Functional quantization and metric entropy for Riemann-Liouville processes

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Abstract

We derive a high-resolution formula for the L^2-quantization errors of Riemann-Liouville processes and the sharp Kolmogorov entropy asymptotics for related Sobolev balls. We describe a quantization procedure which leads to asymptotically optimal functional quantizers. Regular variation of the eigenvalues of the covariance operator plays a crucial role.

Keywords: Functional quantization, metric entropy, Gaussian process, Riemann-Liouville process, optimal quantizer.

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1 Introduction

Functional quantization of stochastic processes can be seen as a discretization of the path-space of a process and the approximation (coding) of a process by finitely many deterministic functions from its path-space. In a Hilbert space setting this reads as follows.

Let $(H, <\cdot,\cdot>)$ be a separable Hilbert space with norm $\|\cdot\|$ and let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to H$ be a random vector taking its values in H with distribution $\mathbb{P}_X$. For $n \in \mathbb{N}$, the L^2-quantization problem for X of level n (or of nat-level $\log n$) consists in minimizing

$$\left(\mathbb{E} \min_{a\in\alpha} \|X-a\|^2\right)^{1/2} = \|\min_{a\in\alpha} \|X-a\|\|_{L^2(\mathbb{P})}$$

over all subsets $\alpha \subset H$ with $\mathrm{card}(\alpha) \le n$. Such a set α is called n-codebook or n-quantizer. The minimal nth quantization error of X is then defined by

$$e_n(X) := \inf\left\{(\mathbb{E}\min_{a\in\alpha}\|X-a\|^2)^{1/2} : \alpha \subset H,\ \mathrm{card}(\alpha) \le n\right\}. \tag{1.1}$$

Under the integrability condition

$$\mathbb{E}\,\|X\|^2 < \infty \tag{1.2}$$

the quantity $e_n(X)$ is finite.

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For a given n-quantizer α one defines an associated closest neighbour projection

$$\pi_\alpha := \sum_{a\in\alpha} a\mathbf{1}_{C_a(\alpha)}$$

and the induced α-quantization (Voronoi quantization) of X by

$$\hat{X}^\alpha := \pi_\alpha(X), \tag{1.3}$$

where $\{C_a(\alpha) : a \in \alpha\}$ is a Voronoi partition induced by α, that is a Borel partition of H satisfying

$$C_a(\alpha) \subset V_a(\alpha) := \{x \in H : \|x-a\| = \min_{b\in\alpha} \|x-b\|\} \tag{1.4}$$

for every $a \in \alpha$. Then one easily checks that, for any random vector $X' : \Omega \to \alpha \subset H$,

$$\mathbb{E}\,\|X - X'\|^2 \geq \mathbb{E}\,\|X - \hat{X}^\alpha\|^2 = \mathbb{E}\,\min_{a\in\alpha}\|X-a\|^2$$

so that finally

$$\begin{aligned} e_n(X) &= \inf\Big\{(\mathbb{E}\,\|X-\hat{X}\|^2)^{1/2} : \hat{X} = f(X), f : H \to H \text{ Borel measurable}, \\ &\qquad \operatorname{card}(f(H)) \leq n\} \\ &= \inf\Big\{(\mathbb{E}\,\|X-\hat{X}\|^2)^{1/2} : \hat{X} : \Omega \to H \text{ random vector}, \operatorname{card}(\hat{X}(\Omega)) \leq n\Big\}. \end{aligned} \tag{1.5}$$

Observe that the Voronoi cells $V_a(\alpha), a \in \alpha$ are closed and convex (where convexity is a characteristic feature of the underlying Hilbert structure). Note further that there are infinitely many α-quantizations of X which all produce the same quantization error and $\hat{X}^\alpha$ is $\mathbb{P}$-a.s. uniquely defined if $\mathbb{P}_X$ vanishes on hyperplanes.

A typical setting for functional quantization is $H = L^2([0,1], dt)$ but is obviously not restricted to the Hilbert space setting. Functional quantization is the natural extension to stochastic processes of the so-called optimal vector quantization of random vectors in $H = \mathbb{R}^d$ which has been extensively investigated since the late 1940's in Signal processing and Information Theory (see [4], [8]). For the mathematical aspects of vector quantization in $\mathbb{R}^d$, one may consult [5], for algorithmic aspects see [15] and "non-classical" applications can be found in [14], [16]. For a first promising application of functional quantization to the pricing of financial derivatives through numerical integration on path-spaces see [17].

We address the issue of high-resolution quantization which concerns the performance of n-quantizers and the behaviour of $e_n(X)$ as $n \to \infty$. The asymptotics of $e_n(X)$ for $\mathbb{R}^d$-valued random vectors has been completely elucidated for non-singular distributions $\mathbb{P}_X$ by the Zador Theorem (see [5]) and for a class of self-similar (singular) distributions by [6]. In infinite dimensions no such global results hold, even for Gaussian processes.

It is convenient to use the symbols $\sim$ and $\stackrel{<}{\sim}$, where $a_n \sim b_n$ means $a_n/b_n \to 1$ and $a_n \stackrel{<}{\sim} b_n$ means $\limsup_{n\to\infty} a_n/b_n \leq 1$. A measurable function $\varphi : (s,\infty) \to (0,\infty)\,(s \geq 0)$ is said to be regularly varying at infinity with index $b \in \mathbb{R}$ if, for every $c > 0$,

$$\lim_{x\to\infty} \frac{\varphi(cx)}{\varphi(x)} = c^b.$$

Now let X be centered Gaussian. Denote by $K_X \subset H$ the reproducing kernel Hilbert space (Cameron-Martin space) associated to the covariance operator

$$C_X : H \to H,\ C_X y := \mathbb{E}\,(<y, X> X) \tag{1.6}$$

of X. Let $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ be the ordered nonzero eigenvalues of C_X and let $\{u_j : j \geq 1\}$ be the corresponding orthonormal basis of $\mathrm{supp}(\mathbb{P}_X)$ consisting of eigenvectors (Karhunen-Loève basis). If $d := \dim K_X < \infty$, then $e_n(X) = e_n\left(\bigotimes_{j=1}^{d} N(0,\lambda_j)\right)$, the minimal nth L^2-quantization error of $\bigotimes_{j=1}^{d} N(0,\lambda_j)$ with respect to the l_2-norm on $\mathbb{R}^d$, and thus we can read off the asymptotic behaviour of $e_n(X)$ from the high-resolution formula

$$e_n(\bigotimes_{j=1}^{d} N(0,\lambda_j)) \sim q(d)\sqrt{2\pi}\left(\Pi_{j=1}^{d}\lambda_j\right)^{1/2d}\left(\frac{d+2}{d}\right)^{(d+2)/4} n^{-1/d} \quad \text{as } n \to \infty \tag{1.7}$$

where $q(d) \in (0,\infty)$ is a constant depending only on the dimension d (see [5]). Except in dimension $d = 1$ and $d = 2$, the true value of $q(d)$ is unknown. However, one knows (see [5]) that

$$q(d) \sim \left(\frac{d}{2\pi e}\right)^{1/2} \quad \text{as } d \to \infty. \tag{1.8}$$

Assume $\dim K_X = \infty$. Under regular behaviour of the eigenvalues the sharp asymptotics of $e_n(X)$ can be derived analogously to (1.7). In view of (1.8) it is reasonable to expect that the limiting constants can be evaluated. The recent high-resolution formula is as follows.

Theorem 1 *([11]) Let X be a centered Gaussian. Assume $\lambda_j \sim \varphi(j)$ as $j \to \infty$, where $\varphi : (s,\infty) \to (0,\infty)$ is a decreasing, regularly varying function at infinity of index $-b < -1$ for some $s \geq 0$. Set, for every $x > s$,*

$$\psi(x) := \frac{1}{x\varphi(x)}.$$

Then

$$e_n(X) \sim \left(\left(\frac{b}{2}\right)^{b-1}\frac{b}{b-1}\right)^{1/2}\psi(\log n)^{-1/2} \quad \text{as } n \to \infty.$$

A high-resolution formula in case $b = 1$ is also available (see [11]). Note that the restriction $-b \leq -1$ on the index of φ is natural since $\sum_{j=1}^{\infty}\lambda_j < \infty$. The minimal L^r-quantization errors of X, $0 < r < \infty$, are strongly equivalent to the L^2-errors $e_n(X)$ (see [2]) and thus exhibit the same high-resolution behaviour.

A related quantization problem is the Kolmogorov metric entropy problem for the closed unit ball

$$U_X := \left\{x \in K_X : \|x\|_{K_X} \leq 1\} = \{x \in \mathrm{supp}(\mathbb{P}_X) : \sum_{j\geq 1}\frac{<x,u_j>^2}{\lambda_j} \leq 1\right\} \tag{1.9}$$

of K_X (Strassen ball). Note that U_X is a compact subset of H. For $n \in \mathbb{N}$, the metric entropy problem for U_X consists in minimizing

$$\max_{x\in U_X}\min_{a\in\alpha}\|x-a\| = \|\min_{a\in\alpha}\|X' - a\|\|_{L^\infty(\mathbb{P})}$$

over all subsets $\alpha \subset H$ with $\mathrm{card}(\alpha) \leq n$, where X' is any H-valued random vector with $\mathrm{supp}(\mathbb{P}_{X'}) = U_X$. The nth entropy number is then defined by

$$e_n(U_X) := \inf\left\{\max_{x\in U_X}\min_{a\in\alpha}\|x-a\| : \alpha \subset H,\ \mathrm{card}(\alpha) \leq n\right\}. \tag{1.10}$$

If $d := \dim K_X < \infty$, then $e_n(U_X) = e_n(\mathcal{E}_d)$, the nth entropy number of the ellipsoid

$$\mathcal{E}_d := \left\{ x \in \mathbb{R}^d : \sum_{j=1}^{d} \frac{x2_j}{\lambda_j} \leq 1 \right\}$$

with respect to the l_2-norm on $\mathbb{R}^d$. Thus we can read off the asymptotic behaviour of $e_n(U_X)$ from the formula

$$e_n(\mathcal{E}) \sim p(d)(\Pi_{j=1}^{d}\lambda_j)^{1/2}(\text{vol }(B_d(0,1)))^{1/d} n^{-1/d} \quad \text{as} \quad n \to \infty \tag{1.11}$$

where the constant $p(d) \in (0,\infty)$ is unknown for $d \geq 3$ and $p(d) \sim q(d), d \to \infty$ (see [9], [5]).

If $\dim K_X = \infty$, the recent solution of the Kolmogorov metric entropy problem for U_X is as follows.

Theorem 2 *([12]) Assume the situation of Theorem 1. Then*

$$e_n(U_X) \sim \left(\frac{b}{2}\right)^{b/2} \varphi(\log n)^{1/2} \quad \textit{as} \quad n \to \infty.$$

This formula is still valid for $b = 1$ and, ignoring the probabilistic interpretation, also for $b \geq 0$ ($00 := 1$) provided $\lambda_j \to 0$ as $j \to \infty$. (see [7], [12]). A different approach via the inverse of $e_n(U_X)$, the Kolmogorov ε-entropy, is due to Donoho [3]. (However, his result does not provide the correct constant.) ¿From Theorems 1 and 2 we conclude that functional quantization and metric entropy are related by

$$e_n(X) \sim \left(\frac{2\log n}{b-1}\right)^{1/2} e_n(U_X) \quad \text{as} \quad n \to \infty. \tag{1.12}$$

The paper is organized as follows. In Section 2 we investigate Riemann-Liouville processes in $H = L^2([0,1],dt)$. For $\rho \in (0,\infty)$, the Riemann-Liouville process $X^\rho = (X_t^\rho)_{t\in[0,1]}$ on $[0,1]$ is defined by

$$X_t^\rho := \int_0^t (t-s)^{\rho-\frac{1}{2}} dW_s \tag{1.13}$$

where W is a standard Brownian motion. We derive a high-resolution formula for X^ρ and correspondingly, the precise entropy asymptotics for fractional Sobolev balls. As a consequence we obtain a new result for fractionally integrated Brownian motions. In Section 3 we describe a quantization procedure which furnishes asymptotically optimal quantizers in the situation of Theorem 1. Here the Karhunen-Loève expansion plays a crucial rôle. In Section 4 we discuss a dimension conjecture.

2 Riemann-Liouville processes

Let $X^\rho = (X_t^\rho)_{t\in[0,1]}$ be the Riemann-Liouville process of index $\rho \in (0,\infty)$ as defined in (1.13). Its covariance function is given by

$$\mathbb{E}\, X_s^\rho X_t^\rho = \int_0^{s\wedge t} (t-r)^{\rho-\frac{1}{2}}(s-r)^{\rho-\frac{1}{2}} dr. \tag{2.1}$$

Using $\rho \wedge \frac{1}{2}$-Hölder continuity of the application $t \mapsto X_t^\rho$ from [0,1] into $L^2(\mathbb{P})$ and the Kolmorogov criterion one checks that X^ρ has a pathwise continuous modification so that we may assume without loss of generality that X^ρ is pathwise continuous. In particular, X^ρ can be seen as a centered Gaussian random vector with values in

$$H = L^2([0,1],dt).$$

The following high-resolution formula relies on a theorem by Vu and Gorenflo [18] on singular values of Riemann-Liouville integral operators

$$R_\beta\, g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds, \qquad \beta \in (0,\infty). \tag{2.2}$$

Theorem 3 *For every* $\rho \in (0,\infty)$,

$$e_n(X^\rho) \sim \pi^{-(\rho+\frac{1}{2})}(\rho+1/2)^\rho (\frac{2\rho+1}{2\rho})^{1/2}\Gamma(\rho+1/2)(\log n)^{-\rho} \quad as \;\; n \to \infty.$$

Proof. For $\beta > 1/2$, the Riemann-Liouville fractional integral operator R_β is a bounded operator from $L^2([0,1],dt)$ into $L^2([0,1],dt)$. The covariance operator

$$C_\rho : L^2([0,1],dt) \to L^2([0,1],dt)$$

of X^ρ is given by the Fredholm transformation

$$C_\rho g(t) = \int_0^1 g(s) E X_s^\rho X_t^\rho ds.$$

Using (2.1), one checks that C_ρ admits a factorization

$$C_\rho = S_\rho S_\rho^*,$$

where

$$S_\rho = \Gamma(\rho+1/2) R_{\rho+\frac{1}{2}}.$$

Consequently, it follows from Theorem 1 in [18] that the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ of C_ρ satisfy

$$\lambda_j \sim \Gamma(\rho+1/2)^2 (\pi j)^{-(2\rho+1)} \;\; \text{as} \;\; j \to \infty. \tag{2.3}$$

Now the assertion follows from Theorem 1 (with $\varphi(x) = \Gamma(\rho+1/2)^2 \pi^{-b} x^{-b}$ and $b = 2\rho+1$). □

An immediate consequence for fractionally integrated Brownian motions on $[0,1]$ defined by

$$Y_t^\beta := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} W_s ds \tag{2.4}$$

for $\beta \in (0,\infty)$ is as follows.

Corollary 1 *For every* $\beta \in (0,\infty)$,

$$e_n(Y^\beta) \sim \pi^{-(\beta+1)}(\beta+1)^{\beta+\frac{1}{2}} (\frac{2\beta+2}{2\beta+1})^{1/2} (\log n)^{-(\beta+\frac{1}{2})} \quad as \;\; n \to \infty.$$

Proof. For $\rho > 1/2$, the Ito formula yields

$$X_t^\rho = (\rho - \frac{1}{2}) \int_0^t (t-s)^{\rho-\frac{3}{2}} W_s ds.$$

Consequently,

$$Y_t^\beta = \frac{1}{\beta\Gamma(\beta)} \beta \int_0^t (t-s)^{\beta+\frac{1}{2}-\frac{3}{2}} W_s ds = \frac{1}{\Gamma(1+\beta)} X_t^{\beta+\frac{1}{2}}.$$

The assertion follows from Theorem 3. □

Remark. The preceding corollary provides new high-resolution formulas for $e_n(Y^\beta)$ in the cases $\beta \in (0,\infty) \setminus \mathbb{N}$.

One further consequence is a precise relationship between the quantization errors of Riemann-Liouville processes and fractional Brownian motions. The fractional Brownian motion with Hurst exponent $\rho \in (0,1]$ is a centered pathwise continuous Gaussian process $Z^\rho = (Z_t^\rho)_{t\in[0,1]}$ having the covariance function

$$\mathbb{E}\, Z_s^\rho Z_t^\rho = \frac{1}{2}(s^{2\rho} + t^{2\rho} - \mid s-t \mid^{2\rho}). \tag{2.5}$$

Corollary 2 *For every $\rho \in (0,1)$,*

$$e_n(X^\rho) \sim \frac{\Gamma(\rho+1/2)}{(\Gamma(2\rho+1)\sin(\pi\rho))^{1/2}} e_n(Z^\rho) \quad \text{as} \quad n \to \infty.$$

Proof. By [11], we have

$$e_n(Z^\rho) \sim \pi^{-(\rho+\frac{1}{2})}(\rho+1/2)^\rho \left(\frac{2\rho+1}{2\rho}\right)^{1/2} (\Gamma(2\rho+1)\sin(\pi\rho))^{1/2}(\log n)^{-\rho}, n \to \infty.$$

Combining this formula with Theorem 3 yields the assertion □

Observe that strong equivalence $e_n(X^\rho) \sim e_n(Z^\rho)$ as $n \to \infty$ is true for exactly two values of $\rho \in (0,1)$, namely for $\rho = 1/2$ where even $e_n(X^{1/2}) = e_n(Z^{1/2}) = e_n(W)$ and, a bit mysterious, for $\rho = 0.81557\ldots$

Now consider the Strassen ball U_ρ of X^ρ. Since the covariance operator C_ρ satisfies $C_\rho = \Gamma(\rho+\frac{1}{2})R_{\rho+\frac{1}{2}}(\Gamma(\rho+\frac{1}{2})R_{\rho+\frac{1}{2}})^*$, one gets

$$\begin{aligned} U_\rho &= \Gamma(\rho+1/2)R_{\rho+\frac{1}{2}}(B_{L^2}(0,1)) \\ &= \left\{R_{\rho+1/2}\,g : g \in L^2([0,1],dt), \int 1_0 g(t)^2 dt \le \Gamma(\rho+1/2)^2\right\}, \end{aligned} \tag{2.6}$$

a fractional Sobolev ball. Theorem 2 and (2.3) yield the solution of the entropy problem for fractional Sobolev balls.

Theorem 4 *For every $\rho \in (0,\infty)$,*

$$\begin{aligned} e_n(U_\rho) &\sim \left(\frac{\rho+\frac{1}{2}}{\pi}\right)^{\rho+\frac{1}{2}} \Gamma(\rho+1/2)(\log n)^{-(\rho+\frac{1}{2})} \\ &\sim \left(\frac{\rho}{\log n}\right)^{1/2} e_n(X^\rho) \quad \text{as} \quad n \to \infty. \end{aligned}$$

3 Asymptotically optimal functional quantizers

Let X be a H-valued random vector satisfying (1.2). For every $n \in \mathbb{N}$, L^2-optimal n-quantizers $\alpha \subset H$ exist, that is

$$(\mathbb{E} \min_{a\in\alpha} \|X-a\|^2)^{1/2} = e_n(X).$$

If card $(\text{supp}(\mathbb{P}_X)) \ge n$, optimal n-quantizers α satisfy $\text{card}(\alpha) = n$, $\mathbb{P}(X \in C_a(\alpha)) > 0$ and the stationarity condition

$$a = \mathbb{E}\,(X \mid \{X \in C_a(\alpha)\}),\ a \in \alpha$$

or what is the same

$$\hat{X}^\alpha = \mathbb{E}\,(X \mid \hat{X}^\alpha) \tag{3.1}$$

for every Voronoi partition $\{C_a(\alpha) : a \in \alpha\}$ (see [10]). In particular, $\mathbb{E}\,\hat{X}^\alpha = \mathbb{E}\,X$.

Now let X be centered Gaussian with $\dim K_X = \infty$. The Karhunen-Loève basis $\{u_j : j \geq 1\}$ consisting of normalized eigenvectors of C_X is optimal for the quantization of Gaussian random vectors (see [10]). So we start with the Karhunen-Loève expansion

$$X \overset{H}{=} \sum_{j=1}^{\infty} \lambda_j^{1/2} Z_j u_j,$$

where $Z_j = < X, u_j > /\lambda_j^{1/2}, j \geq 1$ are i.i.d. $N(0,1)$-distributed random variables. The design of an asymptotically optimal quantization of X is based on optimal quantizing blocks of coefficients of variable (n-dependent) block length. Let $n \in \mathbb{N}$ and fix temporarily $m, l, n_1, \ldots, n_m \in \mathbb{N}$ with $\Pi_{j=1}^m n_j \leq n$, where m denotes the number of blocks, l the block length and n_j the size of the quantizer for the jth block

$$Z^{(j)} := (Z_{(j-1)l+1}, \ldots, Z_{jl}), \quad j \in \{1, \ldots, m\}.$$

Let $\alpha_j \subset \mathbb{R}^l$ be an L^2-optimal n_j-quantizer for $Z^{(j)}$ and let $\widehat{Z^{(j)}} = \widehat{Z^{(j)}}^{\alpha_j}$ be a α_j-quantization of $Z^{(j)}$. Then, define a quantized version of X by

$$\hat{X}^n := \sum_{j=1}^{m} \sum_{k=1}^{l} \lambda_{(j-1)l+k}^{1/2} (\widehat{Z^{(j)}})_k u_{(j-1)l+k}. \tag{3.2}$$

It is clear that $\mathrm{card}(\hat{X}^n(\Omega)) \leq n$. Using (3.1) for $Z^{(j)}$, one gets $\mathbb{E}\,\hat{X}^n = 0$. If

$$\widehat{Z^{(j)}} = \sum_{b \in \alpha_j} b \mathbf{1}_{C_b(\alpha_j)}(Z^{(j)}),$$

then

$$\hat{X}^n = \sum_{a \in \times_{j=1}^m \alpha_j} \Big(\sum_{j=1}^{m} \sum_{k=1}^{l} \lambda_{(j-1)l+k}^{1/2} a_k^{(j)} u_{(j-1)l+k}\Big) \Pi_{j=1}^m \mathbf{1}_{C_{a^{(j)}}(\alpha_j)}(Z^{(j)})$$

where $a = (a^{(1)}, \ldots, a^{(m)}) \in \times_{j=1}^m \alpha_j$. Observe that in general, $\hat{X}^n$ is not a Voronoi quantization of X since it is based on the (less complicated) Voronoi partitions for $Z^{(j)}, j \leq m$. ($\hat{X}^n$ is a Voronoi quantization if $l = 1$ or if $\lambda_{(j-1)l+1} = \ldots = \lambda_{jl}$ for every j.) Using again (3.1) for $Z^{(j)}$ and the independence structure, one checks that $\hat{X}^n$ satisfies a kind of stationarity equation:

$$\mathbb{E}\,(X \mid \hat{X}^n) = \hat{X}^n.$$

Lemma 1 *Let $n \geq 1$. For every $l \geq 1$ and every $m \geq 1$*

$$\mathbb{E}\,\|X - \hat{X}^n\|^2 \leq \sum_{j=1}^{m} \lambda_{(j-1)l+1} e_{n_j}(N(0, I_l))^2 + \sum_{j \geq ml+1} \lambda_j. \tag{3.3}$$

Furthermore, (3.3) stands as an equality if $l = 1$ (or $\lambda_{(j-1)l+1} = \ldots = \lambda_{jl}$ for every j, $l \geq 1$).

Proof. The claim follows from the orthonormality of the basis $\{u_j : j \geq 1\}$. We have

$$\begin{aligned}\mathbb{E}\,\|X-\hat{X}^n\|^2 &= \sum_{j=1}^{m}\sum_{k=1}^{l}\lambda_{(j-1)l+k}\mathbb{E}\mid Z_k^{(j)}-(\widehat{Z^{(j)}})_k\mid^2+\sum_{j\geq ml+1}\lambda_j\\ &\leq \sum_{j=1}^{m}\lambda_{(j-1)l+1}\sum_{k=1}^{l}\mathbb{E}\mid Z_k^{(j)}-\widehat{Z^{(j)}})_k\mid^2+\sum_{j\geq ml+1}\lambda_j\\ &= \sum_{j=1}^{m}\lambda_{(j-1)l+1}e_{n_j}(Z^{(j)})^2+\sum_{j\geq ml+1}\lambda_j.\end{aligned}$$

□

Set

$$C(l) := \sup_{k\geq 1} k^{2/l}e_k(N(0,I_l))^2. \tag{3.4}$$

By (1.7), $C(l) < \infty$. For every $l \in \mathbb{N}$,

$$e_{n_j}(N(0,I_l)^2 \leq n_j^{-2/l}C(l) \tag{3.5}$$

Then one may replace the optimization problem which consists, for fixed n, in minimizing the right hand side of Lemma 1 by the following optimal allocation problem:

$$\min\{C(l)\sum_{j=1}^{m}\lambda_{(j-1)l+1}n_j^{-2/l}+\sum_{j\geq ml+1}\lambda_j : m,l,n_1,\dots,n_m\in\mathbb{N},\Pi_{j=1}^m n_j\leq n\}. \tag{3.6}$$

Set

$$m = m(n,l) := \max\{k\geq 1 : n^{1/k}\lambda_{(k-1)l+1}^{l/2}(\Pi_{j=1}^k\lambda_{(j-1)l+1})^{-l/2k}\geq 1\}, \tag{3.7}$$

$$n_j = n_j(n,l) := [n^{1/m}\lambda_{(j-1)l+1}^{l/2}(\Pi_{i=1}^m\lambda_{(i-1)l+1})^{-l/2m}],\ j\in\{1,\dots,m\}, \tag{3.8}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$ and

$$l = l_n := [(\max\{1,\log n\})^{\vartheta}],\ \vartheta\in(0,1). \tag{3.9}$$

In the following theorem it is demonstrated that this choice is at least asymptotically optimal provided the eigenvalues are regularly varying.

Theorem 5 *Assume the situation of Theorem 1. Consider $\hat{X}^n$ with tuning parameters defined in (3.7)-(3.9). Then $\hat{X}^n$ is asymptotically n-optimal, i.e.*

$$(\mathbb{E}\,\|X-\hat{X}^n\|^2)^{1/2}\sim e_n(X)\ \text{ as }\ n\to\infty.$$

Note that no block quantizer with fixed block length is asymptotically optimal (see [11]). As mentioned above, $\hat{X}^n$ is not a Voronoi quantization of X. If $\alpha_n := \hat{X}^n(\Omega)$, then the Voronoi quantization $\hat{X}^{\alpha_n}$ is clearly also asymptotically n-optimal.

The key property for the proof is the following l-asymptotics of the constants $C(l)$ defined in (3.4). It is interesting to consider also the smaller constants

$$Q(l) := \lim_{k\to\infty} k^{2/l}e_k(N(0,I_l))^2 \tag{3.10}$$

(see (1.7)).

Proposition 1 *The sequences* $(C(l))_{l\geq 1}$ *and* $(Q(l))_{l\geq 1}$ *satisfy*

$$\lim_{l\to\infty} \frac{C(l)}{l} = \lim_{l\to\infty} \frac{Q(l)}{l} = \inf_{l\geq 1} \frac{C(l)}{l} = \inf_{l\geq 1} \frac{Q(l)}{l} = 1.$$

Proof. From [11] it is known that

$$\liminf_{l\to\infty} \frac{C(l)}{l} = 1. \tag{3.11}$$

Furthermore, it follows immediately from (1.7) and (1.8) that

$$\lim_{l\to\infty} \frac{Q(l)}{l} = 1. \tag{3.12}$$

(The proof of the existence of $\lim\limits_{l\to\infty} C(l)/l$ we owe to S. Dereich.) For $l_0, l \in \mathbb{N}$ with $l \geq l_0$, write

$$l = n\, l_0 + m \text{ with } n \in \mathbb{N}, m \in \{0, \ldots, l_0 - 1\}.$$

Since for every $k \in \mathbb{N}$,

$$[k^{l_0/l}]^n\; [k^{1/l}]^m \leq k,$$

one obtains by a block-quantizer design consisting of n blocks of length l_0 and m blocks of length 1 for quantizing $N(0, I_l)$,

$$e_k(N(0,I_l))^2 \leq n e_{[k^{l_0/l}]}(N(0,I_{l_0}))^2 + m e_{[k^{1/l}]}(N(0,1))^2. \tag{3.13}$$

This implies

$$\begin{aligned} C(l) &\leq nC(l_0) \sup_{k\geq 1} \frac{k^{2/l}}{[k^{l_0/l}]^{2/l_0}} + mC(1) \sup_{k\geq 1} \frac{k^{2/l}}{[k^{1/l}]^2} \\ &\leq 4^{1/l_0} n C(l_0) + 4mC(1). \end{aligned}$$

Consequently, using $n/l \leq 1/l_0$,

$$\frac{C(l)}{l} \leq \frac{4^{1/l_0} C(l_0)}{l_0} + \frac{4mC(1)}{l}$$

and hence

$$\limsup_{l\to\infty} \frac{C(l)}{l} \leq \frac{4^{1/l_0} C(l_0)}{l_0}.$$

This yields

$$\limsup_{l\to\infty} \frac{C(l)}{l} \leq \liminf_{l_0\to\infty} \frac{C(l_0)}{l_0} = 1. \tag{3.14}$$

It follows from (3.13) that

$$Q(l) \leq nQ(l_0) + mQ(1).$$

Consequently

$$\frac{Q(l)}{l} \leq \frac{Q(l_0)}{l_0} + \frac{mQ(1)}{l}$$

and therefore

$$1 = \lim_{l\to\infty} \frac{Q(l)}{l} \leq \frac{Q(l_0)}{l_0}.$$

This implies

$$\inf_{l_0 \geq 1} \frac{Q(l_0)}{l_0} = 1. \tag{3.15}$$

Since $Q(l) \leq C(l)$, the proof is complete. $\square$

The n-asymptotics of the number $m(n, l_n)l_n$ of quantized coefficients in the Karhunen-Loève expansion in the quantization $\hat{X}^n$ is as follows.

Lemma 2 *([12], Lemma 4.8) Assume the situation of Theorem 1. Let $m(n, l_n)$ be defined by (3.7) and (3.9). Then*

$$m(n, l_n)l_n \sim \frac{2 \log n}{b} \quad \textit{as} \quad n \to \infty.$$

Proof of Theorem 5. For every $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{j=1}^{m} \lambda_{(j-1)l+1} n_j^{-2/l} &\leq \sum_{j=1}^{m} \lambda_{(j-1)l+1} (n_j + 1)^{-2/l} (\frac{n_j + 1}{n_j})^{2/l} \\
&\leq 4^{1/l} m n^{-2/ml} (\Pi_{j=1}^{m} \lambda_{(j-1)l+1})^{1/m} \\
&\leq 4^{1/l} m \lambda_{(m-1)l+1}.
\end{aligned}$$

Therefore, by Lemma 1 and (3.5),

$$\mathbb{E} \| X - \hat{X}^n \|^2 \leq 4^{1/l} \frac{C(l)}{l} m l \lambda_{(m-1)l+1} + \sum_{j \geq ml+1} \lambda_j$$

for every $n \in \mathbb{N}$. By Lemma 2, we have

$$ml = m(n, l_n)l_n \sim \frac{2 \log n}{b} \quad \text{as} \quad n \to \infty.$$

Consequently, using regular variation at infinity with index $-b < -1$ of the function φ,

$$ml \lambda_{(m-1)l+1} \sim ml \lambda_{ml} \sim \left(\frac{2}{b} \right)^{1-b} \psi(\log n)^{-1}$$

and

$$\sum_{j \geq ml+1} \lambda_j \sim \frac{ml \varphi(ml)}{b-1} \sim \frac{1}{b-1} \left(\frac{2}{b} \right)^{1-b} \psi(\log n)^{-1} \quad \text{as} \quad n \to \infty,$$

where, like in Theorem 1, $\psi(x) = 1/x\varphi(x)$. Since by Proposition 1,

$$\lim_{n \to \infty} \frac{4^{1/l_n} C(l_n)}{l_n} = 1,$$

one concludes

$$\mathbb{E} \| X - \hat{X}^n \|^2 \lesssim \left(\frac{2}{b} \right)^{1-b} \frac{b}{b-1} \psi(\log n)^{-1} \quad \text{as} \quad n \to \infty.$$

The assertion follows from Theorem 1. $\square$

Numerical and computational aspects: As soon as the Karhunen-Loève basis $(u_j)_{j \geq 1}$ of a Gaussian process X is explicit, it is possible to compute the asymptotically optimal functional quantization (3.2) which solves the minimization problem (3.6) as well as its distribution and induced quantization error (at least for a given $\vartheta \in (0, 1)$). This is possible since some optimal (or at least locally optimal) vector quantizations of the $N(0, I_d)$-distribution has been already computed and kept off line. Let us be more specific.

– In 1-dimension, the normal distribution $N(0,1)$ has only one stationary n-quantizer – hence optimal – since its probability density is log-concave (for this result due to Kiefer, see *e.g.* [5]). Deterministic methods to compute these optimal quantizers are based on the stationary equation (3.1). They are very easy to implement, converge very fast with a very high accuracy. The Newton-Raphson algorithm is a possible choice (see [15] for details). Closed forms for the lowest quadratic quantization error $\|Z - \widehat{Z}\|_{L^2(\mathbb{P})}$ and for the distribution of the optimal n-quantization $\widehat{Z}^\alpha$ as a function of the optimal n-quantizer α are also available in [15]. These three quantities have been tabulated up to very high values of n. A file can be downloaded at the URL `www.proba.jussieu.fr/pageperso/pages.html`.

– In higher dimension, one still relies on the stationary equation (3.1) which reads:

$$\mathbb{E}\left(\mathbf{1}_{C_a(\alpha)}(Z)(a - Z)\right) = 0, \qquad a \in \alpha$$

One must keep in mind that the left hand side of the above equation is but the gradient of the (squared) quantization error $\mathbb{E}\|Z - \widehat{Z}^\alpha\|^2$ viewed as a function of the quantizer α (assumed to be of full size n). A stochastic gradient descent based on this integral representation can be implemented easily since the normal distribution $N(0, I_d)$ can be simulated on a computer from (pseudo-)random numbers (*e.g.* by the Box-Muller method). This algorithm is known as the Competitive Learning Vector Quantization (or $CLVQ$) algorithm. It has been extensively investigated both from a theoretical (see *e.g.* [14], [1]) and numerical (see *e.g.* [15] as concerns normally distributed vectors) viewpoints. The algorithm reads as follows: let $(\zeta(t))_{t\geq 1}$ be an i.i.d. sequence of $N(0, I_d)$-distributed random vectors, let $(\gamma_t)_{t\geq 1}$ be a decreasing sequence of positive *gain* parameter satisfying $\sum_t \gamma_t = +\infty$ and $\sum_{t\geq 1} \gamma_t^2 < +\infty$ and let $\alpha(0) \in (\mathbb{R}^d)^n$ denote a starting n-quantizer. Then, at time $t \in \mathbb{N}$, one update the running n-quantizer $\alpha(t-1) := (\alpha_1(t-1), \ldots, \alpha_n(t-1))$ as follows

$$\begin{array}{lrcl}
\text{COMPETITIVE PHASE: select} & i(t) & \in & \operatorname{argmin}\{i : \|\alpha_i(t-1) - \zeta(t)\| = \min_j \|\alpha_j(t-1) - \zeta(t)\|\} \\
\text{LEARNING PHASE:} & \alpha_{i(t)}(t-1) & = & (1-\gamma_t)\alpha_{i(t)}(t-1) + \gamma_t\, \zeta(t) \\
 & \alpha(t)_j & = & \alpha_{j-1}(t-1), \qquad j \neq i(t).
\end{array}$$

Some further details concerning the numerical implementation of this procedure can be found in [15], especially some heuristics concerning the initialization and the specification of the gain parameter sequence usually choosen of the form $\gamma_t = \frac{A}{B+t}$. It converges toward some local minima of the quantization error at a $\sqrt{\gamma_t}$-rate. Some d-dimensional grids ($d = 2$ up to 10) can be downloaded at the above URL for many values of n in the range 2 up to 2 000. These quantizations were carried out to solve numerically multi-dimensional stopping time problems (pricing of American options on baskets, see [16] and the references therein).

The 1-dimensional optimal quantization of the $N(0,1)$-distribution has already been used to produce some optimal *scalar* product functional quantization - *i.e.* based on blocks of fixed length 1- in [17] with some promising applications to the pricing of path-dependent European options in stochastic volatility models (this work is also based on results about diffusion processes from [13]). To be competitive with other methods (Monte Carlo, pde's) one needs to have good performances for not too large values of n. Within this range of values, it is more efficient to perform directly a numerical optimisation of (3.3) (or (3.6)) with $l = 1$ rather than using the theoretical asymptotically optimal parameters (3.7) and (3.8).

As far as numerical implementation of functional quantization with n-varying block size is concerned, some first numerical experiments carried out by Benedikt Wilbertz [19] suggest that it slightly improves the scalar approach for high values of n, say $n \leq 10^6$, simply using up to 3-dimensional n_j-quantizers with some n_j not greater than 100. A similar improvement can vbe obtained for lower values of n (say $n \geq 20\,000$) by using product quantizers made of blocks with mixed diemnsions (1, 2 or 3).

EXAMPLES: The basic example (among Riemann-Liouville processes) is $X^{1/2} = W$ and $H = L^2([0,1], dt)$, where

$$\lambda_j = (\pi(j-1/2))^{-2},\ u_j(t) = \sqrt{2}\sin\left(t/\sqrt{\lambda_j}\right),\ j \geq 1. \tag{3.16}$$

Since for $\delta, \rho \in (0, \infty)$,

$$X^{\delta+\rho} = \frac{\Gamma(\delta+\rho+\frac{1}{2})}{\Gamma(\rho+\frac{1}{2})} R_\delta(X^\rho),$$

one gets expansions of $X^{\delta+\rho}$ from Karhunen-Loève expansions of X^ρ. In particular,

$$X^{\delta+\frac{1}{2}} = \Gamma(\delta+1)\sum_{j=1}^{\infty}\sqrt{\lambda_j}Z_j R_\delta(u_j).$$

However, the functions $R_\delta(u_j), j \geq 1$, are not orthogonal in H so that the nonzero correlation between the components of $(Z^{(j)} - \widehat{Z^{(j)}})$ prevents the previous estimates for $\mathbb{E}\|X - \widehat{X}^n\|^2$ given in Lemma 1 from working in this setting in the general case.

However, when $l = 1$ (scalar product quantizers made up with blocks of fixed length $l = 1$), one checks that these estimates still stand as equalities since orthogonality can now be substituted by the independence of $Z_j - \hat{Z}_j$ and stationarity property (3.1) of the quantizations $\hat{Z}_j, j \geq 1$. It is often good enough for applications to use scalar product quantizers (see [10], [17]). If, for instance $\delta = 1$, then

$$X := X^{3/2} = \sum_{j=1}^{\infty}\sqrt{\lambda_j}Z_j R_1(u_j),$$

where

$$R_1(u_j)(t) = \sqrt{2\lambda_j}(1 - \cos(t/\sqrt{\lambda_j})).$$

Note that $\|R_1(u_j)\|^2 = \lambda_j(3 - 4(-1)^{j-1}\sqrt{\lambda_j}),\ j \geq 1$. Set

$$\hat{X}^n = \sum_{j=1}^{m}\sqrt{\lambda_j}\hat{Z}_j R_1(u_j).$$

The quantization $\widehat{X}^n$ is non Voronoi (it is related to the Voronoi tessellation of W) and satisfies

$$\mathbb{E}\|X - \widehat{X}^n\|^2 = \sum_{j=1}^{m}\lambda 2_j(3 - 4(-1)^{j-1}\sqrt{\lambda_j})e_{n_j}(N(0,1))^2 + \sum_{j\geq m+1}\lambda_j^2(3 - 4(-1)^{j-1}\sqrt{\lambda_j}). \tag{3.17}$$

It is possible to optimize the (scalar product) quantization error using this expression instead of (3.6). As concerns asymptotics, if the parameters are tuned following (3.7)-(3.9) with $l = 1$ and λ_j replaced by

$$\nu_j := \lambda_j^2(3 + 4\sqrt{\lambda_j}) \sim 3\pi^{-4}j^{-4} \quad \text{as} \quad n \to \infty,$$

and using Theorem 3 gives

$$(\mathbb{E}\,\|X - \hat{X}^n\|^2)^{1/2} \lesssim \left(\frac{3(12C(1)+1)}{4}\right)^{1/2} e_n(X) \ \text{ as } \ n \to \infty. \tag{3.18}$$

Numerical experiments seem to confirm that $C(1) = Q(1)$. Since $Q(1) = \pi\sqrt{3}/2$ (see [5], p. 124), the above upper bound is then

$$\left(\frac{3(6\pi\sqrt{3}+1)}{4}\right)^{1/2} = 5.02357\ldots$$

4 Dimension

Let X be a H-valued random vector satisfying (1.2). For $n \in \mathbb{N}$, let $\mathcal{C}_n(X)$ be the (nonempty) set of all L^2-optimal n-quantizers. Introduce the integral number

$$d_n(X) := \min\{\dim \text{span}\,(\alpha)) : \alpha \in \mathcal{C}_n(X)\}. \tag{4.1}$$

It represents the dimension at level n of the functional quantization problem for X. Here span(α) denotes the linear subspace of H spanned by α. In view of Section 3, a reasonable conjecture for Gaussian random vectors is $d_n(X) \sim 2\log n/b$ in regular cases, where $-b$ is the regularity index. We have at least the following lower estimate in the Gaussian case.

Proposition 2 *Assume the situation of Theorem 1. Then*

$$d_n(X) \gtrsim \frac{1}{b^{1/(b-1)}}\,\frac{2\log n}{b} \quad \textit{as} \quad n \to \infty.$$

Proof. For every $n \in \mathbb{N}$, we have

$$d_n(X) = \min\left\{k \geq 0 : e_n(\bigotimes_{j=1}^{k} N(0,\lambda_j))^2 + \sum_{j\geq k+1} \lambda_j \leq e_n(X)^2\right\} \tag{4.2}$$

(see [10]). Define

$$c_n := \min\left\{k \geq 0 : \sum_{j\geq k+1} \lambda_j \leq e_n(X)^2\right\}.$$

Clearly, c_n increases to infinity as $n \to \infty$ and by (4.2), $c_n \leq d_n(X)$ for every $n \in \mathbb{N}$. Using Theorem 1 and the fact that ψ is regularly varying at infinity with index $b-1$, we obtain

$$((b-1)\psi(c_n))^{-1} \sim \sum_{j\geq c_n+1} \lambda_j \sim e2_n(X) \sim \left(\frac{2}{b}\right)^{1-b} \frac{b}{b-1}\,\psi(\log n)^{-1}$$

and thus

$$\psi(c_n) \sim \left(\frac{2}{b}\right)^{1-b} \frac{1}{b}\psi(\log n) \sim \psi\left(\frac{1}{b^{1/(b-1)}}\,\frac{2\log n}{b}\right) \quad \text{as} \quad n \to \infty.$$

Consequently,

$$c_n \sim \frac{1}{b^{1/(b-1)}}\,\frac{2\log n}{b} \quad \text{as} \quad n \to \infty.$$

This yields the assertion. □

For Riemann-Liouville processes one concludes

$$d_n(X^\rho) \gtrsim (2\rho+1)^{-1/2\rho}\,\frac{2\log n}{2\rho+1}$$

(see (2.3)).

For the metric entropy problem one may introduce the numbers $d_n(U_X)$ analogously. Then, in the situation of Theorem 1 it is known that $d_n(U_X) \gtrsim 2\log n/b$ (see [12]). It remains an open question whether $d_n(X) \sim d_n(U_X) \sim 2\log n/b$.

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