 \le \varepsilon^{-k}.$$

Furthermore, a straightforward exercise on distributions theory shows that, for all $\alpha,\beta \in \mathbb{N}^d\setminus\{(0,0)\}$ and $\varepsilon\in(0,1]$, $\partial_{x\,y}^{(\alpha,\beta)}S_\varepsilon = D_\varepsilon^{\alpha,\beta}$ in $\mathcal{D}'(X^2)$.
Consequently,

$$\begin{aligned}\left\|\partial_{x\,y}^{(\alpha,\beta)}S_\varepsilon\right\|_2 = \left\|D_\varepsilon^{\alpha,\beta}\right\|_2 &\le \sum_{n=1}^{+\infty}\frac{1}{n!}\left\|\partial_{x\,y}^{(\alpha,\beta)}L_{n,\varepsilon}\right\|_2\\ &\le \left\|\partial_{x\,y}^{(\alpha,\beta)}H_\varepsilon\right\|_2+\sum_{n=2}^{+\infty}\frac{1}{n!}\left\|\partial_x^\alpha H_\varepsilon\right\|_2\left\|H_\varepsilon\right\|_2^{n-2}\left\|\partial_y^\beta H_\varepsilon\right\|_2\\ &\le \left\|\partial_{x\,y}^{(\alpha,\beta)}H_\varepsilon\right\|_2+\left\|\partial_x^\alpha H_\varepsilon\right\|_2\left\|\partial_y^\beta H_\varepsilon\right\|_2 e^{\|H_\varepsilon\|_2}.\end{aligned}$$

As H is in $\mathcal{G}_{L^2\,ln}(X^2)$, there exists $k\in\mathbb{N}$ such that $\|H_\varepsilon\|_2$, $\|\partial_x^\alpha H_\varepsilon\|_2$, $\left\|\partial_y^\beta H_\varepsilon\right\|_2$ and $\left\|\partial_{x\,y}^{(\alpha,\beta)}H_\varepsilon\right\|_2$ are less than $\ln\left(\varepsilon^{-k}\right)$, for ε small enough. Hence

$$\left\|\partial_{x\,y}^{(\alpha,\beta)}S_\varepsilon\right\|_2 = O\left(\varepsilon^{-k-1}\right)\ \text{ as } \varepsilon\to 0.$$

Finally, $(S_\varepsilon)_\varepsilon$ is in $\mathcal{E}_{L^2}(X^2)$ and we can denote by S its class in $\mathcal{G}_{L^2}(X^2)$.

• We show now that S does not depend on the choice of the representative of H. Let $(H_\varepsilon^1)_\varepsilon$ and $(H_\varepsilon^2)_\varepsilon$ be two representatives of H in $\mathcal{G}_{L^2 ln}(X^2)$. As previously, from $(H_\varepsilon^1)_\varepsilon$, we define $(L_{n,\varepsilon}^1)_\varepsilon$ and $(S_\varepsilon^1)_\varepsilon$, and from $(H_\varepsilon^2)_\varepsilon$, we define $(L_{n,\varepsilon}^2)_\varepsilon$ and $(S_\varepsilon^2)_\varepsilon$. For n greater than 2, we write $L_{n,\varepsilon}^1 - L_{n,\varepsilon}^2$, as in the proof of theorem-definition 38, as a sum of n integrals. After derivating this last expression, we obtain, for $(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d$,

$$\begin{aligned}\left\|\partial_{x\,y}^{(\alpha,\beta)}\left(L_{n,\varepsilon}^1-L_{n,\varepsilon}^2\right)\right\|_2 &\le \left\|\partial_x^\alpha(H_\varepsilon^1-H_\varepsilon^2)\right\|_2\left\|H_\varepsilon^1\right\|_2^{n-2}\left\|\partial_y^\beta H_\varepsilon^1(\cdot,y)\right\|_2\\ &+\left\|\partial_x^\alpha H_\varepsilon^2\right\|_2\left\|H_\varepsilon^1-H_\varepsilon^2\right\|_2\left\|H_\varepsilon^1\right\|_2^{n-3}\left\|\partial_y^\beta H_\varepsilon^1\right\|_2\\ &+\cdots\\ &+\left\|\partial_x^\alpha H_\varepsilon^2\right\|_2\left\|H_\varepsilon^2\right\|_2^{n-2}\left\|\partial_y^\beta(H_\varepsilon^1-H_\varepsilon^2)\right\|_2.\end{aligned}$$

Since $(H_\varepsilon^1)_\varepsilon$ and $(H_\varepsilon^2)_\varepsilon$ are in $\mathcal{G}_{L^2\,ln}(X^2)$, there exists $k\in\mathbb{N}$ such that for $\gamma=(0,0)$, $\gamma=(\alpha,0)$ and $\gamma=(0,\beta)$,

$$\left\|\partial_{x\,y}^\gamma H_\varepsilon^i\right\|_2 \le k\ln|\varepsilon|\,,\ \text{for } \varepsilon \text{ small enough and } i=1,2.$$

As $(H_\varepsilon^1-H_\varepsilon^2)_\varepsilon$ is in $\mathcal{I}_{L^2}(X^2)$, for a given $m\in\mathbb{N}$, there exists $C>0$ such that, for $\gamma=(0,0)$, $\gamma=(\alpha,0)$ and $\gamma=(0,\beta)$,

$$\left\|\partial_{x\,y}^\gamma(H_\varepsilon^1-H_\varepsilon^2)\right\|_2\le C\varepsilon^m,\ \text{for } \varepsilon \text{ small enough.}$$

Then, each term in the right hand side of the estimate of $\left\|\partial_{x\,y}^{(\alpha,\beta)}\left(L_{n,\varepsilon}^1-L_{n,\varepsilon}^2\right)\right\|_2$ is less than $C(k|\ln\varepsilon|)^{n-1}\varepsilon^m$, for ε small enough. Thus

$$\left\|\partial_{x\,y}^{(\alpha,\beta)}\left(L_{n,\varepsilon}^1-L_{n,\varepsilon}^2\right)\right\|_2\le Cn(k|\ln\varepsilon|)^{n-1}\varepsilon^m,\ \text{for } \varepsilon \text{ small enough.}$$

Finally

$$\begin{aligned}\left\|\partial_{x\,y}^{(\alpha,\beta)}\left(S_\varepsilon^1-S_\varepsilon^2\right)\right\|_2 &\le C\varepsilon^m\sum_{n=1}^{+\infty}\frac{1}{(n-1)!}(k|\ln\varepsilon|)^{n-1},\ \text{for } \varepsilon \text{ small enough}\\ &\le C\varepsilon^m e^{k|\ln\varepsilon|},\ \text{for } \varepsilon \text{ small enough.}\end{aligned}$$

It follows that $\left\|\partial_{x\,y}^{(\alpha,\beta)}\left(S_\varepsilon^1-S_\varepsilon^2\right)\right\|_2=O\left(\varepsilon^{m-k}\right)$ as $\varepsilon\to 0$ for all $m\in\mathbb{N}$ and that $(S_\varepsilon^1-S_\varepsilon^2)_\varepsilon$ is in $\mathcal{I}_{L^2}(X^2)$. Consequently, S does not depend on the choice of the representative of H in $\mathcal{G}_{L^2 ln}(X^2)$.

Remark 40 *The generalization of the results presented in subsections 6.1 and 6.2 for the exponential, to any entire function is straightforward by replacing the logarithm scale of growth by an adapted scale.*

6.3 Properties of the exponential of generalized integral operators

Proposition 41 *Let X be an open subset of $\mathbb{R}^d$. For H in $\mathcal{G}_{C\,ln}(X^2)$ or in $\mathcal{G}_{L^2\,ln}(X^2)$, we have*

$$\widehat{H}\circ e^{\widehat{H}}=e^{\widehat{H}}\circ\widehat{H}\ ;\qquad e^{a\widehat{H}}\circ e^{b\widehat{H}}=e^{(a+b)\widehat{H}},\ \text{for all } (a,b)\in\mathbb{R}^2\ ;\quad \frac{d}{dt}\,e^{t\widehat{H}}=\widehat{H}\circ e^{t\widehat{H}}.$$

Moreover, if K is in $\mathcal{G}_{C\,ln}(X^2)$ or in $\mathcal{G}_{L^2\,ln}(X^2)$ and if $\widehat{H}$ and $\widehat{K}$ commute, which amounts to $\int H(\cdot_1,\xi)K(\xi,\cdot_2)\,\mathrm{d}\xi=\int K(\cdot_1,\xi)H(\xi,\cdot_2)\,\mathrm{d}\xi$, then $e^{\widehat{H}}\circ e^{\widehat{K}}=e^{\widehat{H}+\widehat{K}}$.

Proof. By applying the theorem concerning the characterization of generalized integral operators by their kernel, we prove these properties by using the associated kernels. Denote by $Ker(\cdot)$ the kernel of a generalized integral operator. From H in $\mathcal{G}_{C\,ln}(X^2)$ or $\mathcal{G}_{L^2\,ln}(X^2)$, we define S and L_n, for $n\geq 1$, as in Theorem-definitions 38 or 39. Note that, for all integers p, q greater than 1, one has

$$\int_X L_p(\cdot_1,\xi)L_q(\xi,\cdot_2)\,\mathrm{d}\xi=L_{p+q}(\cdot_1,\cdot_2).$$

We are going to prove these properties for H in $\mathcal{G}_{C\,ln}(X^2)$ and the case where H is in $\mathcal{G}_{L^2\,ln}(X^2)$ is treated analogously. We also don't go back to representatives: All the sums written below are well defined and independent of representatives. Integrals are performed on a compact set κ such that the support of H is contained in the interior of κ^2

- One has $\widehat{H}\circ e^{\widehat{H}}=\widehat{H}\circ\widehat{S}+\widehat{H}$ and

$$Ker\left(\widehat{H}\circ\widehat{S}\right)(\cdot_1,\cdot_2)=\int_\kappa H(\cdot_1,\xi)S(\xi,\cdot_2)\,\mathrm{d}\xi=\int_\kappa H(\cdot_1,\xi)\sum_{n=1}^{+\infty}\frac{L_n}{n!}(\xi,\cdot_2)\,\mathrm{d}\xi=\sum_{n=1}^{+\infty}\frac{L_{n+1}}{n!}(\cdot_1,\cdot_2).$$

Thus

$$Ker\left(\widehat{H}\circ e^{\widehat{H}}\right)=\sum_{n=1}^{+\infty}\frac{L_{n+1}}{n!}+H.$$

Knowing that $\int_\kappa H(\cdot_1,\xi)L_n(\xi,\cdot_2)\,\mathrm{d}\xi=L_{n+1}(\cdot_1,\cdot_2)=\int_\kappa L_n(\cdot_1,\xi)H(\xi,\cdot_2)\,\mathrm{d}\xi$, one gets

$$Ker\left(\widehat{H}\circ e^{\widehat{H}}\right)=Ker\left(e^{\widehat{H}}\circ\widehat{H}\right),$$

which implies the result.

- One has $e^{c\widehat{H}}=\widehat{S}_c+Id$, for $c=a,\ b,\ a+b$, with

$$S_c(\cdot_1,\cdot_2)=\sum_{n=1}^{+\infty}\frac{c^n}{n!}\int_{\kappa^{n-1}}H(\cdot_1,\xi_1)\cdots H(\xi_{n-1},\cdot_2)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{n-1}=\sum_{n=1}^{+\infty}\frac{c^n}{n!}L_n(\cdot_1,\cdot_2).$$

We have $e^{a\widehat{H}} \circ e^{b\widehat{H}} = \widehat{S_a} \circ \widehat{S}_b + \widehat{S_a} + \widehat{S}_b + Id$, and

$$\begin{aligned} Ker(\widehat{S_a} \circ \widehat{S}_b)(\cdot_1, \cdot_2) &= \int_\kappa \sum_{n=1}^{+\infty} \frac{a^n}{n!} L_n(\cdot_1, \xi) \sum_{n=1}^{+\infty} \frac{b^n}{n!} L_n(\xi, \cdot_2)\, \mathrm{d}\xi \\ &= \sum_{n=2}^{+\infty} \sum_{k=1}^{n-1} \frac{a^{n-k} b^k}{(n-k)!k!} L_n(\cdot_1, \cdot_2) = \sum_{n=2}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n-1} C_n^k a^{n-k} b^k L_n(\cdot_1, \cdot_2). \end{aligned}$$

Thus

$$\begin{aligned} Ker\left(\widehat{S_a} \circ \widehat{S}_b + \widehat{S_a} + \widehat{S}_b\right) &= \sum_{n=2}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n-1} C_n^k a^{n-k} b^k L_n + \sum_{n=1}^{+\infty} \frac{a^n}{n!} L_n + \sum_{n=1}^{+\infty} \frac{b^n}{n!} L_n \\ &= \sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{k=0}^{n} C_n^k a^{n-k} b^k L_n = \sum_{n=1}^{+\infty} \frac{1}{n!} (a+b)^n = S_{a+b}, \end{aligned}$$

which implies the required property.

- One has $e^{t\widehat{H}} = \widehat{S}_t + Id$ with $S_t = \sum\limits_{n=1}^{+\infty} \frac{t^n}{n!} L_n$, so $\frac{d}{dt} e^{t\widehat{H}} = \frac{d}{dt} \widehat{S}_t = \widehat{\frac{d}{dt} S_t}$. Now $\frac{d}{dt} S_t = \sum\limits_{n=1}^{+\infty} \frac{t^{n-1}}{(n-1)!} L_n$. Furthermore $\widehat{H} \circ e^{t\widehat{H}} = \widehat{H} \circ \widehat{S}_t + \widehat{H}$ and

$$\begin{aligned} Ker\left(\widehat{H} \circ \widehat{S}_t\right)(\cdot_1, \cdot_2) &= \int_\kappa H(\cdot_1, \xi) \sum_{n=1}^{+\infty} \frac{t^n}{n!} L_n(\xi, \cdot_2)\, \mathrm{d}\xi \\ &= \sum_{n=1}^{+\infty} \frac{t^n}{n!} L_{n+1}(\cdot_1, \cdot_2) \\ &= \sum_{n=2}^{+\infty} \frac{t^{n-1}}{(n-1)!} L_n(\cdot_1, \cdot_2) = \frac{d}{dt} S_t(\cdot_1, y) - L_1(\cdot_1, \cdot_2). \end{aligned}$$

As $L_1 = H$, one gets $\widehat{H} \circ e^{t\widehat{H}} = \widehat{\frac{d}{dt} S_t} = \frac{d}{dt} e^{t\widehat{H}}$.

- From T in $\mathcal{G}_{C\,ln}(X^2)$, we define, for $n \geq 1$, S_T and $L_{T,n}$ as in Theorem-definition 38. Take H and K which commute and κ a compact set such that the supports of H and K are contained in the interior of κ^2. On one hand, we have $e^{\widehat{H}} \circ e^{\widehat{K}} = \widehat{S_H} \circ \widehat{S_K} + \widehat{S_H} + \widehat{S_K} + Id$ with

$$\begin{aligned} Ker\left(\widehat{S_H} \circ \widehat{S_K}\right)(\cdot_1, \cdot_2) &= \int_\kappa \sum_{n=1}^{+\infty} \frac{1}{n!} L_{H,n}(\cdot_1, \xi) \sum_{n=1}^{+\infty} \frac{1}{n!} L_{K,n}(\xi, \cdot_2)\, \mathrm{d}\xi \\ &= \sum_{n=2}^{+\infty} \sum_{k=1}^{n-1} \frac{1}{(n-k)!k!} \int_\kappa L_{H,k}(\cdot_1, \xi) L_{K,n-k}(\xi, \cdot_2)\, \mathrm{d}\xi \\ &= \sum_{n=2}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n-1} C_n^k \int_\kappa L_{H,k}(\cdot_1, \xi) L_{K,n-k}(\xi, \cdot_2)\, \mathrm{d}\xi. \end{aligned}$$

Thus

$$Ker\left(\widehat{S_H} \circ \widehat{S_K} + \widehat{S_H} + \widehat{S_K}\right) = \sum_{n=1}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n} C_n^k \int_\kappa L_{H,k}(\cdot_1, \xi) L_{K,n-k}(\xi, \cdot_2)\, \mathrm{d}\xi.$$

On the other hand, we have $e^{\widehat{H}+\widehat{K}} = Id + \widehat{S_{H+K}}$ with $Ker(\widehat{S_{H+K}})(\cdot_1,\cdot_2) = \sum\limits_{n=1}^{+\infty} \frac{1}{n!} L_{H+K,n}(\cdot_1,\cdot_2)$ and, for all n greater than 1,

$$\begin{aligned} L_{H+K,n}(\cdot_1,\cdot_2) &= \int_{\kappa^{n-1}} (H+K)(\cdot_1,\xi_1)\cdots(H+K)(\xi_{n-1},\cdot_2)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{n-1} \\ &= \sum_{k=1}^{n} C_n^k \int_{\kappa} L_{H,k}(\cdot_1,\xi)L_{K,n-k}(\xi,\cdot_2)\,\mathrm{d}\xi. \end{aligned}$$

This last equality follows from a straightforward induction, which uses mainly the fact that H and K commute, which implies that $L_{H,p}$ and $L_{K,q}$ have the same property for all integers $p \geq 1$ and $q \geq 1$. Thus $Ker(\widehat{S_{H+K}})(\cdot_1,\cdot_2) = Ker\left(\widehat{S_H}\circ\widehat{S_K} + \widehat{S_H} + \widehat{S_K}\right)$, which ends the proof.

6.4 Example: A unitary generalized integral operator

In this subsection, we apply the above results to the special case of operators with symmetrical kernel, which are essential in view of forthcoming applications to theoretical physics. Fix X an open subset of $\mathbb{R}^d$.

Proposition-definition 42 *The map $(f,g) \mapsto \int f(x)\bar{g}(x)\,\mathrm{d}x$ from $(\mathcal{G}_{L^2}(X))^2$ to $\overline{\mathbb{C}}$ defines a* generalized scalar product *on $\mathcal{G}_{L^2}(X)$, that is $(\cdot,\cdot)$ is bilinear, positive ((f,f) has a representative $(\varphi_\varepsilon)_\varepsilon$ with $\varphi_\varepsilon \geq 0$ for all $\varepsilon \in (0,1]$) and non degenerate, id est: $(f,f) = 0$ in $\overline{\mathbb{C}}$ implies that $f = 0$ in $\mathcal{G}_{L^2}(X)$.*

Proof. The only non trivial assertion is the last one. Take f such that $(f,f) = 0$ in $\overline{\mathbb{C}}$ and denote by $(f_\varepsilon)_\varepsilon$ one of its representatives. For all $n \in \mathbb{N}$, $\|f_\varepsilon\|_2 = O(\varepsilon^n)$ as $\varepsilon \to 0$, that is $(f_\varepsilon)_\varepsilon$ is in $\mathcal{I}_{l^2}(X)$. Furthermore $(f_\varepsilon)_\varepsilon \in \mathcal{E}_{L^2}(X)$ so $(f_\varepsilon)_\varepsilon \in \mathcal{I}_{l^2}(X) \cap \mathcal{E}_{L^2}(X) = \mathcal{I}_{L^2}(X)$ (this result is proved in [8] by a method analogous to the one employed in the proof of theorem 1.2.3 in [12]), which means that $f = 0$ in $\mathcal{G}_{L^2}(X)$.

Definition 43 *We say that a generalized function H of $\mathcal{G}_{L^2}(X^2)$ is* symmetric *if, for all $(x,y) \in X^2$, $H(x,y) = \overline{H(y,x)}$ in $\overline{\mathbb{C}}$.*

Remark 44 *If H of $\mathcal{G}_{L^2}(X^2)$ is symmetric, then $\widehat{H}$ is symmetric for the generalized scalar product introduced in proposition-definition 42, that is $(\widehat{H}(f),g) = (f,\widehat{H}(g))$ for all $f,\ g \in \mathcal{G}_{L^2}(X^2)$.*

Definition 45 *We say that a generalized operator $A : D(A) \subset \mathcal{G}_{L^2}(X) \to \mathcal{G}_{L^2}(X)$ is* unitary *if, for all $(f,g) \in D(A)^2$, one has $(A(f),A(g)) = (f,g)$.*

Proposition 46 *Let H be a symmetric generalized function in $\mathcal{G}_{L^2\,ln}(X^2)$. The generalized integral operator $e^{it\widehat{H}}$ is unitary.*

Proof. We don't go back to representatives as in subsection 6.3. As $e^{it\widehat{H}} \circ e^{-it\widehat{H}} = Id$, we have just to prove that $\left(e^{it\widehat{H}}(f),g\right) = \left(f,e^{-it\widehat{H}}(g)\right)$ in $\overline{\mathbb{C}}$, for all $f,g \in \mathcal{G}_{L^2}(X)$. Let f and g be in $\mathcal{G}_{L^2}(X)$. By definition of the exponential, it suffices to show that $\left(\widehat{S}_{it}(f),g\right) = \left(f,\widehat{S}_{-it}(g)\right)$

in $\overline{\mathbb{C}}$. We have

$$\begin{aligned}
\left(\widehat{S}_{it}(f), g\right) &= \int_X \widehat{S}_{it}(f)(x)\bar{g}(x)\,\mathrm{d}x \\
&= \int_X \sum_{n=1}^{+\infty} \frac{(it)^n}{n!} \int_X L_n(x,y)f(y)\,\mathrm{d}y\ \bar{g}(x)\,\mathrm{d}x \\
&= \sum_{n=1}^{+\infty} \frac{(it)^n}{n!} \int_X \int_X L_n(x,y)f(y)\bar{g}(x)\,\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

As H is symmetrical, so is L_n for all $n \geq 1$ and, by applying Fubini's theorem, one gets

$$\begin{aligned}
\left(\widehat{S}_{it}(f), g\right) &= \sum_{n=1}^{+\infty} \frac{(it)^n}{n!} \int_X \int_X \bar{L}_n(y,x)f(y)\bar{g}(x)\,\mathrm{d}x\mathrm{d}y \\
&= \int_X f(y) \left(\int_X \sum_{n=1}^{+\infty} \frac{(it)^n}{n!} \bar{L}_n(y,x)\bar{g}(x)\,\mathrm{d}x \right) \mathrm{d}y \\
&= \left(f, \widehat{S}_{-it}(g)\right).
\end{aligned}$$

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