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Abstract. This paper investigates new properties concerning the multifractal structure of a class of statistically self-similar measures. These measures include the well-known Mandelbrot multiplicative cascades, sometimes called independent random cascades. We evaluate the scale at which the multifractal structure of these measures becomes discernible. The value of this scale is obtained through what we call the growth speed in Hölder singularity sets of a Borel measure. This growth speed yields new information on the multifractal behavior of the rescaled copies involved in the structure of statistically self-similar measures. Our results are useful to understand the multifractal nature of various heterogeneous jump processes.

1. Introduction

This paper investigates new properties concerning the multifractal structure of statistically self-similar measures. The class of measures to which our results apply includes the well-known Mandelbrot multiplicative cascades [41], sometimes called independent random cascades. The case of another important class, the random Gibbs measures, is treated in [15].

Multifractal analysis is a field introduced by physicists in the context of fully developed turbulence [24]. It is now widely accepted as a pertinent tool in modeling other physical phenomena characterized by a wild spatial (or temporal) variability [42, 44, 34]. Given a positive measure μ defined on a compact subset of $\mathbb{R}^d$, performing the multifractal analysis of μ consists in computing (or estimating) the Hausdorff dimension $d_\mu(\alpha)$ of Hölder singularities sets E^μ_α. These sets E^μ_α are the level sets associated with the Hölder exponent

$$h_\mu(t) = \lim_{r\to 0^+} \frac{\log \mu(B(t,r))}{\log(r)}$$

(whenever it is defined at t). Thus

$$E^\mu_\alpha = \{t : h_\mu(t) = \alpha\}. \tag{1.1}$$

Of course, these limit behaviors are numerically unreachable, both when simulating model measures or when processing real data. Nevertheless, this difficulty can be circumvented since the Hausdorff dimension $d_\mu(\alpha)$ of E^μ_α can sometimes be numerically estimated by counting at scale 2^{-j} the number of boxes B (in a regular fine grid) such that $\mu(B) \approx 2^{-j\alpha}$. This number can formally be defined, for any

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scale $j \geq 0$, $\varepsilon > 0$ and $\alpha > 0$, by

$$N_j^\varepsilon(\alpha) = \#\{I \in \mathcal{I}_j : b^{-j(\alpha+\varepsilon)} \leq \mu(I) \leq b^{-j(\alpha-\varepsilon)}\},$$

where b is an integer ≥ 2 and $\mathcal{I}_j$ stands for the set of b-adic cubes of generation j contained in the support of μ. Then when some multifractal formalisms are fulfilled, it can be shown that, for some $\alpha > 0$, one has

$$d_\mu(\alpha) = \lim_{\varepsilon\to 0} \lim_{j\to+\infty} \frac{\log N_j^\varepsilon(\alpha)}{\log 2^j}. \tag{1.2}$$

This is the case for instance for the multifractal measures used as models [41]. In this frame it is thus natural to seek for theoretical results giving estimates of the first scale from which a substantial part of the singularity set E_α^μ is discernible when measuring the μ-mass of the elements B of the regular grid. In other words, we search for the first scale $J \geq 0$ such that for every $j \geq J$, one has $N_j^\varepsilon(\alpha) \approx 2^{jd_\mu(\alpha)}$. This is of course important for numerical applications and modelisation.

The properties studied in this paper and in [15] rely on this question. We provide new accurate information on the fine structure of multiplicative cascades, which bring some answers to the above problem. This study also shows that Mandelbrot measures and Gibbs measures have very different behaviors from the statistical self-similarity point of view, while they cannot be distinguished by the form of their multifractal spectra. Finally, our results are critical tools for the Hausdorff dimension estimate of a new class of limsup sets (see (1.6)) involved in multifractal analysis of recent jump processes [10, 12, 13, 14].

A definition of statistical self-similarity. Let us now specify what we mean by statistically self-similar measure in the sequel. Our point of view takes into account a structure which often arises in the construction of random measures generated by multiplicative processes.

Let $\mathcal{I}$ be the set of closed b-adic sub-hypercubes of $[0,1]^d$. A random measure $\mu(\omega)$ on $[0,1]^d$ ($d \geq 1$) is said to be statistically self-similar if there exist an integer $b \geq 2$, a sequence $Q_n(t,\omega)$ of random non-negative functions, and a sequence of random measures $(\mu^I)_{I\in\mathcal{I}}$ on $[0,1]^d$ such that:

1. For every $I \in \mathcal{I}$ and $g \in C([0,1]^d, \mathbb{R})$ one has ($\stackrel{d}{\equiv}$ means equality in distribution)

$$\int_{[0,1]^d} g(u)\mu(\omega)(du) \stackrel{d}{\equiv} \int_{[0,1]^d} g(u)\mu^I(\omega)(du).$$

2. With probability one, for every $I \in \mathcal{I}$ and $g \in C(I, \mathbb{R})$ one has

$$\int_I g(v)\mu(\omega)(dv) = \lambda(I) \int_I Q_n(v,\omega)\, g(v)\mu^I(\omega) \circ f_I^{-1}(dv), \tag{1.3}$$

where f_I stands for a similitude that maps $[0,1]^d$ onto I and λ is the Lebesgue measure.

Property 1. asserts that the measures μ^I and μ have the same probability distribution. Property 2. asserts that, up to the density Q_n, the behavior of the restriction of μ to I is ruled by the rescaled copy μ^I of μ.

Of course, the random density $Q_n(t,\cdot)$ plays a fundamental role, both in the construction of the measure μ, which is often equal to the almost sure weak limit of $Q_n(t,\cdot)\cdot\lambda$, and in the local behavior of μ.

We restrict ourselves to measures with support equal to $[0,1]^d$. Up to technical refinements, our point of view can easily be extended to measures which support is the limit set of more general iterated random similitudes systems ([27, 22, 45, 1, 5]).

Two main classes of measures illustrate conditions 1. and 2. The first one appears in random dynamical systems as Gibbs measures [31, 23, 7]. The other one consists in some $[0,1]^d$-martingales (in the sense of [28, 29]) considered in [9]. This second class of measures is illustrated in particular by independent multiplicative cascades [41, 30] as well as compound Poisson cascades [8] and their extensions [3, 9]. As claimed above, these two classes are quite identical regarding their multifractal structure in the sense that any measure in these classes is ruled by the so-called multifractal formalisms [18, 46]. However, the study of their self-similarity properties reveals the notable differences that exist between them. These differences are consequences of their construction's schemes: For random Gibbs measures, the copies μ^I only depend on the generation of I while they are all different for $[0,1]^d$-martingales (they depend on the interval I).

This difference is quantitatively measured thanks to a notion which is related with the multifractal structure, namely the *growth speed* in the μ^I Hölder singularities sets $E_\alpha^{\mu^I}$ (see Theorem A and B below). This quantity is precisely defined and studied in the rest of the paper for independent random cascades. It yields an estimate of the largest scale from which the observation of the μ^I's mass distribution accurately coincides with the prediction of the multifractal formalism.

New limsup-sets and conditioned ubiquity. As claimed above, the growth speed in Hölder singularities sets naturally appears in the computation of the Hausdorff dimension a new type of limsup-sets, which are themselves related to some heterogeneous jump processes. Particular cases of the jump processes considered in [10, 14] are for instance the sum of Dirac masses $\sum_{j\geq 0}\sum_{0\leq k\leq b^j-1} j^{-2}\mu([kb^{-j},(k+1)b^{-j}])\delta_{kb^{-j}}$, and the Lévy process X in multifractal time μ defined as $\big(X\circ\mu([0,t])\big)_{0\leq t\leq 1}$.

Let μ be a statistically self-similar measure whose support is $[0,1]$. Let $\{x_n\}$ denote a countable set of points and $(\lambda_n)_{n\geq 1}$ be a sequence decreasing to 0 such that $\limsup_{n\to\infty} B(x_n,\lambda_n)=[0,1]$. Typically we think as $\{x_n\}$ as being the sequence of jump points of the processes of [10, 14]. It appears that the multifractal nature of these processes is closely related to the computation of the Hausdorff dimension of the limsup-sets $K(\alpha,\xi)$ defined for $\alpha>0$ and $\xi>1$ and for some sequence (ε_n) converging to 0 by

$$K(\alpha,\xi)=\bigcap_{N\geq 1}\ \bigcup_{n\geq 1:\,\mu([x_n-\lambda_n,x_n+\lambda_n])=\lambda_n^{\alpha+\varepsilon_n}} [x_n-\lambda_n^\xi,x_n+\lambda_n^\xi]. \tag{1.4}$$

Heuristically, $K(\alpha,\xi)$ contains the points that are infinitely often close to a jump point x_n at rate ξ relatively to λ_n, upon the condition that $\nu([x_n-\lambda_n,x_n+\lambda_n])\sim\lambda_n^\alpha$. This condition implies that ν has roughly a Hölder exponent α at scale λ_n. One of the main results of [10, 11] (see also [12]) is the computation of the Hausdorff dimension of $K(\alpha,\xi)$. The value of this dimension is related to the free energy function τ_μ considered in the multifractal formalism for measures in [26, 18]. For every $q\in\mathbb{R}$ and for every integer $j\geq 1$, let us introduce the quantities

$$\tau_{\mu,j}(q)=-\frac{1}{j}\log_b\sum_{I\in\mathcal{I}_j}\mu(I)^q \quad\text{and}\quad \tau_\mu(q)=\liminf_{j\to\infty}\tau_{\mu,j}(q). \tag{1.5}$$

The Legendre transform τ_μ^* of τ_μ at $\alpha > 0$ is defined as $\tau_\mu^*(\alpha) := \inf_{q\in\mathbb{R}} \alpha q - \tau_\mu(q)$.

Under suitable assumptions on (λ_n), we prove in [10, 11] that, for all α such that $\tau_\mu^*(\alpha) > 0$ and all $\xi \geq 1$, with probability one, (dim stands for the Hausdorff dimension)

$$\dim K(\alpha,\xi) = \tau_\mu^*(\alpha)/\xi. \tag{1.6}$$

This achievement is a non-trivial generalization of what is referred to as "ubiquity properties" (see [20] and references therein) of the resonant system ([2]) $\{(x_n,\lambda_n)\}$. The main difficulty here lies in the fact that μ may be a multifractal measure and not just the uniform Lebesgue measure. Results on growth speed in Hölder singularity set are determinant to obtain estimate (1.6).

Growth speed in μ^I's Hölder singularity sets. Let μ be a statistically self-similar positive Borel measure as described above. As we said, multifractal analysis of μ [23, 33, 7, 27, 43, 1, 5] usually considers Hölder singularities sets of the form (1.1) and their Hausdorff dimension $d_\mu(\alpha)$, which is a measure of their size. The method used to compute $d_\mu(\alpha)$ is to find a random measure μ_α (of the same nature as μ) such that μ_α is concentrated on $E_\alpha^\mu \cap E^{\mu_\alpha}_{\tau_\mu^*(\alpha)}$. This measure μ_α is often referred to as an analyzing measure of μ at α. Then, by the Billingsley lemma ([17] pp 136–145), one gets $d_\mu(\alpha) = \tau_\mu^*(\alpha)$, and the multifractal formalism for measures developed in [18] is said to hold for μ at α. Finally, the estimate (1.2) is a direct consequence of the multifractal formalism ([48]) for the large deviation spectrum. Thus the existence of μ_α has wide consequence regarding the possibility of measuring the mass distribution of μ at large enough scales.

In this paper we refine the classical approach by considering, instead of the level sets E_α^μ, the finer level sets $\widetilde{E}^\mu_{\alpha,p}$ and $\widetilde{E}^\mu_\alpha$ defined for a sequence ε_n going down to 0 by

$$\widetilde{E}^\mu_{\alpha,p} = \Big\{t\in[0,1]^d : \forall\, n\geq p,\ b^{-n(\alpha+\varepsilon_n)} \leq \mu\big(I_n(t)\big)\big) \leq b^{-n(\alpha-\varepsilon_n)}\Big\} \tag{1.7}$$

$$\text{and}\quad \widetilde{E}^\mu_\alpha = \bigcup_{p\geq 1} E^\mu_{\alpha,p}. \tag{1.8}$$

It is possible to choose $(\varepsilon_n)_{n\geq1}$ so that with probability one, for all the exponents α such that $\tau_\mu^*(\alpha) > 0$ one has $\mu_\alpha(\widetilde{E}^\mu_\alpha) = \|\mu_\alpha\|$.

Since the sets sequence $\{\widetilde{E}^\mu_{\alpha,p}\}_{p\geq1}$ is non-decreasing and $\mu_\alpha(\widetilde{E}^\mu_\alpha) = \|\mu_\alpha\|$, we can define the growth speed of $\widetilde{E}^\mu_{\alpha,p}$ as the smallest value of p for which the μ_α-measure of $\widetilde{E}^\mu_{\alpha,p}$ reaches a certain positive fraction $f\in(0,1)$ of the mass of μ_α, i.e.

$$GS(\mu,\alpha) = \inf\Big\{p : \mu_\alpha(\widetilde{E}^\mu_{\alpha,p}) \geq f\,\|\mu_\alpha\|\Big\}.$$

For each copy μ^I of μ, the corresponding family of analyzing measures μ^I_α exists and are related with μ^I as μ_α is related with μ. The result we focus on in the following is the asymptotic behavior as the generation of I goes to ∞ of

$$GS(\mu^I,\alpha) = \inf\Big\{p : \mu^I_\alpha(\widetilde{E}^{\mu^I}_{\alpha,p}) \geq f\,\|\mu^I_\alpha\|\Big\}. \tag{1.9}$$

This number yields an estimate of the number of generations needed to observe a substantial amount of the singularity set $E^{\mu^I}_\alpha$. Let

$$\mathcal{N}_n(\mu^I,\alpha) = \#\big\{J\in\mathcal{I}_n : b^{-n(\alpha+\varepsilon_n)} \leq \mu^I(J) \leq b^{-n(\alpha-\varepsilon_n)}\big\}.$$

As a counterpart of controlling $GS(\mu^I,\alpha)$, we shall also control the smallest rank n from which $\mathcal{N}_n(\mu^I,\alpha)$ behaves like $b^{n\tau_\mu^*(\alpha)}$. This rank is defined by

$$GS'(\mu^I,\alpha)=\inf\left\{p:\forall\ n\geq p, b^{n(\tau_\mu^*(\alpha)-\varepsilon_n)}\leq \mathcal{N}_n(\mu^I,\alpha)\leq b^{n(\tau_\mu^*(\alpha)+\varepsilon_n)}\right\}$$

and yields far more precise information than a result like (1.2).

A simplified version of the main results. In this paper, we focus on the one-dimensional case and deal with a slight extension of the first example of $[0,1]$-martingales introduced in [41], called independent random cascades (see Section 3.1). Let us give simplified versions of the main results detailed in Section 3. We start with a recall of the theorem proved in [15].

Theorem A. *Let μ be a random Gibbs measure as in* [15] *(in this case $\mu^I=\mu^J$ if I and J are of the same generation). Suppose that τ_μ is C^2. Let $\beta>0$. There exists a choice of $(\varepsilon_n)_{n\geq 1}$ such that, with probability one, for all $\alpha>0$ such that $\tau_\mu^*(\alpha)>0$, if I is of generation j large enough, then $GS(\mu^I,\alpha)\leq \exp\sqrt{\beta\log j}$.*

The fact that $GS(\mu^I,\alpha)$ behaves like $o(j)$ as $j\to\infty$ is a crucial property needed to establish (1.6) for random Gibbs measures.

Theorem **B** shall be compared with Theorem **A**. Under suitable assumptions, we have (see Theorem 2)

Theorem B. *Let μ be an independent random cascade. Let $\eta>0$. There exists a choice of $(\varepsilon_n)_{n\geq 1}$ such that, with probability one, for all $\alpha>0$ such that $\tau_\mu^*(\alpha)>0$, if I is of generation j large enough, then $GS(\mu^I,\alpha)\leq j\log^\eta j$.*

Consequently we lost the uniform behavior over $\mathcal{I}_j$ of $GS(\mu^I,\alpha)$ like $o(j)$, which was determinant to get (1.6). In fact this "worse" behavior is not surprising, since Theorem B controls simultaneously b^j distinct measures μ^I at each scale j, while Theorem A controls only one measure at each scale. Nevertheless, this technical difficulty can be circumvented, by using a refinement of Theorem B (see Theorem 3 and 6), which is enough to get (1.6).

Theorem C. *Let μ be an independent random cascade. Let $\eta>0$. There exists a choice of $(\varepsilon_n)_{n\geq 1}$ such that for every $\alpha>0$ such that $\tau_\mu^*(\alpha)>0$, with probability one, for μ-almost every t, for j large enough, $GS(\mu^{I_j(t)},\alpha)\leq j\log^{-\eta} j$.*

The paper is organized as follows. Section 2 gives new definitions and establishes two general propositions useful for our main results. In Section 3 independent random cascades are defined in an abstract way. This makes it possible to consider Mandelbrot measures as well as their substitute in the critical case of degeneracy. Then the main results (Theorems 1, 2 and 3) are stated and proved. Theorem 4 is a counterpart of Theorem B in terms of $GS'(\mu^I,\alpha)$. Theorem 5 deals with a problem connected with the estimate of the growth speed in singularities sets, namely the estimation of the speed of convergence of $\tau_{\mu,j}$ towards τ_μ. Eventually, Section 4 is devoted to the version of Theorem 3 needed to get (1.6).

The techniques presented in this paper can be applied to derive similar results for other statistically self-similar $[0,1]$-martingales described in [8, 3, 9].

2. General estimates for the growth speed in singularity sets

2.1. Measure of fine level sets: a neighboring boxes condition. Let $(\Omega, \mathcal{B}, \mathbb{P})$ stand for the probability space on which the random variables in this paper are defined. Fix an integer $b \geq 2$.

Let $\mathcal{A} = \{0, \dots, b-1\}$. For every $w \in \mathcal{A}^* = \bigcup_{j\geq 0} \mathcal{A}^j$ ($\mathcal{A}^0 := \{\emptyset\}$), let I_w be the closed b-adic subinterval of $[0,1]$ naturally encoded by w. If $w \in \mathcal{A}^j$, we set $|w| = j$.

For $n \geq 1$ and $0 \leq k \leq b^n - 1$, $I_{n,k}$ denotes the interval $[kb^{-n}, (k+1)b^{-n})$. If $t \in [0,1)$, $k_{n,t}$ is the unique integer such that $t \in [k_{n,t}b^{-n}, (k_{n,t}+1)b^{-n})$. We denote by $w^{(n)}(t)$ the unique element w of $\mathcal{A}^n$ such that $I_w = [k_{n,t}b^{-n}, (k_{n,t}+1)b^{-n}]$.

With $w \in \mathcal{A}^j$ can be associated a unique number $i(w) \in \{0, 1, \dots, b^j - 1\}$ such that $I_w = [i(w)b^{-j}, (i(w)+1)b^{-j}]$. Then, if $(v, w) \in \mathcal{A}^j$, $\delta(v, w)$ stands for $|i(v) - i(w)|$.

Let μ and m be two positive Borel measures with supports equal to $[0,1]$.

Let $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 0}$ be a positive sequence, $N \geq 1$, and $\alpha \geq 0$.

We consider a slight refinement of the sets introduced in (1.8): For $p \geq 1$, define

$$E^\mu_{\alpha,p}(N, \widetilde{\varepsilon}) = \left\{ t \in [0,1] : \begin{cases} \forall\, n \geq p,\ \forall w \in \mathcal{A}^n \text{ such that } \delta(w, w^{(n)}(t)) \leq N, \\ \forall\, \gamma \in \{-1, 1\},\ \text{one has } b^{\gamma n(\alpha - \gamma\varepsilon_n)} \mu(I_w)^\gamma \leq 1 \end{cases} \right\} \tag{2.1}$$

$$\text{and } E^\mu_\alpha(N, \widetilde{\varepsilon}) = \bigcup_{p\geq 1} E^\mu_{\alpha,p}(N, \widetilde{\varepsilon}). \tag{2.2}$$

This set contains the points t for which, at each scale n large enough, the μ-measure of the $2N+1$ neighbors of I_{n,k_t} belongs to $[b^{-n(\beta+\varepsilon_n)}, b^{-j(\beta-\varepsilon_n)}]$. The information on neighboring intervals is involved in proving (1.6).

For $n \geq 1$ and $\varepsilon, \eta > 0$, let us define the quantity

$$S_n^{N,\varepsilon,\eta}(m, \mu, \alpha) = \sum_{\gamma\in\{-1,1\}} b^{n(\alpha-\gamma\varepsilon)\gamma\eta} \sum_{v,w\in\mathcal{A}^n:\ \delta(v,w)\leq N} m(I_v)\mu(I_w)^{\gamma\eta}. \tag{2.3}$$

The following result is already established in [15], but we give the proof for completeness.

Proposition 1. *Let $(\eta_n)_{n\geq 1}$ be a positive sequence.*
If $\sum_{n\geq 1} S_n^{N,\varepsilon_n,\eta_n}(m, \mu, \alpha) < +\infty$, then $E^\mu_\alpha(N, \widetilde{\varepsilon})$ is of full m-measure.

Remark 1. *Similar conditions were used in* [6] *to obtain a comparison between the multifractal formalisms of* [18] *and* [46].

Proof. For $\gamma \in \{-1, 1\}$ and $n \geq 1$, let us define

$$E^\mu_\alpha(N, \varepsilon_n, \gamma) = \left\{ t \in [0,1] : \begin{cases} \forall\ w \in \mathcal{A}^n \text{ such that } \delta(w, w^{(n)}(t)) \leq N, \\ \text{one has } b^{\gamma n(\alpha-\gamma\varepsilon_n)}\mu(I_w)^\gamma \leq 1 \end{cases} \right\}. \tag{2.4}$$

For $t \in [0,1]$, if there exists (a necessarily unique) $w \in \mathcal{A}^n$ such that $i(w) - i(w^{(n)}(t)) = k$, this word w is denoted $w_k^{(n)}(t)$. For $\gamma \in \{-1, 1\}$, let $S_{n,\gamma} = \sum_{-N\leq k\leq N} m_k$ with

$$m_k = m\left(\left\{ t \in [0,1] : i(w) - i(w^{(n)}(t)) = k \Rightarrow b^{\gamma n(\alpha-\gamma\varepsilon_n)}\mu(I_w)^\gamma > 1 \right\} \right).$$

One clearly has

$$m\Big((E^\mu_\alpha(N, \varepsilon_n, -1))^c \bigcup E^\mu_\alpha(N, \widetilde{\varepsilon}_n, 1))^c \Big) \leq S_{n,-1} + S_{n,1}. \tag{2.5}$$

Fix $\eta_n > 0$ and $-N \le k \le N$. Let $Y(t)$ be random variable defined to be equal to $b^{\gamma n(\alpha-\gamma\varepsilon_n)\eta_n}\mu\big(I_{w_k^{(n)}(t)}\big)^{\gamma\eta_n}$ if $w_k^{(n)}(t)$ exists or 0 otherwise. The Markov inequality applied to $Y(t)$ with respect to m yields $m_k \le \int Y(t)dm(t)$. Since Y is constant over each b-adic interval I_v of generation n, we get

$$m_k \le \sum_{v,w\in\mathcal{A}^n:\, i(w)-i(v)=k} b^{n(\alpha-\gamma\varepsilon_n)\gamma\eta_n} m(I_v)\mu(I_w)^{\gamma\eta_n}.$$

Summing over $|k| \le N$ yields $S_{n,-1} + S_{n,1} \le S_n^{N,\varepsilon_n,\eta_n}(m,\mu,\alpha)$. The conclusion follows from (2.5) and from the Borel-Cantelli Lemma. □

2.2. **Uniform growth speed in singularity sets.** Let Λ be a set of indexes, and Ω^* a measurable subset of Ω of probability 1. Some notations and technical assumptions are needed to state the general result that we shall apply in Section 3. These assumptions describe a common situation in multifractal analysis. In particular the measures in the following sections satisfy these requirements.

• For every $\omega \in \Omega^*$, we consider two sequences of families of measures, namely $\Big(\{\mu_\lambda^w\}_{\lambda\in\Lambda}\Big)_{w\in\mathcal{A}^*}$ and $\Big(\{m_\lambda^w\}_{\lambda\in\Lambda}\Big)_{w\in\mathcal{A}^*}$ (indexed by $\mathcal{A}^*$) such that for every $w \in \mathcal{A}^*$, the elements of the families $\{\mu_\lambda^w\}_{\lambda\in\Lambda}$ and $\{m_\lambda^w\}_{\lambda\in\Lambda}$ are positive finite Borel measures whose support is $[0,1]$. For $\nu \in \{\mu, m\}$, $\{\nu_\lambda^\emptyset\}_{\lambda\in\Lambda}$ is written $\{\nu_\lambda\}_{\lambda\in\Lambda}$.

• We consider an integer $N \ge 1$, a positive sequence $\widetilde{\varepsilon} = (\varepsilon_n)_{n\ge1}$, and a family of positive numbers $(\alpha_\lambda)_{\lambda\in\Lambda}$. Then, remembering (2.4) let us consider for every $j \ge 0$, $w \in \mathcal{A}^j$ and $p \ge 1$ the sets

$$E_{\alpha_\lambda,p}^{\mu_\lambda^w}(N,\widetilde{\varepsilon}) = \bigcap_{n\ge p} E_{\alpha_\lambda}^{\mu_\lambda^w}(N,\varepsilon_n,-1) \cap E_{\alpha_\lambda}^{\mu_\lambda^w}(N,\varepsilon_n,1). \tag{2.6}$$

The sets $\{E_{\alpha_\lambda,n}^{\mu_\lambda^w}(N,\widetilde{\varepsilon})\}_n$ form a non-decreasing sequence. We assume that m_λ^w is concentrated on $\lim_{p\to+\infty} E_{\alpha_\lambda,p}^{\mu_\lambda^w}(N,\widetilde{\varepsilon})$. One defines the growth speed of $E_{\alpha_\lambda,p}^{\mu_\lambda^w}(N,\widetilde{\varepsilon})$ as

$$GS(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda,N,\widetilde{\varepsilon}) = \inf\Big\{p\ge 1 : m_\lambda^w\big(E_{\alpha_\lambda,p}^{\mu_\lambda^w}(N,\widetilde{\varepsilon})\big) \ge 1/2\Big\}. \tag{2.7}$$

This number, which may be infinite, is a measure of the number p of generations needed for $E_{\alpha_\lambda,p}^{\mu_\lambda^w}(N,\widetilde{\varepsilon})$ to recover a certain given fraction (here chosen equal to 1/2) of the measure m_λ^w. Since $\mu_\lambda^w(\lim_{p\to+\infty} E_{\alpha_\lambda,p}^{\mu_\lambda^w}(N,\widetilde{\varepsilon})) = 1$, $GS(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda,N,\widetilde{\varepsilon}) < \infty$.

• We assume that for every positive sequence $\widetilde{\eta} = (\eta_j)_{j\ge0}$, there exist a random vector $\big(U(\widetilde{\eta}),V(\widetilde{\eta})\big) \in \mathbb{R}_+ \times \mathbb{R}_+^{\mathbb{N}}$ and a sequence $(U^w,V^w)_{w\in\mathcal{A}^*}$ of copies of $\big(U(\widetilde{\eta}),V(\widetilde{\eta})\big)$ and finally a sequence $(\psi_j(\widetilde{\eta}))_{j\ge0}$, such that for $\mathbb{P}$-almost every $\omega \in \Omega^*$,

$$\forall\, w \in \mathcal{A}^*,\ \forall n \ge \psi_{|w|}(\widetilde{\eta}),\ \begin{cases} U^w \le \inf_{\lambda\in\Lambda} \|m_\lambda^w\|, \\ V_n^w \ge \sup_{\lambda\in\Lambda} S_n^{N,\varepsilon_n,\eta_n}(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda), \end{cases} \tag{2.8}$$

where $S_n^{N,\varepsilon_n,\eta_n}(m_\lambda^w,\mu_\lambda^w,\alpha_\lambda)$ is defined in (2.3) (if $w \in \mathcal{A}_j$, remember that $|w| = j$). This provides us with a uniform control over $\lambda \in \Lambda$ of the families of measures $(m_\lambda^w,\mu_\lambda^w)_{w\in\mathcal{A}^*}$.

Proposition 2 (**Uniform growth speed in singularity sets**). *Assume that two sequences of positive numbers $\widetilde{\eta} = (\eta_j)_{j\geq 0}$ and $(\rho_j)_{j\geq 0}$ are fixed.*
Let $(\mathcal{S}_j)_{j\geq 0}$ be a sequence of integers such that $\mathcal{S}_j \geq \psi_j(\widetilde{\eta})$. If

$$\sum_{j\geq 0} b^j \mathbb{P}\Big(U(\widetilde{\eta}) \leq b^{-\rho_j}\Big) < \infty \text{ and } \sum_{j\geq 0} b^j b^{\rho_j} \sum_{k\geq \mathcal{S}_j} \mathbb{E}\Big(V_k(\widetilde{\eta})\Big) < \infty, \tag{2.9}$$

then, with probability one, for j large enough, for every $w \in \mathcal{A}^j$ and $\lambda \in \Lambda$, one has $GS(m_\lambda^w, \mu_\lambda^w, \alpha_\lambda, N, \widetilde{\varepsilon}) \leq \mathcal{S}_j$.

Proof. Fix $j \geq 1$ and $w \in \mathcal{A}^j$. As shown in the proof of Proposition 1, for every $n \geq 1$ and every $\lambda \in \Lambda$, one can write

$$m_\lambda^w\Big(\big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, -1)\big)^c \cup \big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, 1)\big)^c\Big) \leq S_n^{N,\varepsilon_n,\eta_n}(m_\lambda^w, \mu_\lambda^w, \alpha_\lambda).$$

Thus, using (2.8), one gets

$$m_\lambda^w\Big(\bigcup_{n\geq \mathcal{S}_j} \big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, -1)\big)^c \cup \big(E_{\alpha_\lambda}^{\mu_\lambda^w}(N, \varepsilon_n, 1)\big)^c\Big) \leq \sum_{n\geq \mathcal{S}_j} V_n^w. \tag{2.10}$$

Let us apply the "statistical self-similar control" (2.9) combined with the Borel-Cantelli lemma. On the one hand, the left part of (2.9) yields $\sum_{j\geq 1} \mathbb{P}\Big(\exists\ w \in \mathcal{A}^j,\ U^w \leq b^{-\rho_j}\Big) < \infty$. Hence, with probability one, for j large enough and for all $w \in \mathcal{A}^j$,

$$\sup_{\lambda\in\Lambda} \|m_\lambda^w\|^{-1} \leq (U^w)^{-1} \leq b^{\rho_j}. \tag{2.11}$$

On the other hand, the right part of (2.9) yields

$$\sum_{j\geq 1} \mathbb{P}\Big(\exists\ w \in \mathcal{A}^j,\ b^{\rho_j} \sum_{n\geq \mathcal{S}_j} V_n^w \geq 1/2\Big) \leq 2 \sum_{j\geq 1} b^j b^{\rho_j} \mathbb{E}\Big(\sum_{n\geq \mathcal{S}_j} V_n^w\Big) < \infty.$$

This implies that with probability one, $b^{\rho_j} \sum_{n\geq \mathcal{S}_j} V_n^w < 1/2$ for every j large enough and for all $w \in \mathcal{A}^j$.

Thus, by (2.11), $\sup_{\lambda\in\Lambda} \frac{\sum_{n\geq \mathcal{S}_j} V_n^w}{\|m_\lambda^w\|} < 1/2$. Combining this with (2.10) and (2.7), one gets that for every $\lambda \in \Lambda$, $GS(m_\lambda^w, \mu_\lambda^w, \alpha_\lambda, N, \widetilde{\varepsilon}) \leq \mathcal{S}_j$. □

3. Main results for independent random cascades

3.1. **Definition.** Let $v = (v_1, \ldots, v_{|v|}) \in \mathcal{A}^*$. We need the following truncation notation: For every $k \in \{1, \ldots, |v|\}$, $v|k$ is the word $(v_1, \ldots, v_k) \in \mathcal{A}^k$, and by convention $v|0$ is the empty word $\emptyset$.

We focus in this paper on the measures introduced in [41] and more recently in [5]. A measure $\mu(\omega)$ is said to be an independent random cascade if it has the following property: There exist a sequence of random positive vectors $\big(W(w) = (W_0(w), \ldots, W_{b-1}(w))\big)_{w\in\mathcal{A}^*}$ and a sequence of random measures $(\mu^w)_{w\in\mathcal{A}^*}$ such that

- **(P1)** for all $v, w \in \mathcal{A}^*$, $\mu(I_{vw}) = \mu^v(I_w) \prod_{k=0}^{|v|-1} W_{v_{k+1}}(v|k)$ $(\mu^\emptyset = \mu)$,
- **(P2)** the random vectors $W(w)$, for $w \in \mathcal{A}^*$, are i.i.d. with a vector $W = (W_0, \ldots, W_{b-1})$ such that $\sum_{k=0}^{b-1} \mathbb{E}(W_k) = 1$,
- **(P3)** for all $v \in \mathcal{A}^*$, $\big(\mu^v(I_w)\big)_{w\in\mathcal{A}^*} \equiv \big(\mu(I_w)\big)_{w\in\mathcal{A}^*}$. Moreover, for every $j \geq 1$, the measures μ^v, $v \in \mathcal{A}^j$, are mutually independent,

• **(P4)** for every $j \geq 1$, the σ-algebras $\sigma\big(W(w):\ w \in \cup_{0\leq k\leq j-1}\mathcal{A}^k\big)$ and $\sigma(\mu^v(I_w):\ v \in \mathcal{A}^j,\ w \in \mathcal{A}^*)$ are independent.

Let $\big(W(w)\big)_{w\in\mathcal{A}^*}$ be as above. For $q \in \mathbb{R}$ define the function

$$\widetilde{\tau}_\mu(q) = -\log_b \mathbb{E}\Big(\sum_{k=0}^{b-1} W_k^q\Big) \in \mathbb{R}\cup\{-\infty\}. \tag{3.1}$$

The two classes of measures we deal with are the following.

Non-degenerate multiplicative martingales when $\widetilde{\tau}_\mu'(1^-) > 0$. With probability one, $\forall\, v \in \mathcal{A}^*$, the sequence of measures $\{\mu_j^v\}_{j\geq 0}$

$$\frac{d\mu_j^v}{d\ell}(t) = b^j \prod_{k=0}^{j-1} W_{t_{k+1}}(v\cdot t|k) \tag{3.2}$$

defined on $[0,1]$ converges weakly, as $n\to\infty$ to a measure μ^v. For $\mu = \mu^\emptyset$:
(1) Properties **(P1)** to **(P4)** are satisfied;
(2) If $\widetilde{\tau}_\mu'(1^-) > 0$, the total masses $\|\mu^v\|$ are almost surely positive, and their expectation is equal to 1 ([30, 21]).

The modified construction in the critical case $\widetilde{\tau}_\mu'(1^-) = 0$. Suppose that $\widetilde{\tau}_\mu'(1^-) = 0$ and $\widetilde{\tau}_\mu(h) > -\infty$ for some $h > 1$. Then, with probability one, for all $v \in \mathcal{A}^*$, the function of b-adic intervals
(3.3)

$$\mu^v(I_w) = \lim_{j\to\infty} -\sum_{u\in\mathcal{A}^j}\Big(\prod_{k=0}^{|w|+j-1} W_{(w\cdot u)_{k+1}}\big(v\cdot(w\cdot u|k)\big)\Big)\log \prod_{k=0}^{|w|+j-1} W_{(w\cdot u)_{k+1}}\big(v\cdot(w\cdot u|k)\big)$$

is well defined and yields a positive Borel measure whose support is $[0,1]$ ([5, 37]).
For $\mu = \mu^\emptyset$,
(1) properties **(P1)** to **(P4)** are satisfied;
(2) $\mathbb{E}(\|\mu\|^h) < \infty$ for $h \in [0,1)$ but $\mathbb{E}(\|\mu\|) = \infty$.

3.2. **Analyzing measures.** In both above cases we define J as the interior of the interval $\{q \in \mathbb{R} : \widetilde{\tau}_\mu'(q)q - \widetilde{\tau}_\mu(q) > 0\}$. One always has $(0,1) \subset J$ and $J \subset (-\infty, 1)$ if $\widetilde{\tau}_\mu'(1^-) = 0$. We assume that:
- If $\widetilde{\tau}_\mu'(1^-) > 0$, J contains the closed interval $[0,1]$,
- If $\widetilde{\tau}_\mu'(1^-) = 0$ then $0 \in J$.

For $q \in J$, $v \in \mathcal{A}^*$, $j \geq 1$, let $\mu_{q,j}^v$ be the measure defined as μ_j^v in (3.2) but with the sequence $\big(W_q(v\cdot w) = (b^{\widetilde{\tau}_\mu(q)}W_0(v\cdot w)^q, \dots, b^{\widetilde{\tau}_\mu(q)}W_{b-1}(v\cdot w)^q)\big)_w$ instead of $\big(W(v\cdot w)\big)_w$. It is proved in [5] that there exists a subset Ω^* of Ω of probability 1 such that $\forall \omega \in \Omega^*$, $\forall\, v \in \mathcal{A}^*$ and $\forall\, q \in J$, the sequence $\mu_{q,j}^v$ converges weakly to a positive measure μ_q^v.

If one denotes $\mu_q^\emptyset = \mu_q$, $Y_q = \|\mu_q^\emptyset\|$, and $Y_q(v) = \|\mu_q^v\|$ for $v \in \mathcal{A}^*$, it is proved in [16, 5] that with probability one the mappings $q \in J \mapsto Y_q(v)$ are analytic and positive. Moreover, [5] proves that $\tau_\mu \equiv \widetilde{\tau}_\mu$ on J almost surely.

Eventually, one can see that $J \supset \mathbb{R}_+$ (resp. $\mathbb{R}_-$) if and only if $\widetilde{\tau}_\mu(hq) - h\widetilde{\tau}_\mu(q) > 0$ for all $q \in \mathbb{R}_+$ (resp. $\mathbb{R}_-$) and $h > 1$, which amounts to saying that $\forall\, q \in \mathbb{R}_+$ (resp. $\mathbb{R}_-$), $\mathbb{E}(Y_q^h) < \infty$ and $h > 1$ (see the proof of Lemma 3).

3.3. **Main results.** Recall that for an independent random cascade μ, if $\widetilde{\tau}'_\mu(1^-) > 0$ we assume that J contains $[0,1]$, and if $\widetilde{\tau}'_\mu(1^-) = 0$ then $J \subset (-\infty, 1)$, and we assume that $0 \in J$.

Theorem 1. *Let μ be an independent random cascade. Let N be an integer ≥ 1 and $\widetilde{\varepsilon} = (\varepsilon_n)_{n\geq 1}$ a sequence of positive numbers going to 0. Assume that $\forall \alpha > 0$, the series $\sum_{n\geq 1} n b^{-n\alpha\varepsilon_n^2}$ converges.*

Then, with probability one, for every $q \in J$, $\tau_\mu(q) = \widetilde{\tau}_\mu(q)$, and the two level sets $E^{\mu_q}_{\tau'_\mu(q)q-\tau_\mu(q)}(N, \widetilde{\varepsilon})$ and $E^{\mu}_{\tau'_\mu(q)}(N, \widetilde{\varepsilon})$ are both of full μ_q-measure.

Remark 2. *The conclusions of Theorem 1 hold as soon as*

$$\exists \eta > 0 \text{ such that for every } n,\ \varepsilon_n \geq n^{-1/2} \log(n)^{1/2+\eta}. \tag{3.4}$$

Theorem 2 (**Growth speed in Hölder singularity sets**)**.** *Under the assumptions of Theorem 1, assume that $(\varepsilon_n)_n$ satisfies (3.4) and that there exists $A > 1$ such that with probability one $A^{-1} \leq W_i$ (resp. $W_i \leq A$) for all $0 \leq i \leq b-1$. Let K be a compact subinterval of $J \cap \mathbb{R}_+$ (resp. $J \cap \mathbb{R}_-$).*

Then, with probability one, for j large enough, for all $q \in K$ and $w \in \mathcal{A}^j$,

$$\max\left(GS\left(\mu_q^w, \mu^w, \tau'_\mu(q), N, \widetilde{\varepsilon}\right), GS\left(\mu_q^w, \mu_q^w, \tau'_\mu(q)q - \tau_\mu(q), N, \widetilde{\varepsilon}\right)\right) \leq \mathcal{S}_j$$

with $\mathcal{S}_j = \left[\exp\left(\left(j\log(j)^\eta\right)^{\frac{1}{1+2\eta}}\right)\right]$.

If there exists $\eta > 0$ such that for every n, $\varepsilon_n \geq \log(n)^{-\eta}$, the above conclusion holds with $\mathcal{S}_j = \left[j\log(j)^{\eta'}\right]$, for any $\eta' > 2\eta$.

Remark 3. *In Theorem 2, the first choice of ε_n corresponds to the "best" choice for the speed of convergence ε_n, and the growth speed $\mathcal{S}_j$ is very slow. To the contrary, the second choice for ε_n is "worse", but as a counterpart $\mathcal{S}_j$ is improved.*

We assume that the number of neighbors N is fixed. In fact, it is not difficult to consider a sequence of neighbors N_n simultaneously with the speed of convergence ε_n. This number N_n can then go to ∞ under the condition that $\log N_n = o(n\varepsilon_n^2)$. Another modification would consist in replacing the fixed fraction f in (1.9) by a fraction f_j going to 1 as j goes to ∞. The choice $f_j = 1 - b^{-s_j}$ with $s_j = j$ is convenient. These two improvements yield technical complications, but comparable results are easily derived from the proofs we propose.

The growth speed obtained in Theorem 2 can be improved by considering results valid only almost surely, for almost every q, μ_q almost-everywhere.

Recall that if $t \in [0,1)$ and $j \geq 1$, $w^{(j)}(t)$ is the unique element w of $\mathcal{A}^j$ such that $t \in [i(w)b^{-j}, (i(w)+1)b^{-j})$.

Theorem 3 (**Improved growth speed**)**.** *Under the assumptions of Theorem 1, fix $\kappa > 0$ and assume that (3.4) holds. For $j \geq 2$, let $\mathcal{S}_j = \left[j\log(j)^{-\kappa}\right]$.*

(1) For every $q \in J$, with probability one, the property $\mathcal{P}(q)$ holds, where $\mathcal{P}(q)$ is: For μ_q-almost every $t \in [0,1)$, if j is large enough, for $w = w^{(j)}(t)$ one has

$$\max\left(GS\left(\mu_q^w, \mu^w, \tau'_\mu(q), N, \widetilde{\varepsilon}\right), GS\left(\mu_q^w, \mu_q^w, \tau'_\mu(q)q - \tau_\mu(q), N, \widetilde{\varepsilon}\right)\right) \leq \mathcal{S}_j.$$

(2) With probability one, for almost every $q \in J$, $\mathcal{P}(q)$ holds.

For $w \in \mathcal{A}^*$, $n \geq 1$ and $q \in J$, let

$$\mathcal{N}_n(\mu^w, \alpha, \varepsilon_n) = \#\left\{b\text{-adic box } I \text{ of scale } n : |I|^{\alpha+\varepsilon_n} \leq \mu^w(I) \leq |I|^{\alpha-\varepsilon_n}\right\}. \tag{3.5}$$

Remember that $\tau_\mu = \widetilde{\tau}_\mu$ on J.

Theorem 4 (**Renewal speed of large deviations spectrum).** *Under the assumptions of Theorem 1, let us also assume that (3.4) holds,* $J=\mathbb{R}$ *(in particular* $\widetilde{\tau}_\mu'(1)>0$*) and there exists* $A>1$ *such that with probability one* $A^{-1}\le W_i\le A$ *for all* $0\le i\le b-1$*. Let* $\mathcal{S}_j$ *be defined as in Theorem 2. Let* K *be a compact subinterval of* $\mathbb{R}$ *and* $\beta=1+\max_{q\in K}|q|$*.*

Then, with probability one, for j *large enough, for all* $q\in K$ *and* $w\in\mathcal{A}^j$*, and for all* $n\ge\mathcal{S}_j$*, one has*

$$Y_q^w b^{n(\widetilde{\tau}_\mu'(q)q-\widetilde{\tau}_\mu(q)-\beta\varepsilon_n)}\le\mathcal{N}_n\big(\mu^w,\widetilde{\tau}_\mu'(q),\varepsilon_n\big)\le Y_q^w b^{n(\widetilde{\tau}_\mu'(q)q-\widetilde{\tau}_\mu(q)+\beta\varepsilon_n)}.$$

For $w\in\mathcal{A}^*$, $n\ge 1$ and $q\in\mathbb{R}$, let us introduce the functions ($\tau_n^\emptyset$ associated with $\mu^\emptyset=\mu$ is simply denoted by τ_n)

$$\tau_n^w(q)=-\frac{1}{n}\log_b\sum_{v\in\mathcal{A}^n}\mu^w(I_v)^q.$$

The speed of convergence obtained in Theorem 5 provides precisions on the estimator of the function τ_μ discussed in [19, 47].

Theorem 5 (**Convergence speed of** τ_n^w **toward** $\widetilde{\tau}_\mu$**).** *Under the assumptions of Theorem 4, let* K *be a compact subinterval of* $\mathbb{R}$*. There exists* $\theta>0$ *and* $\delta\in(0,1)$ *such that, with probability one,*
1. for j *large enough,* $\big|\widetilde{\tau}_\mu(q)-\tau_j(q)\big|\le\big|\log_b Y_q\big|\,j^{-1}+\theta\log(j)j^{-1}$*;*
2. for j *large enough, for every* $n\ge j^\delta$*, for every* $w\in\mathcal{A}^j$*,* $\big|\widetilde{\tau}_\mu(q)-\tau_n^w(q)\big|\le\big|\log_b Y_q^w\big|\,n^{-1}+\theta\log(n)n^{-1}$*, with* $|\log Y_q^w\big|\le\theta\log(j)$*.*

3.4. **Proof of Theorem 1.** Fix K a compact subinterval of J and $\widetilde{\eta}=(\eta_n)_{n\ge1}$ a bounded positive sequence to be specified later. For $\omega\in\Omega^*$ and $q\in K$, let us introduce (recall (2.3))

$$F_n(q)=S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q,\mu,\tau_\mu'(q)\big)\text{ and }G_n(q)=S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q,\mu_q,\tau_\mu'(q)q-\tau_\mu(q)\big).\tag{3.6}$$

We begin by giving estimates for $\mathbb{E}\big(H_n(q)\big)$ and $\mathbb{E}\big(H_n'(q)\big)$ for $H\in\{F,G\}$.

Lemma 1. *Under the assumptions of Theorem 1, if* $\|\widetilde{\eta}\|_\infty$ *is small enough, there exists* $C_K>0$ *such that if* $H\in\{F,G\}$*,*

$$\forall\,q\in K,\ \max\big(\mathbb{E}\big(H_n(q)\big),\mathbb{E}\big(H_n'(q)\big)\big)\le C_K n b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))},\tag{3.7}$$

where $O(\eta_n^2)$ *is uniform over* $q\in K$*.*

The proof of this lemma is postponed to the next subsection.

Let q_0 be the left end point of K. Since $\sup_{q\in K}H_n(q)\le H_n(q_0)+\int_K|H_n'(q)|\,dq$, one has

$$\mathbb{E}\big(\sup_{q\in K}H_n(q)\big)\le C_K(1+|K|)nb^{-n(\varepsilon_n\eta_n+O(\eta_n^2))}.\tag{3.8}$$

Choosing $\eta_n=\varepsilon_n/A$ with A large enough yields $\varepsilon_n\eta_n+O(\eta_n^2)\ge\varepsilon_n^2/2A$. Using the assumptions of Theorem 1, we get the almost sure convergence of $\sum_{n\ge1}\sup_{q\in K}H_n(q)$ for $H\in\{F,G\}$. We conclude with Proposition 1.

3.5. **Proof of Lemma 1.** • **The case** $H = F$**:** For $v, w \in \mathcal{A}^n$, $q \in J$ and $\gamma \in \{-1, 1\}$, one can write

$$\mu_q(I_v)\mu(I_w)^{\gamma\eta_n} = Y_q(v)Y_1(w)^{\gamma\eta_n} b^{n\tau_\mu(q)} \prod_{k=0}^{n-1} W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}.$$

Moreover, it follows from estimates of [5] that for $\|\widetilde{\eta}\|_\infty$ small enough, the quantities

$$C'_K(\widetilde{\eta}) = \sup_{\substack{q\in K,\ \gamma\in\{-1,1\} \\ n\geq 1,\ v,w\in\mathcal{A}^n}} \Big(\mathbb{E}\Big(\Big|\frac{d}{dq}Y_q(v)Y_1(w)^{\gamma\eta_n}\Big|\Big) + \mathbb{E}\big(Y_q(v)Y_1(w)^{\gamma\eta_n}\big)\Big)$$

$$\text{and} \quad C''_K(\widetilde{\eta}) = \sup_{\substack{q\in K,\ \gamma\in\{-1,1\} \\ n\geq 1,\ v,w\in\mathcal{A}^n,\ 0\leq k\leq n-1}} \frac{\mathbb{E}\big(\big|\frac{d}{dq}W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big|\big)}{\mathbb{E}\left(W_{v_{k+1}}(v)^q W_{w_{k+1}}(w)^{\gamma\eta_n}\right)}$$

are finite. Hence, due to the definition of $F_n(q)$ and the fact that $\widetilde{\tau}$ is continuously differentiable on J, there exists a constant $C_K(\widetilde{\eta})$ such that for every $q \in K$, $\max\big(\mathbb{E}\big(F_n(q)\big), \mathbb{E}\big(F'_n(q)\big)\big) \leq C_K(\widetilde{\eta})T_n(q)$, where

$$T_n(q) = n\, b_n(q) \sum_{\substack{\gamma\in\{-1,1\}, \\ v,w\in\mathcal{A}^n,\ \delta(v,w)\leq N}} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big),$$

where $b_n(q) = b^{n(\tau_\mu(q)+\gamma\eta_n(\tau'_\mu(q)-\gamma\varepsilon_n))}$. Let us make the following important remark.

Remark 4. *If v and w are words of length n, and if $\bar{v}$ and $\bar{w}$ stand for their prefixes of length $n-1$, then $\delta(\bar{v}, \bar{w}) > k$ implies $\delta(v, w) > bk$. It implies that, given two integers $n \geq m > 0$ and two words v and w in $\mathcal{A}^n$ such that $b^{m-1} < \delta(v, w) \leq b^m$, there are two prefixes $\bar{v}$ and $\bar{w}$ of respectively v and w of common length $n-m$ such that $\delta(\bar{v}, \bar{w}) \leq 1$. Moreover, for these words $\bar{v}$ and $\bar{w}$, there are at most b^{2m} pairs (v, w) of words in $\mathcal{A}^n$ such that $\bar{v}$ and $\bar{w}$ are respectively the prefixes of v and w.*

Due to Remark 4 and the form of $T_n(q)$, there exists a constant C'_K such that for all $q \in K$ and $n \geq 1$

$$T_n(q) \leq C'_K n b_n(q) \sum_{\substack{\gamma\in\{-1,1\}, \\ v,w\in\mathcal{A}^n,\ \delta(v,w)\leq 1}} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big).$$

The situation is thus reducible to the case $N = 1$. Now $T_n(q) \leq n\big(T_{n,1}(q)+T_{n,2}(q)\big)$,

$$\text{where} \begin{cases} T_{n,1}(q) = b_n(q) \displaystyle\sum_{\gamma\in\{-1,1\},\, v\in\mathcal{A}^n} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^{q+\gamma\eta_n}\big), \\ T_{n,2}(q) = b_n(q) \displaystyle\sum_{\substack{\gamma\in\{-1,1\},\, v,w\in\mathcal{A}^n, \\ \delta(v,w)=1}} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big). \end{cases}$$

One immediately gets

$$T_{n,1}(q) = \sum_{\gamma\in\{-1,1\}} b^{n\left(\tau_\mu(q)+\gamma\eta_n(\tau'_\mu(q)-\gamma\varepsilon_n)-\tau_\mu(q+\gamma\eta_n)\right)} = 2b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))},$$

where $O(\eta_n^2)$ is uniform over $q \in K$ if $\|\widetilde{\eta}\|_\infty$ is small enough (the twice continuous differentiability of τ_μ has been used).

Let g_k and d_k respectively stand for the word consisting of k consecutive zeros and the word consisting of k consecutive $b-1$. The estimation of $T_{n,2}(q)$ is achieved by using the following identity:

$$\bigcup_{m=0}^{n-1} \bigcup_{u\in\mathcal{A}^m} \bigcup_{r\in\{0,\dots,b-2\}} \big\{(u.r.d_{n-1-m}, u.(r+1).g_{n-1-m})\big\}, \tag{3.9}$$

One has $T_{n,2}(q) = \mathcal{T}_n(q,-1) + \mathcal{T}_n(q,1)$ where for $\gamma \in \{-1,1\}$

$$\begin{aligned}
\mathcal{T}_n(q,\gamma) &= b_n(q) \sum_{\substack{v,w\in\mathcal{A}^n\\ \delta(v,w)=1}} \prod_{k=0}^{n-1} \mathbb{E}\big(W_{v_{k+1}}(v|k)^q W_{w_{k+1}}(w|k)^{\gamma\eta_n}\big)\\
&= b_n(q) \sum_{m=0}^{n-1} \sum_{u\in\mathcal{A}^m} \sum_{r=0}^{b-2} \Theta_{n-1-m}(r) \prod_{k=0}^{m-1} \mathbb{E}\big(W_{u_{k+1}}(u|k)^{q+\gamma\eta_n}\big)\\
&= b_n(q) \sum_{m=0}^{n-1} b^{-m\tau_\mu(q+\gamma\eta_n)} \sum_{r=0}^{b-2} \Theta_{n-1-m}(r),
\end{aligned}$$

and where $\Theta_m(r)$ is defined by

$$\begin{aligned}
\Theta_m(r) &= \mathbb{E}\big(W_r^q W_{r+1}^{\gamma\eta_n}\big)\big(\mathbb{E}(W_{b-1}^q)\big)^m \big(\mathbb{E}(W_0^{\gamma\eta_n})\big)^m\\
&\qquad + (\mathbb{E}\big(W_r^{\gamma\eta_n} W_{r+1}^q\big)\big(\mathbb{E}(W_0^q)\big)^m \big(\mathbb{E}(W_{b-1}^{\gamma\eta_n})\big)^m.
\end{aligned}$$

All the components of W are positive almost surely. Thus, by definition (3.1) of $\widetilde{\tau}_\mu(q) = \tau_\mu(q)$, there is a constant $c_K \in (0,1)$ such that for all $q \in K$ one has $\max\big(\mathbb{E}(W_0^q), \mathbb{E}(W_{b-1}^q)\big) \le c_K b^{-\tau_\mu(q)}$. Moreover, if $\|\widetilde{\eta}\|_\infty$ is small enough, $\max\big(\mathbb{E}(W_0^{\gamma\eta_n}), \mathbb{E}(W_{b-1}^{\gamma\eta_n})\big) \le (c_K^{-1}+1)/2$ (this maximum goes to 1 when $\|\eta\|_\infty \to 0$). This yields (since $c_K(c_K^{-1}+1) = c_K+1$)

$$\begin{aligned}
\Theta_m(r) &\le \big(\mathbb{E}\big(W_r^q W_{r+1}^{\gamma\eta_n}\big) + (\mathbb{E}\big(W_r^{\gamma\eta_n} W_{r+1}^q\big)\big)\big((c_K+1)/2\big)^m b^{-m\tau_\mu(q)}\\
&\le C_K\,\big((c_K+1)/2\big)^m b^{-m\tau_\mu(q)}.
\end{aligned}$$

Consequently we get

$$\begin{aligned}
\mathcal{T}_n(q,\gamma) &\le C_K b^{n(\tau_\mu(q)+\gamma\eta_n(\tau'_\mu(q)-\gamma\varepsilon_n))} b^{-(n-1)\tau_\mu(q+\gamma\eta_n)}\\
&\qquad\qquad \times \sum_{m=0}^{n-1} \big((c_K+1)/2\big)^m b^{m(\tau_\mu(q+\gamma\eta_n)-\tau_\mu(q))}\\
&\le C_K b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))} \sum_{m=0}^{n-1} \big((c_K+1)/2\big)^m b^{m(\tau_\mu(q+\gamma\eta_n)-\tau_\mu(q))}.
\end{aligned}$$

The function τ_μ is continuously differentiable. Hence the sum $\sum_{m=0}^{n-1}((c_K+1)/2)^m$ $b^{m(\tau_\mu(q+\gamma\eta_n)-\tau_\mu(q))}$ is uniformly bounded over $n \ge 0$ and $q \in K$ if $\|\widetilde{\eta}\|_\infty$ is small enough. Finally, if $\|\widetilde{\eta}\|_\infty$ is small enough we also have $\mathcal{T}_n(q,\gamma) \le C_K b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))}$. Going back to $T_n(q)$, we get $T_n(q) \le C_K n b^{-n(\varepsilon_n\eta_n+O(\eta_n^2))}$, $\forall q \in K$. This shows (3.7).

• **The case $H = G$:** The proof follows similar lines as for $F_n(q)$. The only new point it requires is the boundedness of $\sup_{q\in K}\mathbb{E}(Y_q^{-h})$ for some $h > 0$. In fact, we shall need the following stronger property in the proof of Theorem 2.

Lemma 2. *1. For every compact subinterval K of J, there exists $h > 0$ such that $\mathbb{E}\big(\sup_{q\in K} Y_q^{-h}\big) < \infty$.*
2. Assume that there exists $A > 1$ such that with probability one, $A^{-1} \leq W_i$ (resp. $W_i \leq A$) for all $0 \leq i \leq b-1$. Then, for every compact subinterval K of $J \cap \mathbb{R}_+$ (resp. $J \cap \mathbb{R}_-$) there exist two constants depending on K, $C_K > 0$ and $\gamma_K \in (0,1)$ such that for all $x > 0$ small enough,

$$\mathbb{P}(\inf_{q\in K} Y_q \leq x) \leq \exp\big(-C_K x^{-\gamma_K/(1-\gamma_K)}\big).$$

Proof. 1. Fix K a compact subinterval of $J \cap \mathbb{R}_+$ (resp. $J \cap \mathbb{R}_-$). For $w \in \mathcal{A}^*$ one can define $Z_K(w) = \inf_{q\in K} Y_q(w)$ ($Z_K(\emptyset)$ is denoted Z_K). We learn from [16, 5] that this infimum is positive since $q \mapsto Y_q(w)$ is almost surely positive and continuous.

Let us also define $W_K(w) = \inf_{q\in K, 0\leq i\leq b-1} W_{q,i}(w)$ ($W_K(\emptyset)$ is denoted W_K). Since we assumed that J contains a neighborhood of 0, there exists $h > 0$ such that the moment of negative order $-h$ of this random variable $W_K(w)$ is finite.

Moreover, with probability one, $\forall\, q \in J$ one has $Y_q = \sum_{i=0}^{b-1} W_{q,i}(\emptyset) Y_q(i)$, hence

$$Z_K \geq W_K \sum_{i=0}^{b-1} Z_K(i). \tag{3.10}$$

By construction, the random variables $Z_K(i)$, $0 \leq i \leq b-1$, are i.i.d. with Z_K, and they are independent of the positive random variable W_K. Consequently, the Laplace transform of Z_K, denoted $L : t \geq 0 \mapsto \mathbb{E}(\exp(-tZ_K))$, satisfies the inequality

$$L(t) \leq \mathbb{E}\Big(\prod_{i=0}^{b-1} L(W_K t)\Big) \quad (t \geq 0). \tag{3.11}$$

Then, since $\mathbb{E}(W_K^{-h}) < \infty$, using the approach of [43] to study the behavior at ∞ of Laplace transforms satisfying an inequality like (3.11) (see also [4] and [40]) one obtains $\mathbb{E}(Z_K^{-h}) < \infty$.
2. It is a simple consequence of the proof of Theorem 2.5 in [40] (see also the proof of Corollary 2.5 in [27]) and of the fact that in this case, the random variable W_K in (3.11) is lower bounded by a positive constant.

□

3.6. **Proof of Theorem 2.** Fix K a compact subinterval of J. The computations performed to prove Theorem 1 yield (3.8). Thus there are two constants $C > 0$ and $\beta > 0$ as well as a sequence $\widetilde{\eta} = (\eta_n)_{n\geq 1} \in \mathbb{R}_+^{\mathbb{N}^*}$ such that for every $j, n \geq 1$, $q \in K$ and $w \in \mathcal{A}^j$,

$$\mathbb{E}\Big(\sup_{q\in K} S^{N,\varepsilon_n,\eta_n}\big(\mu_q^w, \mu^w, \tau_\mu'(q)\big)\Big) \leq Cnb^{-\beta n\varepsilon_n^2}. \tag{3.12}$$

In order to apply Proposition 2, let us define

- $\Lambda = K$, $\{(m_\lambda^w, \mu_\lambda^w)\}_{w\in\mathcal{A}^*,\lambda\in\Lambda} = \{\mu_q^w, \mu^w\}_{w\in\mathcal{A}^*, q\in K}$ and $\{\alpha_\lambda\}_{\lambda\in\Lambda} = \{\tau_\mu'(q)\}_{q\in K}$,
- For $w \in \mathcal{A}^*$ and $n \geq 1$,

$$U^w = \inf_{q\in K} \|\mu_q^w\| \quad \text{and} \quad V_n^w = \sup_{q\in K} S^{N,\varepsilon_n,\eta_n}\big(\mu_q^w, \mu^w, \tau_\mu'(q)\big), \tag{3.13}$$

- For every $j \geq 1$, $\psi_j(\widetilde{\eta}) = 1$ and $\rho_j = \log(j)^{1+\eta}$.

• Fix $\eta > 0$ and $\eta' > 2\eta$. For every $j \geq 1$, we set $\mathcal{S}_j = \big[\exp\big(\big(j\log(j)^{\eta}\big)^{\frac{1}{1+2\eta}}\big)\big]$ if $\log(j)^{-\eta} \geq \varepsilon_j \geq j^{-1/2}\log(j)^{1/2+\eta}$ and $\mathcal{S}_j = \big[j\log(j)^{\eta'}\big]$ if $\varepsilon_j \geq \log(j)^{-\eta}$.

Now, on the one hand, Lemma 2.2 implies that

$$u_j := b^j \mathbb{P}(U^w \leq b^{-\rho_j}) \leq b^j \exp\big(-C_K b^{\frac{\gamma_K}{1-\gamma_K}(\log j)^{(1+\eta)}}\big). \tag{3.14}$$

Moreover, $\sum_{j\geq 1} u_j < \infty$. On the other hand, for some $\alpha > 0$, for any $w \in \mathcal{A}^*$ one has

$$v_j := \sum_{n \geq \mathcal{S}_j} \mathbb{E}(V_n^w) \leq \sum_{n\geq \mathcal{S}_j} Cnb^{-\beta n \varepsilon_n^2} = O\big(b^{-j\log(j)^{\alpha}}\big).$$

The sequence ρ_j has been chosen so that $\sum_{j\geq 1} b^j b^{\rho_j} v_j < \infty$. Consequently, Proposition 2 yields the desired upper bound for the growth speed $GS\big(\mu_q^w, \mu^w, \tau'_\mu(q), N, \widetilde{\varepsilon}\big)$.

Changing the measures $\{(m_q^w, \mu_q^w)\}_{w\in\mathcal{A}^*, q\in K}$ into $\{\mu_q^w, \mu_q^w\}_{w\in\mathcal{A}^*, q\in K}$ and the exponents $\{\tau'_\mu(q)\}_{q\in K}$ into $\{\tau'_\mu(q)q - \tau_\mu(q)\}_{q\in K}$, the same arguments yield the conclusion for $GS\big(\mu_q^w, \mu_q^w, \tau'_\mu(q)q - \tau_\mu(q)), N, \widetilde{\varepsilon}\big)$.

3.7. **Proof of Theorem 3.** We only prove the results for the control of $GS\big(\mu_q^w, \mu^w, \tau'_\mu(q), N, \widetilde{\varepsilon}\big)$ by $\mathcal{S}_j$, since $GS\big(\mu_q^w, \mu_q^w, \tau'_\mu(q)q - \tau_\mu(q)), N, \widetilde{\varepsilon}\big)$ is controlled by using the same approach.

(1) Recall that $(\Omega, \mathcal{B}, \mathbb{P})$ denotes the probability space on which the random variables in this Section 3 are defined. Let us consider on $\mathcal{B}\otimes\mathcal{B}([0,1])$ the so-called Peyrière probability $\mathcal{Q}_q$ [30]

$$\mathcal{Q}_q(A) = \mathbb{E}\Big(\int_{[0,1]} \mathbf{1}_A(\omega, t)\, \mu_q(t)\Big) \quad (A \in \mathcal{B}\otimes\mathcal{B}([0,1])).$$

It is important to notice that by construction $\mathcal{Q}_q$-almost surely means $\mathbb{P}$-almost surely, $\mu_q(\omega)$-almost everywhere.

Fix $\widetilde{\eta}$ as in the proof of Theorem 2. Also, for $j \geq 1$ let $\rho_j = \log(j)^{1+\eta}$, and let $\mathcal{S}_j = \big[j\log(j)^{-\kappa}\big]$. Now, for $j \geq 0$ and $n \geq 1$ define on $\Omega \times [0,1)$ the random variables

$$\begin{aligned} \mathbf{U}^{(j)}(\omega, t) &= \|\mu_q^{w^{(j)}(t)}(\omega)\|. \\ \text{and } \mathbf{V}_n^{(j)}(\omega, t) &= S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^{w^{(j)}(t)}(\omega), \mu^{w^{(j)}(t)}(\omega), \tau'_\mu(q)\big). \end{aligned}$$

We can use the proof of Proposition 2 to claim that it is enough to prove that

$$\exists\, h \in (0,1],\ \sum_{j\geq 0} \mathcal{Q}_q\big(\mathbf{U}^{(j)} \leq b^{-\rho_j}\big) < \infty \text{ and } \sum_{j\geq 0} b^{\rho_j h} \mathbb{E}_{\mathcal{Q}_q}\big(\big(\sum_{n\geq\mathcal{S}_j} \mathbf{V}_n^{(j)}\big)^h\big) < \infty.$$

The main difference with the proofs of Proposition 2 and Theorem 2 is that here we do not seek for a result valid uniformly over the w of the same generation j, but only for a result valid for $w^{(j)}(t)$, for μ-almost every t. As a consequence we must control only one pair of random variables $(\mathbf{U}^{(j)}, \mathbf{V}^{(j)})$ on each generation instead of b^j. This allows to slow down $\mathcal{S}_j$.

Fix $h \in (0,1)$. Since $x \mapsto x^h$ is sub-additive on $\mathbb{R}_+$, one has

$$\mathbb{E}_{\mathcal{Q}_q}\Big(\Big(\sum_{n\geq\mathcal{S}_j} \mathbf{V}_n^{(j)}\Big)^h\Big) \leq \sum_{n\geq\mathcal{S}_j} \mathbb{E}_{\mathcal{Q}_q}\Big(\big(\mathbf{V}_n^{(j)}\big)^h\Big).$$

For $\omega\in\Omega^*$, $j\geq 1$ and $n\geq 1$, by definition of the measures μ_q and μ_q^w, and since $\big(\mu_q^{w^{(j)}(t)}(\omega),\mu^{w^{(j)}(t)}(\omega)\big)$ does not depend on $t\in I_w$, one has

$$\begin{aligned}\int_{[0,1]}\big(\mathbf{V}_n^{(j)}(\omega,t)\big)^h\mu_q(\omega)(dt)&=\sum_{w\in\mathcal{A}^j}\int_{I_w}\big(\mathbf{V}_n^{(j)}(\omega,t)\big)^h\mu_q(\omega)(dt)\\ &=\sum_{w\in\mathcal{A}^j}\prod_{k=0}^{j-1}W_{q,w_{k+1}}(w|k)\int_{I_w}\big(\mathbf{V}_n^{(j)}(\omega,t)\big)^h\mu_q^w(\omega)\circ f_{I_w}^{-1}(dt)\\ &=\sum_{w\in\mathcal{A}^j}\Big(\prod_{k=0}^{j-1}W_{q,w_{k+1}}(w|k)\Big)\big(V_n^w\big)^h\,\|\mu_q^w\|,\end{aligned}$$

where $V_n^w=S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^w(\omega),\mu^w(\omega),\tau_\mu'(q)\big)$ is defined as in the proof of Theorem 2. The above sum is a random variable on $(\Omega,\mathcal{B},\mathbb{P})$. In addition, in each of its terms, the product is independent of $\big(V_n^w\big)^h\,\|\mu_q^w\|$. Moreover, the probability distribution of $\big(V_n^w\big)^h\,\|\mu_q^w\|$ does not depend on w. Consequently, using the martingale property of $\|\mu_{q,n}\|$, one gets

$$\mathbb{E}_{\mathcal{Q}_q}\left(\big(\mathbf{V}_n^{(j)}\big)^h\right)=\mathbb{E}\left(\big(V_n^w\big)^h\,\|\mu_q^w\|\right),$$

where $w\in\mathcal{A}^j$. Let $p=1/(1-h)$. The Hölder inequality yields

$$\mathbb{E}\left(\big(V_n^w\big)^h\,\|\mu_q^w\|\right)\leq\big(\mathbb{E}\big(V_n^w\big)\big)^h\,\mathbb{E}\big(\|\mu_q\|^p\big)^{1/p}.$$

Finally, p is fixed close enough to 1 so that $\mathbb{E}\big(\|\mu_q\|^p\big)<\infty$ (see the proof of Lemma 3 for the existence of such a p). Then (3.12) yields $\sum_{j\geq 1}b^{\rho_j h}\sum_{n\geq\mathcal{S}_j}\big(\mathbb{E}\big(V_n^w\big)\big)^h<\infty$, hence the conclusion.

Similar computations as above show that for every $j\geq 1$,

$$\mathcal{Q}_q\big(\mathbf{U}^{(j)}\leq b^{-\rho_j}\big)=\mathbb{E}\big(\mathbf{1}_{\{Y_q\leq b^{-\rho_j}\}}Y_q\big)\leq b^{-\rho_j}\mathbb{P}(Y_q\leq b^{-\rho_j}).$$

It follows from Lemma 2.1 that for some $h>0$ one has $\mathbb{P}(Y_q\leq x)=O(x^h)$ as $x\to 0$. This implies $\sum_{j\geq 1}\mathcal{Q}_q\big(\mathbf{U}^{(j)}\leq b^{-\rho_j}\big)<\infty$.

(2) The proof is similar to the one of (1). It is enough to prove the result for a compact subinterval K of J instead of J. Fix such an interval K. The idea is now to consider on $\big(K\times\Omega\times[0,1],\mathcal{B}(K)\otimes\mathcal{B}\otimes\mathcal{B}([0,1])\big)$ the probability distribution $\mathcal{Q}_K$

$$\mathcal{Q}_K(A)=\int_K\big(\mathbb{E}_{\mathcal{Q}_q}\mathbf{1}_A(q,\omega,t)\big)\,\frac{dq}{|K|}\quad\big(A\in\mathcal{B}(K)\otimes\mathcal{B}\otimes\mathcal{B}([0,1])\big).$$

Then $\mathbf{U}^{(j)}(q,\omega,t)$ and $\mathbf{V}_n^{(j)}(q,\omega,t)$ are redefined as $\mathbf{U}^{(j)}(q,\omega,t)=\|\mu_q^{w^{(j)}(t)}(\omega)\|$ and $\mathbf{V}_n^{(j)}(q,\omega,t)=S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^{w^{(j)}(t)}(\omega),\mu^{w^{(j)}(t)}(\omega),\tau_\mu'(q)\big)$.

Since there exists $p>1$ such that $M=\sup_{q\in K}\mathbb{E}\big(\|\mu_q\|^p\big)^{1/p}<\infty$ (see the proof of Lemma 3), the computations performed above yield

$$\sum_{j\geq 0}b^{\rho_j h}\sum_{n\geq\mathcal{S}_j}\mathbb{E}_{\mathcal{Q}_K}\left(\big(\mathbf{V}_n^{(j)}\big)^h\right)\leq|K|\,M\sum_{j\geq 1}b^{\rho_j h}\sum_{n\geq\mathcal{S}_j}\Big(\mathbb{E}\big(V_n^w\big)\Big)^h<\infty.$$

Finally, $\sum_{j\geq 0}\mathcal{Q}_K\big(\mathbf{U}^{(j)}\leq b^{-\rho_j}\big)\leq|K|\sum_{j\geq 1}b^{-\rho_j}\mathbb{P}\big(\inf_{q\in K}Y_q\leq b^{-\rho_j}\big)$, which is finite by item 1. of Lemma 2.

3.8. **Proof of Theorem 4.** We assume without loss of generality that K contains the point 1. Define $q_K = \max\{|q| : q \in K\}$. Recall that for $j \geq 0$ and $n \geq 1$, if $(w,v) \in \mathcal{A}^j \times \mathcal{A}^n$ and $q \in K$ then

$$\mu^w(I_v)^q = \mu_q^w(I_v) b^{-n\widetilde{\tau}_\mu(q)} \frac{Y(wv)^q}{Y_q(wv)}.$$

As a consequence,

$$Y_q(w) b^{-n\widetilde{\tau}_\mu(q)} \inf_{q\in K, v\in\mathcal{A}^n} \frac{Y(wv)^q}{Y_q(wv)} \leq b^{-n\tau_n^w(q)} \tag{3.15}$$

$$\text{and} \quad b^{-n\tau_n^w(q)} \leq Y_q(w) b^{-n\widetilde{\tau}_\mu(q)} \sup_{q\in K, v\in\mathcal{A}^n} \frac{Y(wv)^q}{Y_q(wv)}. \tag{3.16}$$

Let us fix $\delta \in (0,1)$ and $\theta > 0$ such that the conclusions of Propositions 3 and 4 below hold. Then, with probability one, for j large enough, for every $w \in \mathcal{A}^j$, $q \in K$ and $n \geq j^\delta$, one has $b^{-n\tau_n^w(q)} \leq Y_q(w) b^{-n\widetilde{\tau}_\mu(q)} n^{(q_K+1)\theta}$. This yields

$$\begin{aligned} \mathcal{N}_n(\mu^w, \widetilde{\tau}'_\mu(q), \varepsilon_n) &\leq b^{n\left(\widetilde{\tau}'_\mu(q)q - \tau_n^w(q) + sgn(q)q\varepsilon_n\right)} \\ &\leq Y_q(w) b^{n\left(\widetilde{\tau}'_\mu(q)q - \widetilde{\tau}_\mu(q) + sgn(q)q\varepsilon_n\right)} n^{(q_K+1)\theta}. \end{aligned}$$

On the other hand, due to Theorem 2 and Proposition 3, there exists $\theta > 0$ such that, with probability one, for j large enough, for all $w \in \mathcal{A}^*$ and $q \in K$

$$\mu_q^w\Big(E^{\mu^w}_{\widetilde{\tau}'_\mu(q),\mathcal{S}_j}(0,\widetilde{\varepsilon}) \bigcap E^{\mu_q^w}_{\widetilde{\tau}'_\mu(q)q-\widetilde{\tau}_\mu(q),\mathcal{S}_j}(0,\widetilde{\varepsilon})\Big) \geq \|\mu_q^w\|/2,$$

which equals $Y_q(w)/2$. Thus $b^{n(\widetilde{\tau}'_\mu(q)q - \widetilde{\tau}_\mu(q) - \varepsilon_n)} Y_q(w)/2 \leq \mathcal{N}_n(\mu^w, \widetilde{\tau}'_\mu(q), \varepsilon_n)$ for every $n \geq \mathcal{S}_j$. Moreover, for j large enough one has $j^\delta \leq \mathcal{S}_j$, and then for n large enough $\sup_{q\in K} sgn(q)q\varepsilon_n + (q_K+1)\theta \log_b(n)/n$ is controlled by $(1+q_K)\varepsilon_n$. The conclusion follows.

3.9. **Proof of Theorem 5.** Let us begin with three technical lemmas.

Lemma 3. *Let us assume that $J = \mathbb{R}$. For every compact subinterval K of $\mathbb{R}$, there exist $C_K, c_K > 0$ such that*

$$\text{for every } x \geq 1,\ \sup_{q\in K} \mathbb{P}(Y_q \geq x) \leq C_K \exp\big(-c_K x\big).$$

Proof. Let us begin with the following properties involved in the proofs of several statements of Section 3: it is known (see [30, 21]) that if $q \in J$ and $h > 1$ one has $\mathbb{E}(Y_q^h) < \infty$ if and only if $\mathbb{E}\Big(\sum_{i=0}^{b-1} W_{q,i}^h\Big) < 1$, that is $\widetilde{\tau}_\mu(qh) - h\widetilde{\tau}_\mu(q) > 0$. Moreover, one deduces from the proofs of Theorem III.B. and Theorem VI.A.b. of [4]: for every compact subinterval K of J, there exists $h > 1$ such that $\sup_{q\in K} \mathbb{E}(Y_q^h) < \infty$.

The property $J = \mathbb{R}$ is equivalent to the fact that the mapping $q \mapsto \widetilde{\tau}_\mu(q)/q$ is increasing on $\mathbb{R}_-^*$ and $\mathbb{R}_+^*$. As a consequence, one has $\widetilde{\tau}_\mu(qh) - h\widetilde{\tau}_\mu(q) > 0$ for all $q \in \mathbb{R}$ and $h > 1$, that is $\mathbb{E}(Y_q^h) < \infty$. Also, one has $\|W_{q,i}\|_\infty \leq 1$ for all $q \in \mathbb{R}$ and $0 \leq i \leq b-1$.

Let us now fix K a compact subset of $\mathbb{R}$. Let us then introduce $t_k(q) = \mathbb{E}(Y_q^k)/k!$ for $q \in K$ and $k \geq 1$, and $t_0(q) = 1$. The proof of Theorem 4.1 (a) in [35] yields that for every $k \geq 2$, for every $q \in K$,

$$t_k(q) \leq c_K \sum_{(k_0,\dots,k_{b-1}):\, 0\leq k_i\leq k-1 \text{ and } k_0+\cdots+k_{b-1}=k} \ \prod_{i=0}^{b-1} t_{k_i}(q),$$

where $c_K = \sup_{q\in K}\sup_{k\geq 2}\big(1-b^{-\widetilde{\tau}_\mu(kq)+k\widetilde{\tau}_\mu(q)}\big)^{-1}$. One sees that $c_K=\sup_{q\in K}\big(1-b^{-\widetilde{\tau}_\mu(2q)+2\widetilde{\tau}_\mu(q)}\big)^{-1}<\infty$. Hence, if $\widetilde{t}_k=\sup_{q\in K} t_k(q)$, one has

$$\forall\, k\geq 2,\quad \widetilde{t}_k \leq\ c_K \sum_{(k_0,\dots,k_{b-1}):\, 0\leq k_i\leq k-1 \text{ and } k_0+\cdots+k_{b-1}=k} \prod_{i=0}^{b-1} \widetilde{t}_{k_i}.$$

Since $\widetilde{t}_0=\widetilde{t}_1=1$, Lemma 2.6 of [25] yields $\limsup_{k\to\infty}\widetilde{t}_k^{\frac{1}{k}}<\infty$. This implies the existence of a constant $C>0$ such that

$$\forall\, k\geq 1,\quad \sup_{q\in K}\mathbb{E}(Y_q^k)\leq C^k k!.$$

Now, fix $c_K\in(0,C^{-1})$. For $x>0$ one has

$$\begin{aligned}\sup_{q\in K}\mathbb{P}(Y_q\geq x)\leq e^{-c_K x}\sup_{q\in K}\mathbb{E}\big(e^{c_K Y_q}\big) \ &\leq\ e^{-c_K x}\sum_{k=0}^{\infty} c_K^k \sup_{q\in K}\mathbb{E}(Y_q^k)/k!\\ &\leq\ (1-c_K C)^{-1}e^{-c_K x}.\end{aligned}$$

□

Remark 5. *We are not able to control $\mathbb{P}(\sup_{q\in K} Y_q\geq x)$ at ∞. This is the reason why next Lemmas 4 and 5 are needed to prove Proposition 4.*

For $n\geq 1$ let Q_n be the set of dyadic numbers of generation n.

Lemma 4. *Let K be a compact subinterval of J. Let $\eta>0$. There exists $\alpha\in(0,1)$ and $\delta\in(0,1)$ such that, with probability one,*
1. *for j large enough, $\forall\, w\in\mathcal{A}^j$, $\forall\, n\geq[j^{1+\eta}]$, $\forall\, q,q'\in Q_n$ such that $|q-q'|=2^{-n}$, one has $\big|Y_q^w-Y_{q'}^w\big|\leq|q'-q|^\alpha$.*
2. *for j large enough, $\forall n\geq j^\delta$, $\forall\, w\in\mathcal{A}^j$, $\forall v\in\mathcal{A}^n$, $\forall\, m\geq[n^{1+\eta}]$, for all $q,q'\in Q_m$ such that $|q'-q|=2^{-m}$, one has $\big|Y_q^{wv}-Y_{q'}^{wv}\big|\leq|q'-q|^\alpha$.*

Proof. By Theorem VI.A.b. *i)* of [4], $\exists\, h>1$, $C_K>0$ such that

$$\text{(3.17)}\qquad \text{for all } (q,q')\in K^2,\ \mathbb{E}\left(|Y_q-Y_{q'}|^h\right)\leq C_K|q-q'|^h.$$

For $n\geq 1$, let $\widetilde{Q}_n$ be the set of pairs $(q,q')\in Q_n$ such that $|q-q'|=2^{-n}$, and let $\alpha\in(0,(h-1)/h)$. Using (3.17) and the Markov inequality, one gets

$$\begin{aligned} p_n\ &:=\ \mathbb{P}\Big(\exists\,(q,q')\in\widetilde{Q}_n,\ |Y_q-Y_{q'}|\geq|q-q'|^\alpha\Big)\\ &\leq\ \sum_{(q,q')\in\widetilde{Q}_n}\mathbb{P}\Big(|Y_q-Y_{q'}|\geq|q-q'|^\alpha\Big)\leq 2|K|2^nC_K2^{n\alpha h}2^{-nh}.\end{aligned}$$

Let us fix $\eta>0$ and $\delta\in((1+\eta)^{-1},1)$. $\sum_{j\geq1}b^j\sum_{n\geq[j^{1+\eta}]}p_n<\infty$ implies item **1.** of Lemma 4 by the Borel-Cantelli lemma. Also, item **2.** follows from the fact that $\sum_{j\geq 1}b^j\sum_{n\geq j^\delta}b^n\sum_{m\geq[n^{1+\eta}]}p_m<\infty$. □

Lemma 5. *Under the assumptions of Theorem 4, let K be a compact subinterval of $J=\mathbb{R}$. Let $\eta>0$. There exist $\delta\in(0,1)$ and $\theta>1$ such that, with probability one, for j large enough,*
1. $\forall\, w\in\mathcal{A}^j$, $\sup_{q\in Q_{[j^{1+\eta}]}\cap K} Y_q^w\leq j^\theta$.
2. $\forall\, n\geq j^\delta$, $\forall\,(v,w)\in\mathcal{A}^n\times\mathcal{A}^j$, $\sup_{q\in Q_{[n^{1+\eta}]}\cap K} Y_q^{wv}\leq\ n^\theta$.

Proof. Fix $\theta > 1+\eta$. For $q \in K$ and $j \geq 1$ define $p_j(q) = \mathbb{P}(Y_q \geq j^\theta)$. By Lemma 3,

$$\forall\, j \geq 1,\ \ \mathbb{P}\Big(\sup_{q \in Q_{[j^{1+\eta}]} \cap K} Y_q \geq j^\theta \Big) \ \leq \ \sum_{q \in Q_{[j^{1+\eta}]} \cap K} p_j(q)$$
$$\leq \ p_j := 2C_K |K| 2^{j^{1+\eta}} \exp\big(- c_K j^{\theta_K} \big).$$

We let the reader verify that $\sum_{j\geq 1} b^j p_j < \infty$ and $\sum_{j \geq 1} b^j \sum_{n \geq j^\delta} b^n p_n < \infty$ if $\delta \in (\theta_K^{-1}, 1)$. This yields items 1. and 2. of Lemma 5. □

Let us now finish the proof of Theorem 5. It is a consequence of (3.15) and of the next Propositions 3 and 4.

Proposition 3. *Under the assumptions of Theorem 2, let K be a compact subinterval of $J \cap \mathbb{R}_+$ (resp. $J \cap \mathbb{R}_-$). There exist $\theta > 0$ and $\delta \in (0,1)$ such that, with probability one, for j large enough*
1. $\forall\, w \in \mathcal{A}^j$, one has $\inf_{q \in K} Y_q^w \geq j^{-\theta}$.
2. $\forall n \geq j^\delta$, $\forall\, w \in \mathcal{A}^j$, $\forall v \in \mathcal{A}^n$, one has $\inf_{q\in K} Y_q^{wv} \geq n^{-\theta}$.

Proof. Fix $\theta > 1$ such that $\theta_K = \theta \frac{\gamma_K}{1-\gamma_K} > 1$, where γ_K is as in Lemma 2. Let us also define $p_j = \mathbb{P}\left(\inf_{q\in K} Y_q < j^{-\theta}\right)$.

We let the reader verify, using Lemma 2, that $\sum_{j \geq 1} b^j p_j < \infty$ and if $\delta \in (\theta_K^{-1}, 1)$ then $\sum_{j\geq 1} b^j \sum_{n \geq j^\delta} b^n p_n < \infty$. This yields 1. and 2. □

Proposition 4. *Under the assumptions of Theorem 4, let K be a compact subinterval of J. There exist $\theta > 0$ and $\delta \in (0,1)$ such that, with probability one, for j large enough*
1. $\forall\, w \in \mathcal{A}^j$, one has $\sup_{q\in K} Y_q^w \leq j^\theta$.
2. $\forall n \geq j^\delta$, $\forall\, w \in \mathcal{A}^j$, $\forall v \in \mathcal{A}^n$, one has $\sup_{q \in K} Y_q^{wv} \leq n^\theta$.

Proof. We assume without loss of generality that the end points of K are dyadic numbers.

It is standard (see the proof of Kolmogorov theorem in [32]) that Lemma 4 implies that there exists a constant $C_K > 0$ such that, with probability one
1. for j large enough, $\forall\, w \in \mathcal{A}^j$, $\forall\, q, q' \in K$ such that $|q - q'| \leq 2^{-[j^{1+\eta}]}$, one has $\left|Y_q^w - Y_{q'}^w\right| \leq C_K |q' - q|^\alpha$.
2. for j large enough, $\forall n \geq j^\delta$, $\forall\, (v, w) \in \mathcal{A}^n \times \mathcal{A}^j$, for all $q, q' \in K$ such that $|q' - q| \leq 2^{-[n^{1+\eta}]}$, one has $\left|Y_q^{wv} - Y_{q'}^{wv}\right| \leq C_K |q' - q|^\alpha$.

Then, Lemma 5 concludes the proof. □

4. The version of Theorem 3 needed to get (1.6)

Now, let $\{(x_n, \lambda_n)\}_{n \geq 1}$ be a sequence in $[0,1] \times (0,1]$ such that $\lim_{n\to\infty} \lambda_n = 0$. For every $t \in (0,1)$, $k \geq 1$ and $r \in (0,1)$ let us define

$$\mathcal{B}_{k,r}(t) = \left\{ B(x_n, \lambda_n) : \ t \in B(x_n, r\lambda_n),\ \lambda_n \in (b^{-(k+1)}, b^{-k}] \right\}$$

($B(y, r)$ denotes the closed interval centered at y with radius r). Notice that this set may be empty. Then, if $\xi > 1$ and $B(x_n, \lambda_n) \in \mathcal{B}_{k,1/2}(t)$, let $\mathcal{B}_k^\xi(t)$ be the set of b-adic intervals of maximal length included in $B(x_n, \lambda_n^\xi)$. The next result is key to build a generalized Cantor set of Hausdorff dimension $\geq \tau_\mu^*(\alpha)/\xi$ in the set $K(\alpha, \xi)$ (1.4).

Theorem 6. *Suppose that* $\limsup_{n\to\infty} B(x_n, \lambda_n/4) = (0,1)$. *Let* μ *be an independent random cascade. Fix* $\kappa > 0$. *For* $j \geq 2$ *let* $\mathcal{S}_j = j\log(j)^{-\kappa}$ *and* $\rho_j = \log(j)^\alpha$ *with* $\alpha > 1$. *Assume that (3.4) holds.*

For every $q \in J$ *and* $\xi > 1$, *with probability one, the property* $\mathcal{P}(\xi, q)$ *holds, where* $\mathcal{P}(\xi, q)$ *is: For* μ_q*-almost every* t, *there are infinitely many* $k \geq 1$ *such that* $\mathcal{B}_{k,1/2}(t) \neq \emptyset$ *and there exists* $u \in \{v \in \mathcal{A}^* : \exists\, I \in \mathcal{B}_k^\xi(t),\ I = I_v\}$ *such that*

$$(4.1) \qquad GS\big(\mu_q^u, \mu_q^u, \tau_\mu'(q)q - \tau_\mu(q), N, \widetilde{\varepsilon}\big) \leq \mathcal{S}_{|u|}, \quad \text{and} \quad \|\mu_q^u\| \geq b^{-\rho_{|u|}}.$$

Remark 6. *The control of* $GS\big(\mu_q^w, \mu^w, \tau_\mu'(q), N, \widetilde{\varepsilon}\big)$ *is not useful in deriving (1.6).*

The result in [10] *concerning ubiquity conditioned by Mandelbrot measures invokes a slightly different version of Theorem 6. The proof of this other version is easily deduced from that of Theorem 6.*

Proof. For $k \geq 1$ and $w \in \mathcal{A}^{k+3}$, notice that $\mathcal{B}_{k,1/4}(t) \subset \mathcal{B}_{k,1/2}(s)$ for all $t, s \in I_w$. Let $\mathcal{R}_w = \{n : \exists t \in I_w,\ B(x_n, \lambda_n) \in \mathcal{B}_{k,1/4}(t)\}$. Define $n(w) = \inf\big\{n : x_n = \min\{x_m : m \in \mathcal{R}_w\}\big\}$ if $\mathcal{R}_w \neq \emptyset$ and $n(w) = 0$ otherwise.

If $\xi > 1$ and $n(w) > 0$, let $u(w)$ be the word encoding the b-adic interval of maximal length included in $B(x_n, \lambda_n^\xi)$ and whose left end point is minimal. If $\xi > 1$ and $n(w) = 0$, let $u(w)$ be the word of generation $[\xi|w|]$ with prefix w and its $[\xi|w|] - |w|$ last digits equal to 0.

Now, $w^{(j)}(t)$ being defined as in the statement of Theorem 3, we prove a slightly stronger result than Theorem 6: For every $q \in J$ and $\xi > 1$, with probability one, the property $\widetilde{\mathcal{P}}(\xi, q)$ holds, where $\widetilde{\mathcal{P}}(\xi, q)$ is: For μ_q-almost every t, if j is large enough, for all $k \geq j$ such that $n\big(w_{k+3}(t)\big) > 0$, $u = u(w_{k+3}(t))$ satisfies (4.1).

In the sequel we denote $u(w_{k+3}(t))$ by $u_{k,\xi}(t)$.

We fix $\xi > 1$ and $q \in K$.

For $j \geq 0$ and $n \geq 1$ define on $\Omega \times [0,1)$ the random variables

$$\begin{aligned} \mathbf{U}^{(j)}(\omega, t) &= \|\mu_q^{u_{j,\xi}(t)}(\omega)\|. \\ \text{and} \quad \mathbf{V}_n^{(j)}(\omega, t) &= S_n^{N,\varepsilon_n,\eta_n}\big(\mu_q^{u_{j,\xi}(t)}(\omega), \mu_q^{u_{j,\xi}(t)}(\omega), q\tau_\mu'(q) - \tau_\mu'(q)\big). \end{aligned}$$

We can use the proof of Proposition 2 to deduce that it is enough to prove

$$(4.2) \qquad \sum_{j\geq 1} \mathcal{Q}_q\Big(\Big\{\exists\, k \geq j,\ b^{\rho_{|u_{k,\xi}(t)|}} \sum_{n \geq \mathcal{S}_{|u_{k,\xi}(t)|}} \mathbf{V}_n^{(k)}(\omega, t) \geq 1/2\Big\}\Big) < \infty$$

$$(4.3) \ \text{and} \qquad \sum_{j\geq 1} \mathcal{Q}_q\Big(\Big\{\exists\, k \geq j,\ \mathbf{U}^{(j)}(\omega, t) \leq b^{-\rho_{|u_{k,\xi}(t)|}}\Big\}\Big) < \infty.$$

Since there exist $c > c' > 0$ such that $c'\xi k \leq |u_{k,\xi}(t)| \leq c\xi k$ for all t, denoting $\bar{k} = [c\xi k] + 1$ and $\widetilde{k} = [c'\xi k]$, in order to get (4.2) and (4.3), it is enough to show that

$$\begin{cases} \mathcal{T} = \sum_{j\geq 1}\sum_{k\geq j} \mathcal{Q}_q\big(b^{\rho_{\bar{k}}} \sum_{n \geq \mathcal{S}_{\widetilde{k}}} \mathbf{V}_n^{(k)}(\omega, t) \geq 1/2\big) < \infty, \\ \mathcal{T}' = \sum_{j\geq 1}\sum_{k\geq j} \mathcal{Q}_q(\mathbf{U}^{(k)} \leq b^{-\rho_{\bar{k}}}) < \infty. \end{cases}$$

Notice that $\mathcal{T} \leq 2^h \sum_{j\geq 1}\sum_{k\geq j}\sum_{n\geq \mathcal{S}_{\widetilde{k}}} b^{\rho_{\bar{k}} h} \mathbb{E}_{\mathcal{Q}_q}\Big(\big(\mathbf{V}_n^{(k)}\big)^h\Big)$ if $h \in (0,1)$.

Mimicking the computations performed in the proof of Theorem 3, one gets

$$\int_{[0,1]} \big(V_n^{u_{k,\xi}(t)}(\omega)\big)^h \mu_q(\omega)(dt) = \sum_{w \in \mathcal{A}^{k+3}} \Big(\prod_{k=0}^{k-1} W_{q,w_{k+1}}(w|k)\Big)\big(V_n^{u(w)}\big)^h \|\mu_q^w\|.$$

Using the independences as well as p and h as in the proof of Theorem 3, one obtains

$$\mathbb{E}_{\mathcal{Q}_q}\big(\big(\mathbf{V}_n^{(k)}\big)^h\big) \le \big(\mathbb{E}\big(V_n^{u(w)}\big)\big)^h\,\mathbb{E}\big(\|\mu_q^w\|^p\big)^{1/p}$$

where w is any element of $\mathcal{A}^*$. Then our choice for ρ_j and $\mathcal{S}_j$ ensures that $\mathcal{T}$ is finite.

Now, for any $h'>0$ one has $\mathcal{T}' \le \sum_{j\ge 1}\sum_{k\ge j} b^{-\rho_{\tilde{k}}h}\mathbb{E}_{\mathcal{Q}_q}\big(\big(\mathbf{U}^{(k)}\big)^{-h'}\big)$. A computation similar to the previous one yields, with the same h and p, $\mathbb{E}_{\mathcal{Q}_q}\big(\big(\mathbf{U}^{(k)}\big)^{-h'}\big) \le \big(\mathbb{E}\big(Y_q^{u(w)}\big)^{-h'/h}\big)^h\big(\|\mu_q^w\|^p\big)^{1/p}$ for any element w of $\mathcal{A}^*$. If h' is chosen small enough, by Lemma 2 the right hand side is bounded independently of k and the conclusion follows from our choice for ρ_j. □

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