Symmetry results for semilinear elliptic systems in the whole space

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SYMMETRY RESULTS FOR SEMILINEAR ELLIPTIC SYSTEMS IN THE WHOLE SPACE

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1 Introduction

In this paper we study the radial symmetry of classical solutions of elliptic systems of the following type

$$\begin{cases} \Delta u_i + f_i(r, u_1, \ldots, u_n) = 0 & \text{in} \quad \mathbb{R}^N, \ i = 1, \ldots, n, \\ u_i > 0 & \text{in} \quad \mathbb{R}^N, \\ u_i(x) \to 0 & \text{as} \quad r = |x| \to \infty, \end{cases} \tag{1}$$

where $n \geq 1$, $N \geq 2$ are arbitrary integers.

In the case of a bounded domain, related results for autonomous systems were established by Troy [17] (see also de Figueiredo [4], Shaker [16]). Under additional hypotheses on the asymptotic behaviour of the solutions at infinity, in the spirit of Gidas, Ni and Nirenberg [11], a symmetry result in $\mathbb{R}^N$ was obtained by Shaker. We remark that the case of a single equation has been extensively studied since the work of Gidas, Ni and Nirenberg (see for instance C. Li [12], Y. Li and W.-M. Ni [13]).

In a recent paper, D.G. de Figueiredo and J. Yang [8] studied the symmetry of the positive solutions of systems of two equations, under some restrictive hypotheses on the nonlinearities (see Section 2.1).

Using variational methods, de Figueiredo and Yang also proved existence and decay at infinity of positive solutions of such systems. More general results about existence and decay can be found in [15], as well as an application of our symmetry result to the existence of a ground state of the system.

We note $u = (u_1, \ldots, u_n) \in \mathbb{R}^n_+ = (0, \infty)^n$ and

$$A(r, u^1, \ldots, u^n) = \left(\frac{\partial f_i}{\partial u_j}(r, u^i) \right)_{1 \leq i,j \leq n},$$

for $r \geq 0$ and $u^i \in I\!R^n_+,\ 1 \leq i \leq n$. We suppose that $f_i \in C^1([0,\infty)\times I\!R^n_+, I\!R)$ for $i = 1,\dots,n$ (this condition can be weakened, for example, f_i can be supposed to be only Liptschiz).

We use the following hypotheses.

(H1) $\dfrac{\partial f_i}{\partial r}(r,u) \leq 0$ for all $(r,u) \in I\!R^{n+1}_+$ and $i = 1,\dots,n$;

(H2) The system is *cooperative* (or quasimonotone), that is,

$$\frac{\partial f_i}{\partial u_j}(r,u) \geq 0$$

for all $(r,u) \in I\!R^{n+1}_+$ and all $i,j \in \{1,\dots,n\}$, $i \neq j$;

(H3) There exist constants $\varepsilon > 0$ and $R_1 > 0$ such that the system is *fully coupled* in the set

$$\mathcal{O} = \{(r,u) \mid r > R_1,\ u \in I\!R^n_+,\ |u| < \varepsilon\},$$

that is, for any $I, J \subset \{1,\dots,n\}$, with $I \cap J = \emptyset$, $I \cup J = \{1,\dots,n\}$, there exist $i_0 \in I$ and $j_0 \in J$ such that

$$\frac{\partial f_{j_0}}{\partial u_{i_0}}(r,u) > 0,$$

for all $(r,u) \in \mathcal{O}$;

(H4) All n-principal minors of $-A(r,u^1,\dots,u^n)$ have nonnegative determinants for $(r,u^i) \in \mathcal{O},\ 1 \leq i \leq n$. We recall that the n-principal minors of a matrix $(m_{ij})_{1\leq i,j\leq n}$ are the submatrices $(m_{ij})_{1\leq i,j\leq k}$, for $k = 1,\dots,n$.

Assumption (H2) is widely used for elliptic systems. In particular, Troy and Shaker proved their results under (H2). Condition (H3) means that the system cannot be reduced to two independent systems. It is this fact that will force all functions u_i to be radially symmetric with respect to the same origin. Finally, (H4) is the natural generalisation of the hypotheses at infinity, used for single equations. Actually, in the scalar case

$$\Delta u + f(r,u) = 0,$$

(H2) - (H4) reduce to $\dfrac{\partial f}{\partial u}(r,u) \leq 0$ for small u and large r, which is exactly the assumption considered by Y. Li and W.-M. Ni in [13] (see also C. Li [12]).

Note that the functions f_i are not assumed to be defined on points which have a zero coordinate.

Our main result is given by the following theorem.

Theorem 1 *Suppose $f_1, \ldots, f_n$ satisfy* (H1)-(H4)*, and let $u = (u_1, \ldots, u_n)$ be a classical solution of (1). Then there exists a point $x_0 \in I\!R^N$ such that the functions u_i are radially symmetric with respect to the origin x_0, that is, $u_i(x) = u_i(|x - x_0|)$, $i = 1, \ldots, n$. Moreover,*

$$\frac{du_i}{dr} < 0 \ \ \textit{for all} \ \ r = |x - x_0| > 0.$$

Section 2 is devoted to the proof of Theorem 1. In Section 2.1 we give the proof in a simpler setting of two autonomous equations, where the main ideas are made more explicit. In this case we are able to give a full generalisation of hypothesis (H3). We even state a theorem which does not include this hypothesis. Finally, in Section 3 we discuss our assumptions and give simple examples of nonexistence of positive solutions when some of them are not satisfied.

2 Proof of the Main Theorem

2.1 The Case of Two Equations

In this section we prove the symmetry result for classical solutions of the system

$$\left\{ \begin{array}{rll} \Delta u + g(u,v) = 0 & \text{in} & I\!R^N \\ \Delta v + f(u,v) = 0 & \text{in} & I\!R^N \\ u, v > 0 & \text{in} & I\!R^N \\ u(x), v(x) \to 0 & \text{as} & |x| \to \infty, \end{array} \right. \tag{2}$$

with $f, g \in C^1([0,\infty) \times [0,\infty), I\!R)$. We suppose that

(i) $\dfrac{\partial g}{\partial v}(u,v)$ and $\dfrac{\partial f}{\partial u}(u,v)$ are non-negative for all $(u,v) \in [0,\infty) \times [0,\infty)$;

(ii) $\dfrac{\partial g}{\partial u}(0,0) < 0$ and $\dfrac{\partial f}{\partial v}(0,0) < 0$;

(iii) $\det A > 0$, where

$$A = \begin{pmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \\ \\ \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} (0,0).$$

In order to avoid some technicalities, here we have strengthened our hypotheses (H1)-(H4). Of course, by using the method of Section 2.2, all results in Section 2.1 can be shown to hold under (H1)–(H4). We note that (ii) and (iii) are exactly the conditions under which the linearized system at zero satisfies the maximum principle (see [6] and [10]).

In [8] de Figueiredo and Yang considered the case

$$g(u,v) = -u + g_1(v)\,, \quad f(u,v) = -v + f_1(u)$$

where $f_1(0) = g_1(0) = f_1'(0) = g_1'(0) = 0$, f_1 and g_1 are positive and convex in $I\!R_+$, and have power-like growth at zero and infinity. Note that in this case $-A$ is the identity matrix. The result of de Figueiredo and Yang concerned only exponentially decreasing solutions of (2). None of these features is present in our work.

We have the following result.

Theorem 2 *Assume (i), (ii) and (iii) hold. Then there exist two points* $x_0, x_1 \in I\!R^N$ *such that* $u(x) = u(|x - x_0|)$ *and* $v(x) = v(|x - x_1|)$*. Moreover,*

$$\frac{du}{dr_1} < 0 \quad \textit{and} \quad \frac{dv}{dr_2} < 0,$$

for all $r_1 = |x - x_0| > 0$ *and* $r_2 = |x - x_1| > 0$.

If $x_0 \neq x_1$, it follows from Theorem 2 that u changes its values on sets where v is constant and vice versa. Therefore, if $x_0 \neq x_1$ and both functions u and v are effectively present in one of the equations in (2), then this equation cannot be satisfied. We deduce that, in case v (resp. u) appears in a non-zero term in the first (resp. the second) equation in (2), then the solutions are symmetric with respect to the same origin.

Sufficient conditions for $x_0 = x_1$ in Theorem 2 which do not depend on the particular choice of the solutions are for example :

(iv)$'$ either $\dfrac{\partial g}{\partial v}$ or $\dfrac{\partial f}{\partial u}$ is strictly positive in a neighbourhood of $(0,0)$, except possibly on $\{u = 0\} \cup \{v = 0\}$;

(iv)$''$ either $\dfrac{\partial g}{\partial v}$ or $\dfrac{\partial f}{\partial u}$ does not depend on one of its variables and is not identically zero in every neighbourhood of $(0,0)$.

Proof of Theorem 2. As in many other works, in order to prove symmetry of solutions we apply the "moving planes" method. For all $\lambda \in I\!R$ we define the hyperplane $T_\lambda = \{x \in I\!R^N \mid x_1 = \lambda\}$ and set $\Sigma_\lambda = \{x \in I\!R^N \mid x_1 > \lambda\}$. Our goal is to show that the solutions of (2) are symmetric with respect to T_λ, for some $\lambda \in I\!R$. Then we can finish the proof, as explained in the beginning of Section 2.2.

Let u and v be solutions of (2). For any point $x \in \Sigma_\lambda$ we denote with x^λ its reflexion with respect to T_λ. We introduce the functions $u_\lambda(x) = u(x^\lambda)$, $v_\lambda(x) = v(x^\lambda)$, $U_\lambda(x) = u_\lambda(x) - u(x)$ and $V_\lambda(x) = v_\lambda(x) - v(x)$, all of them defined in Σ_λ. The change of variables $x \to x^\lambda$ leaves the equations in (2) unchanged so we can substract these equations from the corresponding ones for u_λ and v_λ. We obtain

$$\begin{aligned} \Delta U_\lambda + g(u_\lambda(x), v_\lambda(x)) - g(u(x), v(x)) &= 0 \\ \Delta V_\lambda + f(u_\lambda(x), v_\lambda(x)) - f(u(x), v(x)) &= 0 \end{aligned}$$

in Σ_λ. Consequently, by Taylor's expansion,

$$\Delta U_\lambda + \frac{\partial g}{\partial u}(\xi_1(x,\lambda), v(x))U_\lambda + \frac{\partial g}{\partial v}(u_\lambda(x), \eta_1(x,\lambda))V_\lambda = 0 \tag{3}$$

$$\Delta V_\lambda + \frac{\partial f}{\partial u}(\xi_2(x,\lambda), v_\lambda(x))U_\lambda + \frac{\partial f}{\partial v}(u(x), \eta_2(x,\lambda))V_\lambda = 0, \tag{4}$$

where

$$\begin{aligned} \xi_i(x,\lambda) &\in (\min\{u(x), u_\lambda(x)\}, \max\{u(x), u_\lambda(x)\})\,, \\ \eta_i(x,\lambda) &\in (\min\{v(x), v_\lambda(x)\}, \max\{v(x), v_\lambda(x)\})\,, \quad i = 1, 2. \end{aligned}$$

We apply the "moving planes" method in three steps.

Step 1 *There exists $\lambda^* > 0$ such that $U_\lambda \geq 0$ and $V_\lambda \geq 0$ in Σ_λ, for all $\lambda \geq \lambda^*$.*

Let us prove the claim in Step 1 for U_λ. Assume for contradiction that for all $\lambda > 0$ there exists a point $x \in \Sigma_\lambda$ such that $U_\lambda(x) < 0$.

First, by using (ii) we choose $\varepsilon_0 > 0$ such that $\dfrac{\partial g}{\partial u}(u, v) < 0$ and $\dfrac{\partial f}{\partial v}(u, v) < 0$ if $|u| + |v| < \varepsilon_0$. Then we fix $\bar{\lambda} > 0$ such that $u(x) + v(x) < \varepsilon_0$, when $|x| > \bar{\lambda}$. Next, we observe that for all $\lambda > 0$ the function U_λ attains its infimum in Σ_λ, since it takes negative values in Σ_λ, vanishes on $T_\lambda = \partial\Sigma_\lambda$, and tends to zero at infinity (note that $|x| \to \infty$ is equivalent to $|x^\lambda| \to \infty$, for λ fixed). We fix $\lambda \geq \bar{\lambda}$ and take $x_0 = x_0(\lambda) \in \Sigma_\lambda$ such that

$$U_\lambda(x_0) = \min_{x \in \Sigma_\lambda} U_\lambda(x) < 0.$$

Then $\Delta U_\lambda(x_0) \geq 0$, so it follows from (3) that

$$\frac{\partial g}{\partial u}(\xi_1(x_0,\lambda), v(x_0))U_\lambda(x_0) \leq -\frac{\partial g}{\partial v}(u_\lambda(x_0), \eta_1(x_0,\lambda))V_\lambda(x_0). \tag{5}$$

Since $U_\lambda(x_0) < 0$ implies $\xi_1(x_0, \lambda) \leq u(x_0)$, we see that the left-hand side of (5) is strictly positive. This implies $V_\lambda(x_0) < 0$. Therefore, there exists $x_1 = x_1(\lambda) \in \Sigma_\lambda$ such that

$$V_\lambda(x_1) = \min_{x \in \Sigma_\lambda} V_\lambda(x) < 0.$$

Because of (4) we can repeat the above argument, showing that $U_\lambda(x_1) < 0$ and

$$\frac{\partial f}{\partial v}(u(x_1), \eta_2(x_1, \lambda))V_\lambda(x_1) \leq -\frac{\partial f}{\partial u}(\xi_2(x_1, \lambda), v_\lambda(x_1))U_\lambda(x_1). \tag{6}$$

We set

$$\alpha(\lambda) = \frac{\partial g}{\partial u}(\xi_1(x_0, \lambda), v(x_0)) < 0\ ,\ \beta(\lambda) = \frac{\partial g}{\partial v}(u_\lambda(x_0), \eta_1(x_0, \lambda)) \geq 0, \tag{7}$$

$$\gamma(\lambda) = \frac{\partial f}{\partial u}(\xi_2(x_1, \lambda), v_\lambda(x_1)) \geq 0\ ,\ \delta(\lambda) = \frac{\partial f}{\partial v}(u(x_1), \eta_2(x_1, \lambda)) < 0. \tag{8}$$

By using (5) and (6) we obtain

$$\begin{aligned} U_\lambda(x_0) &\geq -\frac{\beta(\lambda)}{\alpha(\lambda)}V_\lambda(x_0) \\ &\geq -\frac{\beta(\lambda)}{\alpha(\lambda)}V_\lambda(x_1) \\ &\geq \frac{\beta(\lambda)\gamma(\lambda)}{\alpha(\lambda)\delta(\lambda)}U_\lambda(x_1) \\ &\geq \frac{\beta(\lambda)\gamma(\lambda)}{\alpha(\lambda)\delta(\lambda)}U_\lambda(x_0). \end{aligned}$$

The last quantity is strictly greater than $U_\lambda(x_0)$, provided that

$$a(\lambda) := \alpha(\lambda)\delta(\lambda) - \beta(\lambda)\gamma(\lambda) > 0.$$

Since U_λ and V_λ are both negative at x_0 and x_1, we have

$$u_\lambda(x_0) < u(x_0)\ ,\ \xi_1(x_0, \lambda) \leq u(x_0)\ ,\ \eta_1(x_0, \lambda) \leq v(x_0)\ ,$$

$$v_\lambda(x_1) < v(x_1)\ ,\ \xi_2(x_1, \lambda) \leq u(x_1)\ ,\ \eta_2(x_1, \lambda) \leq v(x_1).$$

The solutions of (2) decay at infinity, so these inequalities imply

$$\lim_{\lambda \to \infty} a(\lambda) = \det A > 0,$$

which leads to a contradiction, for λ sufficiently large and greater than $\bar{\lambda}$. Step 1 is completed.

We set

$$\lambda_0 = \inf\{\lambda \in I\!R \mid U_\mu \geq 0 \ \text{ and } \ V_\mu \geq 0 \ \text{ in } \Sigma_\mu \ \text{ for all } \ \mu \geq \lambda\}.$$

Step 1 implies that $\lambda_0 < +\infty$. On the other hand $\lambda_0 = -\infty$ is impossible, since $U_\lambda(0) < 0$ for any $\lambda < -R$, with R chosen so that $\max_{|x_1| \geq R} u(x) < u(0)$. Hence λ_0 is finite.

Step 2 *Either* $U_{\lambda_0} \equiv 0$ *or* $V_{\lambda_0} \equiv 0$ *in* Σ_{λ_0}.

Since all objects we consider are continuous with respect to λ, we already know that $U_{\lambda_0} \geq 0$ and $V_{\lambda_0} \geq 0$ in Σ_{λ_0}. Then it follows from (3), (4) and (i) that in Σ_{λ_0}

$$\Delta U_{\lambda_0} + \frac{\partial g}{\partial u}(\xi_1(x,\lambda_0), v(x))U_{\lambda_0} = -\frac{\partial g}{\partial v}(u_{\lambda_0}(x), \eta_1(x,\lambda_0))V_{\lambda_0} \leq 0 \tag{9}$$

and

$$\Delta V_{\lambda_0} + \frac{\partial f}{\partial v}(u(x), \eta_2(x,\lambda_0))V_{\lambda_0} = -\frac{\partial f}{\partial u}(\xi_2(x,\lambda_0), v_{\lambda_0}(x))U_{\lambda_0} \leq 0 \tag{10}$$

The strong maximum principle, applied to (9), implies that either $U_{\lambda_0} \equiv 0$ in Σ_{λ_0} or $U_{\lambda_0} > 0$ in Σ_{λ_0}, with $\dfrac{\partial U_{\lambda_0}}{\partial x_1} > 0$ on T_{λ_0}. By (10) the same holds for V_{λ_0}.

Therefore, we only have to exclude the situation when both U_{λ_0} and V_{λ_0} are strictly positive in Σ_{λ_0}, and have strictly positive normal derivatives on T_{λ_0}. Let us suppose this is the case.

By the definition of λ_0, there exist sequences $\{\lambda_k\}_{k=1}^{\infty}$ and $\{x_k\}_{k=1}^{\infty} \subset I\!R^N$ such that $\lambda_k < \lambda_0$, $\lim_{k\to\infty} \lambda_k = \lambda_0$, $x_k \in \Sigma_{\lambda_k}$ and either U_{λ_k} or V_{λ_k} takes a negative value at x_k. For example, let $U_{\lambda_k}(x_k) < 0$. We rename x_k so that

$$U_{\lambda_k}(x_k) = \min_{x \in \Sigma_{\lambda_k}} U_{\lambda_k}(x) < 0.$$

We distinguish two cases.

Case 1 The sequence $\{x_k\}$ contains a bounded subsequence.

This case is treated in a standard way. We extract a subsequence of $\{x_k\}$ which converges to a point $x_0 \in \overline{\Sigma_{\lambda_0}}$. Since $U_{\lambda_0}(x_0) \leq 0$, necessarily $x_0 \in T_{\lambda_0}$. But x_k is point of interior minimum of U_{λ_k}, so $\nabla U_{\lambda_k}(x_k) = 0$. Hence $\nabla U_{\lambda_0}(x_0) = 0$. This contradicts $\dfrac{\partial U_{\lambda_0}}{\partial x_1}(x_0) > 0$.

Case 2 $|x_k| \to \infty$ as $k \to \infty$.

This case has been a basic issue in applying the "moving planes" method in unbounded domains since the first work on symmetry in $I\!R^N$ by Gidas, Ni, and Nirenberg. Fortunately, the machinery that we set up in Step 1 adapts to this case. Indeed, exactly as in Step 1 we can show that there exists an integer k_0 such that $V_{\lambda_k}(x_k) < 0$ when $k \geq k_0$. Likewise, there exists $k_1 \geq k_0$ such that $U_{\lambda_k}(y_k)$ is negative when $k \geq k_1$, with y_k chosen so that

$$V_{\lambda_k}(y_k) = \min_{y \in \Sigma_{\lambda_k}} V_{\lambda_k}(y) < 0.$$

Inequalities (5) and (6) hold, so we finally obtain

$$\frac{\tilde{a}(k)}{\alpha(k)\delta(k)} U_{\lambda_k}(x_k) \geq 0,$$

where

$$\tilde{a}(k) = \alpha(k)\delta(k) - \beta(k)\gamma(k),$$

with $\alpha(k)$, $\beta(k)$, $\gamma(k)$, $\delta(k)$ defined analogously to (7) and (8) :

$$\alpha(k) = \frac{\partial g}{\partial u}(\xi_1(x_k, \lambda_k), v(x_k)), \text{ etc.}$$

It is easy to see that

$$\lim_{k \to \infty} \tilde{a}(k) = \det A > 0,$$

and we obtain a contradiction for k sufficiently large. This argument completes Step 2.

Step 3 *Conclusion.*

Let for example $U_{\lambda_0} \equiv 0$. Then $U_\lambda > 0$ in Σ_λ for all $\lambda > \lambda_0$, so it is straightforward to see that

$$\operatorname{sign}(x_1 - \lambda_0) \frac{\partial u}{\partial x_1}(x_1, x') \leq 0 \tag{11}$$

for all $x = (x_1, x') \in I\!R \times I\!R^{N-1}$. Next, we observe that the function v satisfies the single equation

$$\Delta v + \bar{f}(x, v) = 0,$$

with $\bar{f}(x, v) = f(u(x), v)$. It follows from our hypotheses that $\dfrac{\partial \bar{f}}{\partial v}(x, v)$ is negative for small v and large $|x_1|$. We have, in view of (i) and (11),

$$\operatorname{sign}(x_1 - \lambda_0) \frac{\partial \bar{f}}{\partial x_1}(x_1, x', v) = \operatorname{sign}(x_1 - \lambda_0) \frac{\partial f}{\partial u}(u(x), v) \frac{\partial u}{\partial x_1}(x_1, x') \leq 0$$

for all $x \in I\!R^N, v \in I\!R$. This is exactly what we need in order to apply the results for single equations (see [12], [13]), which permit us to conclude that there exists some λ_0', with $\lambda_0' \le \lambda_0$, such that v is symmetric with respect to $T_{\lambda_0'}$. Alternatively, to prove this we could use the reasonings in Steps 1 and 2 combined with moving planes coming from $-\infty$.

Finally, since $U_\lambda > 0$ and $V_\lambda > 0$ in Σ_λ for $\lambda > \lambda_0$, by using (i) we see that Hopf's lemma, applied to (3), yields

$$\frac{\partial U_\lambda}{\partial x_1}(\lambda, x') > 0 \ \text{ for all } \ \lambda > \lambda_0,\ x' \in I\!R^{N-1}.$$

Analogously, since $U_\lambda < 0$ and $V_\lambda < 0$ in Σ_λ for $\lambda < \lambda_0'$, we infer from (4)

$$\frac{\partial V_\lambda}{\partial x_1}(\lambda, x') < 0 \ \text{ for all } \ \lambda < \lambda_0',\ x' \in I\!R^{N-1}.$$

Since

$$\frac{\partial U_\lambda}{\partial x_1}(\lambda, x') = -2\frac{\partial u}{\partial x_1}(\lambda, x') \ \text{ and } \ \frac{\partial V_\lambda}{\partial x_1}(\lambda, x') = -2\frac{\partial v}{\partial x_1}(\lambda, x'),$$

the proof of Theorem 2 is complete.

2.2 The General Case

This section contains the proof of Theorem 1. In order to show that all u_i are radially symmetric with respect to the same origin, it is enough to establish that, given an arbitrary direction $\gamma \in I\!R^N \setminus \{0\}$, there exists $\lambda = \lambda(\gamma)$ such that all u_i are symmetric with respect to the hyperplane

$$T_\lambda = \{x \in I\!R^N \mid x \cdot \gamma = \lambda\}.$$

Indeed, it is easy to see that in this case u_i are radial with respect to the origin $\bigcap_{i=1}^{n} T_{\lambda(e_i)}$.

We fix a direction γ. We denote with $x \to x^\lambda$ the reflection with respect to T_λ, and with U_i^λ, $i = 1, \dots, n$, the difference functions

$$U_i^\lambda(x) = u_i(x^\lambda) - u_i(x),$$

defined in $\Sigma_\lambda = \{x \in I\!R^N \mid x \cdot \gamma > \lambda\}$.

As in section 2.1, the proof is carried out in three steps. In the first step we show that

$$\Lambda = \inf\{\lambda > 0 \mid U_i^\mu \ge 0 \text{ in } \Sigma_\mu \text{ for } i = 1, \dots, n \text{ and all } \mu \ge \lambda\}$$

is well-defined, that is, $\Lambda < +\infty$. In the second step we prove that either $\Lambda = 0$ or $\Lambda > 0$ and $U_i^\Lambda \equiv 0$ for all $i = 1, \dots, n$.

The conclusion of Theorem 1 then follows easily (see Step 3).

Step 1 $\Lambda < +\infty$.

Since the functions $u_i, i = 1, \ldots, n$, tend to zero at infinity, we can fix some large $R_0 \geq R_1$ such that $|u| < \varepsilon$ in $\mathbb{R}^N \setminus B_{R_0}$ (ε and R_1 are defined in (H3)). We take $\lambda^* > R_0$ for which

$$\max_{\substack{1\leq i\leq n \\ x\in \overline{B}^{\lambda}_{R_0}}} u_i(x) < \min_{\substack{1\leq i\leq n \\ x\in \overline{B}_{R_0}}} u_i(x),$$

for all $\lambda > \lambda^*$, where $\overline{B}^{\lambda}_{R_0} = \{x \mid x^{\lambda} \in \overline{B}_{R_0}\}$. It follows that $U_i^{\lambda} > 0$ in $\overline{B}^{\lambda}_{R_0} \subset \Sigma_{\lambda}$, for all $\lambda > \lambda^*$. Let us show that $U_i^{\lambda} > 0$ in the remaining part $\Sigma_{\lambda} \setminus \overline{B}^{\lambda}_{R_0}$. By writing equations (1) at x and x^{λ} and by using Taylor's expansion, we obtain that the functions U_i^{λ} satisfy the following system of linear partial differential equations

$$\Delta U_i^{\lambda} + \frac{\partial f}{\partial r}(\eta)(r^{\lambda} - r) + \sum_{1\leq j\leq n} \frac{\partial f_i}{\partial u_j}(r, \xi_{i1}, \ldots, \xi_{in}) U_j^{\lambda} = 0, \;\; i = 1, \ldots, n, \quad (12)$$

where $\eta = \eta(x, \lambda) \in \mathbb{R}^{n+1}_+$ and

$$\xi_{ij} = \xi_{ij}(x, \lambda) \in (\min(u_j(x), u_j(x^{\lambda})), \max(u_j(x), u_j(x^{\lambda})).$$

We have $|x^{\lambda}| = r^{\lambda} < r = |x|$ for $x \in \Sigma_{\lambda}$, so from (H1) we obtain the following system of inequalities for U_i^{λ}

$$\Delta U_i^{\lambda} + \sum_{1\leq j\leq n} \frac{\partial f_i}{\partial u_j}(r, \xi_{i1}, \ldots, \xi_{in}) U_j^{\lambda} \leq 0, \;\; i = 1, \ldots, n. \quad (13)$$

Next, we show that in (H4) the n-principal minors can be supposed to be strictly positive. To do so, we shift the diagonal coefficients of the matrix A, by making the following change of functions

$$\overline{U}_i^{\lambda} = \frac{U_i^{\lambda}}{g},$$

where

$$g(x) = \begin{cases} |x|^{-(N-2)/2} + 1 & \text{if} \quad N \geq 3 \\ \ln(\ln(|x| + 27)) & \text{if} \quad N = 2. \end{cases}$$

Simple computations yield $g \geq 1$ and $\Delta g < 0$ in $\mathbb{R}^N \setminus \{0\}$. This change of functions is classical in the scalar case, see [13]. See also [2] for some special types of systems in two dimensions.

It is easy to see that the new functions $\overline{U}_i^\lambda$ satisfy the following system

$$\Delta \overline{U}_i^\lambda + 2\frac{\nabla g}{g}\nabla \overline{U}_i^\lambda + \sum_{1\le j\le n}\left(\frac{\partial f_i}{\partial u_j}(r,\xi_{i1},\dots,\xi_{in}) + \delta_{ij}\frac{\Delta g}{g}\right)\overline{U}_j^\lambda \le 0, \tag{14}$$

for $i = 1,\dots,n$.

Our objective is to show that $\overline{U}_i^\lambda \ge 0$ (so $U_i^\lambda \ge 0$) in Σ_λ, for all $\lambda > \lambda^*$. Suppose for contradiction that there exist $\lambda > \lambda^*$ and $i_0 \in \{1,\dots,n\}$, for which $\inf_{\Sigma_\lambda} \overline{U}_{i_0}^\lambda < 0$. We set $J = \{j \mid \overline{U}_j^\lambda \ge 0 \text{ in } \Sigma_\lambda\} \subsetneqq \{1,\dots,n\}$ (J may be empty), and $I = \{1,\dots,n\}\setminus J$ (note that $i_0 \in I$). We consider only the inequalities in (14) which correspond to indices $i \in I$. Since $\overline{U}_j^\lambda \ge 0$ in Σ_λ for $j \in J$, by (H2) these inequalitites continue to hold if one cancels all terms containing $\overline{U}_j^\lambda$, with $j \in J$. We get, up to a permutation of the indices, a set of inequalities of type (14), for $i = 1,\dots,p$, where $p = |I|$. We note that the permutation does not affect assumptions (H2), (H4), that is, they remain valid for the submatrix $\left(\frac{\partial f_i}{\partial u_j}\right)_{1\le i,j\le p}$. Indeed, this is trivial for (H2), while for (H4) this fact follows from Lemma 2.2 in [6]. For the reader's convenience, we give here the statement of this lemma.

Lemma 1 *Let $M = (m_{ij})_{1\le i,j\le n}$ be a matrix such that $m_{ij} \le 0$ for $i \ne j$. Assume that all n-principal minors of M have positive determinants. Then*

(i) *all minors of M obtained by dropping lines and columns of the same order have positive determinants ;*

(ii) *if M_{ij} is the minor obtained by dropping the i^{th} line and the j^{th} column of M, we have*

$$(-1)^{i+j}\det M_{ij} \ge 0.$$

Since $\inf_{\Sigma_\lambda} \overline{U}_i^\lambda < 0$ for all $i = 1,\dots,p$, $\overline{U}_i^\lambda > 0$ in $\overline{B}_{R_0}^\lambda$, $\overline{U}_i^\lambda \to 0$ at infinity (here we use $g \ge 1$), we may take $x_1,\dots,x_p \in \Sigma_\lambda \setminus \overline{B}_{R_0}^\lambda$ such that

$$\overline{U}_i^\lambda(x_i) = \min_{\Sigma_\lambda} \overline{U}_i^\lambda < 0$$

(this implies $\Delta\overline{U}_i^\lambda(x_i) \ge 0$ and $\nabla\overline{U}_i^\lambda(x_i) = 0$). By writing the equations in (14) at $x_1,\dots,x_p$ respectively, we obtain

$$\sum_{1\le j\le p}\left(\frac{\partial f_i}{\partial u_j}(r,\xi_{i1},\dots,\xi_{ip}) + \delta_{ij}\frac{\Delta g}{g}\right)\overline{U}_j^\lambda(x_j) \le 0,\ i = 1,\dots,p, \tag{15}$$

where we used the fact that $\overline{U}_j^\lambda(x_j) \le \overline{U}_j^\lambda(x_i)$. The last system can be written in terms of matrices as

$$M\overline{U} = Y, \tag{16}$$

where $Y = (y_1, \dots, y_p)$, $M = (m_{ij})_{1\le i,j\le p}$, with

$$y_i \ge 0,\ m_{ij} = -\left(\frac{\partial f_i}{\partial u_j}(r, \xi_{i1}, \dots, \xi_{ip}) + \delta_{ij}\frac{\Delta g}{g}\right),\ i, j = 1, \dots, p,$$

and $\overline{U} = (\overline{U}_1^\lambda(x_1), \dots, \overline{U}_p^\lambda(x_p))$. Since $x_1, \dots, x_p \in \Sigma_\lambda \setminus \overline{B}_{R_0}^\lambda$, we have $r_i > R_0$, so, using the choice of λ^*, as in Section 2.1 we can see that $\xi_{ij}(x_k) \in (0, \varepsilon)$. Besides, we know that $\dfrac{\Delta g}{g} < 0$. Therefore, assumptions (H2) and (H4) yield $m_{ij} \le 0$ for $i \ne j$, so all n-principal minors of M have positive determinants.

Since M is invertible ($\det M > 0$), relation (16) yields $\overline{U} = M^{-1}Y$. Since $y_i \ge 0$, $i = 1, \dots, p$, it follows from Cramer's formula and statement (ii) of Lemma 1 that $\overline{U}_i^\lambda(x_i) \ge 0$, $i = 1, \dots, p$. But we have taken x_i to be such that $\overline{U}_i^\lambda(x_i) < 0$ – a contradiction.

Hence $\Lambda \le \lambda^* < +\infty$.

Step 2 *Either* $\Lambda = 0$, *or* $\Lambda > 0$ *and* $\overline{U}_i^\Lambda \equiv 0$, $i = 1, \dots, n$.

We argue by contradiction. Suppose $\Lambda > 0$ and $\overline{U}_{i_0}^\Lambda \not\equiv 0$, for some index $i_0 \in \{1, \dots, n\}$. By the definition of Λ, we see that $U_i^\Lambda \ge 0$ for all $i = 1, \dots, n$. The strong maximum principle, applied to each equation in (14), implies that either $U_i^\Lambda > 0$ or $U_i^\Lambda \equiv 0$ in Σ_Λ.

We claim that $U_i^\Lambda > 0$ in Σ_Λ for all $i = 1, \dots, n$. We know that $U_{i_0}^\Lambda > 0$. By (H3) there exists $j_0 \in \{1, \dots, n\} \setminus \{i_0\}$ such that

$$\frac{\partial f_{j_0}}{\partial u_{i_0}} > 0 \ \text{ in } \ \mathcal{O}. \tag{17}$$

In case $U_{j_0}^\Lambda \equiv 0$, by using the j_0^{th} inequality in (14) we get

$$\sum_{j\ne j_0} \frac{\partial f_{j_0}}{\partial u_j}(r, \xi_{j_0 1}, \dots, \xi_{j_0 n}) U_j^\Lambda \le 0,$$

which contradicts $U_{i_0}^\Lambda > 0$ for $|x| > R_0$, $x \notin \overline{B}_{R_0}^\Lambda$, in view of (H2) and (17). Hence $U_{j_0}^\Lambda > 0$. By (H3) we can choose $k_0 \in \{1, \dots, n\} \setminus \{i_0, j_0\}$ such that either $\dfrac{\partial f_{i_0}}{\partial u_{k_0}}$ or $\dfrac{\partial f_{j_0}}{\partial u_{k_0}}$ is positive in $\mathcal{O}$. As above, this means that $U_{k_0}^\Lambda > 0$. By repeating the same argument n times we conclude that we have $U_i^\Lambda > 0$ and $\overline{U}_i^\Lambda > 0$ in Σ_Λ, for all $i = 1, \dots, n$.

By the definition of Λ, there exists a sequence $\lambda_k \nearrow \Lambda$ such that

$$\min_{1 \le i \le n} \inf_{\Sigma_{\lambda_k}} \overline{U}_i^{\lambda_k} < 0.$$

By the argument in Step 1, and up to an extraction of a subsequence, we can construct $\{x_k\} \subset \Sigma_{\lambda_k}$ such that

$$\overline{U}_{i_0}^{\lambda_k}(x_k) = \min_{\Sigma_{\lambda_k}} \overline{U}_{i_0}^{\lambda_k} < 0,$$

for some $i_0 \in \{1, \dots, n\}$.

There are two cases to consider.

Case 1 $x_k \to \overline{x}$.

Since $\overline{U}_{i_0}^\Lambda > 0$ in Σ_Λ and $\overline{U}_{i_0}^\Lambda(\overline{x}) \le 0$, necessarily $\overline{x} \in T_\Lambda$, and $\overline{U}_{i_0}^\Lambda(\overline{x}) = 0$. Furthermore, $\nabla \overline{U}_{i_0}^\Lambda(\overline{x}) = 0$.

Since $\overline{U}_i^\Lambda > 0$ in Σ_Λ, $i = 1, \dots, n$, the i_0^{th} equation in (14) yields

$$\Delta \overline{U}_{i_0}^\Lambda + 2\frac{\nabla g}{g} \nabla \overline{U}_{i_0}^\Lambda + \left(\frac{\partial f_{i_0}}{\partial u_{i_0}} + \frac{\Delta g}{g} \right) \overline{U}_{i_0}^\Lambda \le 0 \ \text{ in } \ \Sigma_\Lambda.$$

Hopf's Lemma provides a contradiction.

Case 2 $|x_k| \to +\infty$.

In this case we have $x_k \in \Sigma_{\lambda_k} \setminus \overline{B}_{R_0}^{\lambda_k}$, for sufficiently large k, so the same argument as in Step 1 provides a contradiction.

Step 3 *Conclusion.*

We have obtained the following : either $\Lambda = 0$, or $\Lambda > 0$ and $U_i^\Lambda \equiv 0$ in Σ_Λ, $i = 1, \dots, n$. If $\Lambda > 0$, we are done. If $\Lambda = 0$, we have $U_i^0 \ge 0$ in Σ_0 and, repeating steps 1 and 2 in the opposite direction $(-\gamma)$, we prove that either there exists $\Lambda' > 0$ such that $U_i^{\Lambda'} \equiv 0$ or $U_i^0 \le 0$, $i = 1, \dots, n$. In both cases there is some λ_0 for which the functions u_i are symmetric with respect to the hyperplane T_{λ_0}. It is then standard to show that all u_i are radially symmetric with respect to some point $x_0 \in I\!R^N$. It is also easy to prove, as in section 2.1, that $\dfrac{du_i}{dr} < 0$ for $r = |x - x_0| > 0$.

The proof of Theorem 1 is complete.

3 Discussion

For simplicity in this section we consider the model case of two equations which we already described in section 2.1.

First we point out that Theorem 2 fails if we consider non-autonomous systems or systems of three or more equations. A counterexample is provided by the system

$$\begin{cases} \Delta u - u + u^p & = & 0 \\ \Delta v - v + v^p & = & 0 \\ \Delta w - w + u + v^2 & = & 0. \end{cases}$$

where $1 < p < \frac{N+2}{N-2}$. By taking $u = u(|x|)$ to be the unique positive (exponentially decreasing) solution of the first equation and setting $v = u(|x-x_0|)$, with $x_0 \neq 0$, we see that w cannot be symmetric.

Next, we are going to show that if all hypotheses (i)-(iii) are satisfied, except one of (ii) and (iii), in which the inverse inequality is strict, then no positive solution of (2) can exist.

Let u and v be solutions of (2) and let us put

$$\alpha = \frac{\partial g}{\partial u}(0,0)\ ,\ \beta = \frac{\partial g}{\partial v}(0,0)$$

$$\gamma = \frac{\partial f}{\partial u}(0,0)\ ,\ \delta = \frac{\partial f}{\partial v}(0,0).$$

First we suppose that $\alpha > 0$. We distinguish two cases.

Case 1 $\beta > 0$.

In this case Taylor's expansion yields

$$\Delta u + \alpha u + \beta v + o(u+v) = 0,$$

where $o(t)$ is a quantity such that $\dfrac{o(t)}{t} \to 0$ as $t \to 0$. Without loss of generality we suppose $f(0,0) = g(0,0) = 0$ (otherwise (2) has no solutions).

We fix $\varepsilon_0 > 0$ such that

$$|o(t)| \leq \frac{1}{2}\min\{\alpha, \beta\}|t|$$

if $|t| \leq \varepsilon_0$. Then we take $R_1 > 0$ such that $u(x) + v(x) < \varepsilon_0$, if $|x| \geq R_1$. We obtain that

$$\Delta u + \frac{1}{2}\alpha u + \frac{1}{2}\beta v \leq 0$$

in $I\!R^N \setminus B_{R_1}$. Hence

$$\Delta u + \frac{1}{2}\alpha u \leq 0 \quad \text{in } I\!R^N \setminus B_{R_1}.$$

A well-known sufficient condition for the maximum principle (see for instance [14]) implies that the maximum principle holds for the operator $\Delta + \frac{1}{2}\alpha$ in the annulus $C(R_2) = \{x \in I\!\!R^N \ : \ R_1 < |x| < R_2\}$, for any $R_2 > R_1$. This leads to a contradiction when R_2 is sufficiently large, for example, when R_2 is taken so that
$$\lambda_1(-\Delta, C(R_2)) < \frac{\alpha}{4}.$$

Case 2 $\beta = 0$.
In this case (iii) implies $\delta > 0$. By Taylor's expansion and (i) we obtain

$$\left\{ \begin{array}{rcl} \Delta u + \alpha u + o(u+v) & = & 0 \\ \Delta v + \delta v + o(u+v) & \leq & 0. \end{array} \right.$$

Hence
$$\Delta(u+v) + \min\{\alpha, \delta\}(u+v) + o(u+v) \leq 0,$$
and we conclude as in case 1.

Finally, suppose that $\det A < 0$. In this case (i) and (ii) imply that $\beta > 0$ and $\gamma > 0$. Again by Taylor's expansion we obtain

$$\left\{ \begin{array}{rcl} \Delta u + \alpha u + \beta v + o(u+v) & = & 0 \\ \Delta v + \gamma u + \delta v + o(u+v) & = & 0. \end{array} \right.$$

Consequently,
$$\left\{ \begin{array}{rcl} \Delta u + (\alpha - \varepsilon) u + (\beta - \varepsilon) v & \leq & 0 \\ \Delta v + (\gamma - \varepsilon) u + (\delta - \varepsilon) v & \leq & 0 \end{array} \right. \tag{18}$$
in $I\!\!R^N \setminus B_{R_1}$, for some large ball B_{R_1}, where $\varepsilon > 0$ is chosen so that the "perturbed" matrix A_ε has a negative determinant.

A sufficient condition for the maximum principle to hold for a linear system like (18) was derived in [7]. This condition implies that the maximum principle holds for the operator $\vec{\Delta} + A_\varepsilon$ in $C(R_2)$, for all $R_2 > R_1$. On the other hand, it was proved in [6] that a necessary condition for the maximum principle to hold in $C(R_2)$ is

$$\det\left[\lambda_1(-\Delta, C(R_2)) Id - A_\varepsilon\right] > 0,$$

which leads to a contradiction if R_2 is chosen so that

$$[\lambda_1(-\Delta, C(R_2))]^2 + \lambda_1(-\Delta, C(R_2)) \mathrm{tr} A_\varepsilon < -\det A_\varepsilon.$$

Let us also remark that the case when one of the quantities in (ii) and (iii) is equal to zero and (H4) does not hold appears to be quite difficult. Of course in that case there can be positive solutions. By now only very partial symmetry results are available (see [1], [3], [18] for scalar equations and [2], [9] for systems).

Finally, we note that in recent years there have been some results on maximum principles for non-cooperative systems. We do not know if a symmetry result can be proved in this case. We intend to investigate this question in the future.

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