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To cite this version:


HAL Id: hal-00004072

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Submitted on 26 Jan 2005

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A simple approximation algorithm for WIS based on the approximability in $k$-partite graphs

Jérôme Monnot*

Résumé

Dans cette note, nous montrons comment une solution optimale ou une solution approchée du stable pondéré dans les graphes $k$-partis peut permettre de récupérer une bonne solution approchée du stable pondéré dans les graphes généraux. Plus précisément, un stable pondéré optimal dans les graphes bipartis nous permet d’obtenir une $\frac{2}{\Delta(G)}$-approximation et, plus généralement, une $\rho$ solution approchée dans les graphes $k$-partis fournit une $\frac{k}{\Delta(G)}\rho$ solution approchée dans les graphes généraux.

Mots-clefs : Algorithme approché, coloration, stable pondéré, graphe $k$-parti.

Abstract

In this note, we show how optimal or approximate weighted independent sets in $k$-partite graphs may yield to a good approximate weighted independent set in general graphs. Precisely, optimal solutions in bipartite graphs do yield to a $\frac{2}{\Delta(G)}$-approximation and, more generally, $\rho$-approximate solutions in $k$-partite graphs yield to a $\frac{k}{\Delta(G)}\rho$-approximation in general graphs.

Key words : Approximation algorithms, Coloring, Weighted Independent Set, $k$-partite graphs.

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1 Introduction

In the Maximum Weighted Independent Set problem (WIS, for short), we are given a simple, connected, undirected and loop-free graph $G = (V,E)$ on $n$ vertices, with maximum degree $\Delta(G)$. Each vertex $v$ in $V$ is labelled with a positive weight $w(v) \geq 0$. For a subset $S \subseteq V$ of vertices, we denote by $w(S) = \sum_{v \in S} w(v)$ the sum of the weights of the elements in $S$. The goal of WIS is to find an independent set $S$ (that is a subset of pairwise non-adjacent vertices) in $G$ that maximizes $w(S)$. When the weight of each vertex is equal to one, this problem is usually called Maximum Independent Set problem (IS, for short).

WIS is known to be NP-hard in general graphs, but also for certain classes of graphs (see Garey and Johnson [5]). On the other hand, for a variety of graphs from both practical and theoretical frameworks, which includes the Perfect graphs family, this problem is polynomial, Grotschel et al. [6]. Very recently, Alekseev and Lozin, [1] provided a complete characterization of the $(p,q)$-colorable graphs for which WIS is polynomial, where we recall that a graph is $(p,q)$-colorable if it can be partitioned into at most $p$ cliques and $q$ independent sets. The main result states that WIS is polynomial on $(p,q)$-colorable graphs if and only if $q \leq 2$ (under the assumption $P \neq NP$). In particular, since a bipartite graph is a perfect graph as well as a $(0,2)$-colorable graph, then the Maximum Weighted Independent Set problem is polynomial on bipartite graphs.

Because WIS is one of the most important problem from both a practical and a theoretical point of view, many approximation results have been found out by several authors (see, for instance Hochbaum [9], Halldórsson and Lau [7] for performance ratio only depending on $\Delta(G)$, Halldórsson [8], Halldórsson and Lau [7] and Demange and Paschos [4]). Very recently, Sakai et al. [13] studied the behavior of several greedy strategies on WIS. In particular, they proved that one of these greedy algorithms is a $\frac{1}{\Delta(G)}$-approximation and that this ratio is tight. This algorithm selects, as long as the current graph $G_i$ is not empty, a vertex $v$ maximizing $\frac{w(v)}{d_G_i(v)+1}$ (where $d_G_i(v)$ is the degree of $v$ in the current graph $G_i$), and then deletes $v$ and its neighborhood from the current graph.

Results of this paper. In this paper, we show how an optimum weighted independent set in bipartite graphs and a $\rho$-approximation of WIS in $k$-partite graphs respectively allows to obtain a $\frac{2}{\Delta(G)}$-approximation and a $\frac{k}{\Delta(G)}\rho$-approximation in general graphs. In order to build a $k$-partite graph from a given graph, we use the notion of coloring, that is a partition of the vertices into independent sets (see, Paschos [12], for a survey on the approximability of the coloring problem). Already in the past, Hochbaum [9], exploits this notion of coloring to obtain a $\frac{2}{\Delta(G)}$-approximation of WIS, but in a complete different way. The algorithm of [9] is based on a preprocessing due to Nemhauser and Trotter [11] which provides two disjoint subgraphs, including an independent set; it then computes
on the other subgraph a coloring from which it selects the best independent set which is finally added to the first subgraph. More recently, Halldórsson and Lau [7] proposed an elegant algorithm which consists in partitioning $G$ into at most $\lceil \Delta(G) + 1 \rceil / 3$ subgraphs $G_i$ of degree at most 2; then, for each $G_i$, an optimum weighted independent set in $G_i$ is computed and finally, the best of these solutions is returned. The performance ratio of this algorithm is $1/((\Delta(G) + 1)/3)$, that is better than $2/\Delta(G)$ as soon as $\Delta(G) \geq 7$ or $\Delta(G) = 5$. Our algorithm computes, for every $k$-partite graph built on the coloring, an optimum or an approximate solution; it then returns the best one as a solution of the initial problem.

2 The Algorithm

In the first algorithm, we only use the notion of bipartite graph.

\begin{algorithm}
\begin{algorithmic}[1]
\State 1 Find a coloring $S = (S_1, \ldots, S_\ell)$ by using a polynomial-time algorithm $A$;
\State 2 For any $1 \leq i < j \leq \ell$ do
\State \hspace{1em} 2.1 Find an optimal independent set $S_{i,j}$ of bipartite graph induced by $S_i \cup S_j$;
\State 3 Return $S = \arg\max\{w(S_{i,j}) : 1 \leq i < j \leq \ell\}$;
\end{algorithmic}
\end{algorithm}

This algorithm is trivially polynomial since we apply at most $O(\ell^2)$ times a polynomial procedure.

**Theorem 2.1** Algorithm 1 is a $\frac{2}{3}$-approximation for WIS in general graphs.

**Proof.** Let $I = (G, w)$ be an instance of WIS and let $S^*$ be an optimal solution with value $\text{opt}(I) = w(S^*)$. We set $S^*_i = S^* \cap S_i$ for $1 \leq i \leq \ell$ where $S = (S_1, \ldots, S_\ell)$ is the coloring provided by algorithm $A$.

For any $i, j$ with $j > i$, the following key result holds:

$$w(S) \geq w(S^*_i) + w(S^*_j) \quad (1)$$

Let us explain why this result is true: on the one hand, the set $S^*_i \cup S^*_j$ is an independent set of the bipartite graph induced by $S_i \cup S_j$ and, on the other hand, by construction of the
algorithm, $S_{i,j}$ is an optimal solution of this bipartite graph. Thus, since $w(S) \geq w(S_{i,j})$, we deduce the expected result.

Summing the inequalities (1) for $i = 1$ to $i = \ell - 1$ and $j = i + 1$ to $j = \ell$, we obtain:

$$
\sum_{i=1}^{\ell-1} \sum_{j=i+1}^{\ell} w(S) \geq \sum_{i=1}^{\ell-1} \sum_{j=i+1}^{\ell} (w(S_i^*) + w(S_j^*)) \\
\geq \sum_{i=1}^{\ell-1} (\ell - i) w(S_i^*) + \sum_{j=1}^{\ell-1} \sum_{i=j+1}^{\ell} w(S_j^*) \\
\geq \sum_{i=1}^{\ell-1} (\ell - i) w(S_i^*) + \sum_{j=2}^{\ell} \sum_{i=1}^{j-1} w(S_j^*) \\
\geq (\ell - 1) w(S_1^*) + \sum_{i=2}^{\ell-1} (\ell - i) w(S_i^*) + \sum_{i=1}^{\ell-2} (i - 1) w(S_i^*) + (\ell - 1) w(S_1^*) \\
\geq (\ell - 1) \sum_{i=1}^{\ell} w(S_i^*)
$$

Thus, since $\sum_{i=1}^{\ell} w(S_i^*) = w(S^*)$, and $\sum_{i=1}^{\ell-1} \sum_{j=i+1}^{\ell} w(S) = \frac{\ell(\ell-1)}{2} w(S)$, we obtain:

$$
\frac{\ell(\ell-1)}{2} w(S) \geq (\ell - 1) \text{opt}(I)
$$

and the result follows. \[\blacksquare\]

We know that we can easily obtain a coloring using at most $\Delta(G) + 1$ colors (see Berge [2]): starting from an arbitrary coloring, pick up each vertex $v$ with a color at least $\Delta(G) + 2$ and recolor it with a compatible color among $\{1, \ldots, \Delta(G) + 1\}$ (this is always possible, since $v$ has at most $\Delta(G)$ neighbors). Thus, using Theorem 2.1 and the inequality $\ell \leq \Delta(G) + 1$, we deduce:

**Corollary 2.2** WIS is $\frac{2}{\Delta(G)+1}$-approximable and this ratio is tight.

**Proof.** We show that this ratio is tight, even in the basis case of the maximum independent set problem (i.e., $\forall v \in V, w(v) = 1$). We consider a graph $G = (V,E)$ on $2n(\Delta + 1)$ vertices which are partitioned into a coloring $S = (S_1, \ldots, S_{\Delta+1})$ where $S_i = \{v_{i,1}, \ldots, v_{i,2n}\}$. Moreover, there is an edge between every couple of vertices $(v_{i,k}, v_{j,n+k})$ and $(v_{i,n+k}, v_{j,k})$ where $j \neq i$ and $k = 1, \ldots, n$.

Observe that $G$ is $\Delta$-regular. Assume that the algorithm $A$ produces the coloring $S = (S_1, \ldots, S_{\Delta+1})$; then, the Algorithm 1 returns $S_1$ with a size $2n$ whereas an optimal solution is given by the set $\{v_{i,j} : 1 \leq i \leq \Delta + 1, 1 \leq j \leq n\}$ with a size $(\Delta + 1)n$. \[\blacksquare\]

Remark that this instance also allows us to prove that even a more sophisticated algorithm has no better performance. In this algorithm, we first apply Algorithm 1 and then compute the best weighted independent set on every bipartite graph induced
by $S_{i_1,j_1} \cup S_{i_2,j_2}$, where the sets $S_{i_1,j_1}$ and $S_{i_2,j_2}$ describe the optimal independent sets produced by Algorithm 1.

We can slightly improve this bound to $\frac{2}{\Delta(G)}$, by using the Brook’s theorem, [3] and the constructive proof of Lovasz [10].

**Corollary 2.3** WIS is $\frac{2}{\Delta(G)}$-approximable, when $G$ is $K_{\Delta(G)+1}$-free.

One way to improve this bound is to consider $k$-partite graphs instead of bipartite graphs, where $k$ is a universal constant. Thus, the new algorithm consists in modifying the step 2.1 by finding an optimal independent set $S_{i_1,\ldots,i_k}$ on the $k$-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ for each $i_1 < \ldots < i_k$. Some algebra shows that this algorithm is a $k^{\ell}$-approximation for WIS. Unfortunately, such an algorithm does not run in polynomial time (even for $k = 3$), since computing an optimal independent set in $k$-partite graphs is **NP-hard** (see the characterization of the complexity of WIS in $(p,q)$-colorable graphs in [1]). In those circumstances, we replace an optimal solution by an approximate solution, and the whole algorithm writes now:

---

**Algorithm 2**

1. Find a coloring $S = (S_1,\ldots,S_\ell)$ by using a polynomial-time algorithm $A$;
2. For any $1 \leq i_1 < \ldots < i_k \leq \ell$ do
   2.1 Find an approximate independent set $S_{i_1,\ldots,i_k}$ of the $k$-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ using an algorithm $B$;
3. Return $S = \arg \max \{ w(S_{i_1,\ldots,i_k}) : 1 \leq i_1 < \ldots < i_k \leq \ell \}$;

---

This algorithm is polynomial as soon as the algorithm $B$ runs in polynomial time ($k$ being a constant not depending on the instance size).

**Theorem 2.4** If algorithm $B$ is a $\rho$-approximation of WIS in $k$-partite graphs, then Algorithm 2 is an $k^{\ell}\rho$-approximation for WIS in general graphs.

**Proof.** Let $B$ be an algorithm which yields to a $\rho$-approximation of WIS in $k$-partite graphs, and let $S^*$ be an optimal solution on a given graph $G$. As previously, we set $S^*_i = S^* \cap S_i$ for $i \leq \ell$ where $S = (S_1,\ldots,S_\ell)$ is the coloring provided by the algorithm $A$. 

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The inequality 1 becomes: for any $i_1, \ldots, i_k$ with $i_k > \ldots > i_1$, we have

$$w(S) \geq \rho \sum_{j=1}^{k} w(S_{i_j}^*) \quad (3)$$

In order to see that, just remark that $S_{i_1}^* \cup \ldots \cup S_{i_k}^*$ is a feasible independent set on the $k$-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ (we denoted by $G'$ this graph); since $S_{i_1}, \ldots, i_k$ is a $\rho$-approximation on $G'$, we get $w(S_{i_1}, \ldots, i_k) \geq \rho \text{opt}(G') \geq \rho(w(S_{i_1}^*) + \ldots + w(S_{i_k}^*))$.

Summing up inequalities (3) for all $i_1, \ldots, i_k$ such that $1 \leq i_1 < \ldots < i_k \leq \ell$, we obtain:

$$\frac{\ell(\ell - 1) \ldots (\ell - k + 1)}{k(k - 1) \ldots 2} w(S) \geq \rho \sum_{i=1}^{\ell} \frac{(\ell - 1) \ldots (\ell - k + 1)}{(k - 1) \ldots 2} w(S_{i}^*) \quad (4)$$

Actually, when summing the inequalities (3), the term $w(S)$ appears exactly as many times as the number of choices of $k$ elements among $\ell$, and each $w(S_{i}^*)$ appears as many times as the number of choices of $k-1$ elements among $\ell - 1$. Finally, since $\sum_{i=1}^{\ell} w(S_{i}^*) = w(S^*)$, the result follows. ■

This theorem becomes interesting when having some good bounds of the approximability of WIS in $k$-partite graphs. For instance, using Theorem 2.1, we obtain the bound $\rho = \frac{2}{3}$ for tripartite graphs; unfortunately, this does not allow to improve the bound of Theorem 2.1. Thus, we aim at strictly improving this bound of $\frac{2}{3}$ in order to improve the best performance ratio of $\frac{2}{\Delta(G)}$ or $1/[(\Delta(G) + 1)/3]$ when $\Delta(G)$ is small (i.e., $\Delta(G) = 3, 4, 6$).

Références


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