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Abstract: We consider Sinai's random walk in random environment. We prove that infinitely often (i.o.) the size of the concentration neighborhood of this random walk is almost surely bounded. As an application we get that i.o. the maximal distance between two favorite sites is almost surely bounded.

1 Introduction and results

In this paper we are interested in Sinai's walk i.e a one dimensional random walk in random environment with three conditions on the random environment: two necessaries hypothesis to get a recurrent process (see Solomon [1975]) which is not a simple random walk and an hypothesis of regularity which allows us to have a good control on the fluctuations of the random environment. The asymptotic behavior of such walk was discovered by Sinai [1982] : this walk is sub-diffusive and at an instant n it is localized in the neighborhood of a well defined point of the lattice. The correct almost sure behavior of this walk, originally studied by Deheuvels and Révész [1986], have been checked by the remarkable precise results of Hu and Shi [1998]. We denote Sinai's walk $(X_n, n \in \mathbb{N})$, let us define the local time $\mathcal{L}$, at k $(k \in \mathbb{Z})$ within the interval of time $[1, T]$ $(T \in \mathbb{N}^*)$ of $(X_n, n \in \mathbb{N})$

$$\mathcal{L}(k, T) \equiv \sum_{i=1}^{T} \mathbb{I}_{\{X_i = k\}}. \tag{1.1}$$

$\mathbb{I}$ is the indicator function (k and T can be deterministic or random variables). Let $V \subset \mathbb{Z}$, we denote

$$\mathcal{L}(V, T) \equiv \sum_{j \in V} \mathcal{L}(j, T) = \sum_{i=1}^{T} \sum_{j \in V} \mathbb{I}_{\{X_i = j\}}. \tag{1.2}$$

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Now, let us introduce the following random variables

$$\mathcal{L}^*(n) = \max_{k\in\mathbb{Z}} (\mathcal{L}(k,n))\,, \ \mathbb{F}_n = \{k \in \mathbb{Z},\ \mathcal{L}(k,n) = \mathcal{L}^*(n)\}\,, \tag{1.3}$$

$$Y_n = \inf_{x\in\mathbb{Z}} \min\{k > 0\ :\ \mathcal{L}([x-k, x+k], n) \geq n/2\}\,. \tag{1.4}$$

$\mathcal{L}^*(n)$ is the maximum of the local times (for a given instant n), $\mathbb{F}_n$ is the set of all the favourite sites and Y_n is the size of the interval where the walk spends more than a half of its time. The first almost sure results on the local time are given by Révész [1989], he notices and shows in a special case that $\mathcal{L}^*$ can be very big (see also Révész [1988]), then Shi [1998] proves the result in the general case (we recall this result here : Theorem 1.2). About $\mathbb{F}_n$, in Hu and Shi [2000] it is proven, that the maximal favorite site is almost surely transient and that it has the same almost sure behavior as the walk itself (see also Shi [2001]). Until now, the random variable Y_n has not been studied a lot for Sinai's walk. In Andreoletti [2005] it is proven, that in probability, this random variable is very small comparing to the typical fluctuations of Sinai's walk. Here we are interested in the almost sure behavior of Y_n. We prove that the "liminf" of this random variable is almost surely bounded. We will see that the result we give for Y_n implies the result of Révész about $\mathcal{L}^*$ and have interesting consequence on the favorite sites.

1.1 Definition of Sinai's walk

Let $\alpha = (\alpha_i, i \in \mathbb{Z})$ be a sequence of i.i.d. random variables taking values in $(0,1)$ defined on the probability space $(\Omega_1, \mathcal{F}_1, Q)$, this sequence will be called random environment. A random walk in random environment (denoted R.W.R.E.) $(X_n, n \in \mathbb{N})$ is a sequence of random variable taking value in $\mathbb{Z}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

• for every fixed environment α, $(X_n, n \in \mathbb{N})$ is a Markov chain with the following transition probabilities, for all $n \geq 1$ and $i \in \mathbb{Z}$

$$\begin{aligned} &\mathbb{P}^\alpha\left[X_n = i+1 | X_{n-1} = i\right] = \alpha_i, \\ &\mathbb{P}^\alpha\left[X_n = i-1 | X_{n-1} = i\right] = 1 - \alpha_i \equiv \beta_i. \end{aligned} \tag{1.5}$$

We denote $(\Omega_2, \mathcal{F}_2, \mathbb{P}^\alpha)$ the probability space associated to this Markov chain.

• $\Omega = \Omega_1 \times \Omega_2$, $\forall A_1 \in \mathcal{F}_1$ and $\forall A_2 \in \mathcal{F}_2$, $\mathbb{P}\left[A_1 \times A_2\right] = \int_{A_1} Q(dw_1) \int_{A_2} \mathbb{P}^{\alpha(w_1)}(dw_2)$.

The probability measure $\mathbb{P}^\alpha\left[.|\, X_0 = a\right]$ will be denoted $\mathbb{P}^\alpha_a\left[.\right]$, the expectation associated to $\mathbb{P}^\alpha_a$: $\mathbb{E}^\alpha_a$, and the expectation associated to Q: $\mathbb{E}_Q$.

Now we introduce the hypothesis we will use in all this work. The two following hypothesis are the necessaries hypothesis

$$\mathbb{E}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] = 0, \tag{1.6}$$

$$\mathrm{Var}_Q\left[\log \frac{1-\alpha_0}{\alpha_0}\right] \equiv \sigma^2 > 0. \tag{1.7}$$

Solomon [1975] shows that under 1.6 the process $(X_n, n \in \mathbb{N})$ is $\mathbb{P}$ almost surely recurrent and 1.7 implies that the model is not reduced to the simple random walk. In addition to 1.6 and 1.7 we will consider the following hypothesis of regularity, there exists $0 < \eta_0 < 1/2$ such that

$$\sup\{x,\ Q\left[\alpha_0 \geq x\right] = 1\} = \sup\{x,\ Q\left[\alpha_0 \leq 1 - x\right] = 1\} \geq \eta_0. \tag{1.8}$$

We call *Sinai's random walk* the random walk in random environment previously defined with the three hypothesis 1.6, 1.7 and 1.8.

1.2 Main results

Theorem 1.1. *Assume 1.6, 1.7 and 1.8 hold, there exists* $c_1 \equiv c_1(Q) > 0$ *such that*

$$\mathbb{P}\left[\liminf_n Y_n \leq c_1\right] = 1. \tag{1.9}$$

This first result prove that, almost surely, one can find a subsequence such that the size of the neighborhood where the walk spend more than a half of its time is bounded from above by a constant depending only on the distribution of the random environment.
As a corollary we get the following result originally due to Révész [1989] but which proof have been performed in the general case by Shi [1998] :

Theorem 1.2. *Assume 1.6, 1.7 and 1.8 hold, there exists* $c_2 \equiv c_2(Q) > 0$ *such that*

$$\mathbb{P}\left[\limsup_n \frac{\mathcal{L}^*(n)}{n} \geq c_2\right] = 1. \tag{1.10}$$

In Andreoletti [2005] we were also interested in the size of the interval, centered on the point of localisation defined by Sinai [1982], where the walk spends an arbitrary proportion of time n (see, for example, Theorem 3.1 in Andreoletti [2005]). It is proven that the size of this intervall is once again negligible comparing to the typical fluctuation of the walk. Here we are interested in the following random variable, let $0 \leq \beta < 1$

$$Y_{n,\beta} = \inf_{x \in \mathbb{Z}} \min\left\{k > 0\ :\ \mathcal{L}([x-k, x+k], n) \geq \beta n\right\}, \tag{1.11}$$

notice that $Y_n \equiv Y_{n,1/2}$, we get the following result

Theorem 1.3. *Assume 1.6, 1.7 and 1.8 hold, there exists* $c_3 \equiv c_3(Q) > 0$ *such that for all* $0 \leq \beta < 1$

$$\mathbb{P}\left[\liminf_n Y_{n,\beta} \leq c_3(1-\beta)^{-2}\right] = 1. \tag{1.12}$$

We notice that, when β get close to one, meaning that we look for the size of an interval where the local time is close to n, the size of this interval grows like $1/(1-\beta)^2$. Of course this result implies Theorem 1.1. We will explain in detail this $1/(1-\beta)^2$ dependence.
As an application we get the following result about the maximal distance between two favorite sites,

Theorem 1.4. *Assume 1.6, 1.7 and 1.8 hold, there exists* $c_4 \equiv c_4(Q) > 0$ *such that*

$$\mathbb{P}\left[\liminf_n \max_{(x,y) \in \mathbb{F}_n^2} |x-y| \leq c_4\right] = 1. \tag{1.13}$$

We get that infinitely often the maximal distance between two favorite sites is almost surely bounded, notice that this implies also that, almost surely, there is only a finite number of favorite sites at step n infinitely often.

1.3 About the proof of the results

We have used a similar method of Andreoletti [2005], and also an extension for Sinai's walk of Propositon 3.1 of Gantert and Shi [2002]. We will give the details of proof in such a way the reader understand the $(1-\beta)^{-2}$ dependance occurring in Theorem 1.3. However some details of proof, already present in Andreoletti [2005], have not been repeated here.

This paper is organized as follows. In section 2 we give the proof of Theorems 1.1 to 1.3, in section 3 we prove Theorem 1.4, finally in section 4 we point out remarks and open questions. In the appendix we give the needed estimate for the environment and detail the proof of some of it.

2 Proof of Theorems 1.1-1.3

We have point out that Theorem 1.3 implies the two other (1.1 and 1.2), so the main part of this section is to prove this Theorem. Notice, that Theorem 1.1 is a special case of Theorem 1.3 taking $\beta = 1/2$, at the end of the section we will explain why we also get Theorem 1.2.

To prove Theorem 1.3 we begin with the following elementary remark : By definition we have

$$\liminf_n Y_{n,\beta} \le c_3(1-\beta)^{-2} \iff \bigcap_N \bigcup_{n\ge N} \left\{ Y_{n,\beta} \le c_3(1-\beta)^{-2} \right\}, \tag{2.1}$$

denote $\tilde{c}_3(\beta) \equiv c_3(1-\beta)^{-2}$ and $\theta_\beta(x) = [x - \tilde{c}_3(\beta), x + \tilde{c}_3(\beta)]$, we have the inclusion

$$\left\{ \max_x \mathcal{L}\left(\theta_\beta(x), n\right) \ge \beta n \right\} \subseteq \left\{ Y_{n,\beta} \le \tilde{c}_3(\beta) \right\}, \tag{2.2}$$

so we get that

$$\begin{aligned} \mathbb{P}\left[\liminf_n Y_{n,\beta} \le \tilde{c}_3(\beta)\right] &\ge \mathbb{P}\left[\bigcap_N \bigcup_{n\ge N} \left\{ \max_x \mathcal{L}\left(\theta_\beta(x), n\right) \ge \beta n \right\}\right] \\ &\equiv \mathbb{P}\left[\limsup_n \frac{\max_x \mathcal{L}\left(\theta_\beta(x), n\right)}{n} \ge \beta\right]. \end{aligned} \tag{2.3}$$

To get the result it is enough to prove the two following Propositions :

Proposition 2.1. *Let $(\phi(n), n)$ be a strictly positive sequence such that $\lim_{n\to\infty} \phi(n) = +\infty$, for all $0 \le \beta < 1$ we have*

$$\mathbb{P}\left[\limsup_n \frac{\max_x \mathcal{L}\left(\theta_\beta(x), n\right)}{\phi(n)} = const \in [0, \infty]\right] = 1. \tag{2.4}$$

and

Proposition 2.2. *For all $0 \le \beta < 1$ we have*

$$\mathbb{P}\left[\frac{\max_x \mathcal{L}\left(\theta_\beta(x), n\right)}{n} \ge \beta\right] > 0. \tag{2.5}$$

Notice that Proposion 2.1 is a simple extension for Sinai's walk of Proposition 3.1 of Gantert and Shi [2002], as one can find the details of the proof in the referenced paper, we just explain why it works in our case :

2.1 Proof of Proposition 2.2

Define $f(\alpha, (X_m)) = \limsup_n \frac{\max_x \mathcal{L}(\theta_\beta(x), n)}{\phi(n)}$, following the method of Gantert and Shi [2002] it is enough to prove the two following facts : *Fact* 1 for Q-a.a. α $f(\alpha, (X_m))$ is constant for $\mathbb{P}^\alpha$-a.a. realizations of (X_n, n) and *Fact* 2 $f(\alpha) \equiv f(\alpha, (X_m))$ is a constant for Q-a.a. α. The key point for the proof of this two facts is that for all $x \in \mathbb{Z}$ ($T_x < +\infty$ $\mathbb{P}^\alpha$-a.s for Q-a.a. α) because Sinai's walk is $\mathbb{P}$-a.s recurrent. So we can apply the three steps of the proof of Gantert and Shi [2002] (pages 168-169) : the two first provide Fact 1, the third one Fact 2. Notice that here we need a result for $\max_x \mathcal{L}\left(\theta_\beta(x), n\right)$, with $\theta_\beta(x)$ a finite interval, whereas in Gantert and Shi [2002] $\max_x \mathcal{L}(x, n)$ is studied, however this difference does not change the computations. ∎

2.2 Proof of Proposition 2.12

To prove this Proposition we use a quite similar method of Andreoletti [2005], first let us recall the following decomposition of the measure $\mathbb{P}$, let $\mathcal{C}_n \in \sigma(X_i, i \leq n)$ and $G_n \subset \Omega_1$, we have :

$$\mathbb{P}[\mathcal{C}_n] \equiv \int_{\Omega_1} Q(d\omega) \int_{\mathcal{C}_n} d\mathbb{P}^{\alpha(\omega)} \tag{2.6}$$

$$\geq \int_{G_n} Q(d\omega) \int_{\mathcal{C}_n} d\mathbb{P}^{\alpha(\omega)}. \tag{2.7}$$

So assume that for all $\omega \in G_n$ and n, $\int_{\mathcal{C}_n} d\mathbb{P}^{\alpha(\omega)} \equiv d_1(\omega, n) > 0$ and assume that $Q[G_n] \equiv d_2(n) > 0$ we get that for all n

$$\mathbb{P}[\mathcal{C}_n] \geq d_2(n) \times \min_{w \in G_n} (d_1(w, n)) > 0. \tag{2.8}$$

So choosing $\mathcal{C}_n = \{\max_x \mathcal{L}(\theta_\beta(x), n) \geq \beta n\}$, we have to extract from Ω_1 a subset G_n sufficiently small to get that $\min_{w \in G_n}(d_1(w, n)) > 0$ (Proposition 2.12) but sufficiently large to have $d_2(n) > 0$ (Proposition 2.11) . The largest part of the proof is to construct such a G_n (Section 2.2.1 and Appendix B).

2.2.1 Construction of G_n (arguments for the random environment)

For completeness we begin with some basic notions originally introduced by Sinai [1982].
The random potential and the valleys
Let

$$\epsilon_i \equiv \log \frac{1 - \alpha_i}{\alpha_i}, \ i \in \mathbb{Z}, \tag{2.9}$$

define :

Definition 2.3. The random potential $(S_m, \ m \in \mathbb{Z})$ associated to the random environment α is defined in the following way: for all k and j, if $k > j$

$$S_k - S_j = \begin{cases} \sum_{j+1 \leq i \leq k} \epsilon_i, & k \neq 0, \\ -\sum_{j \leq i \leq -1} \epsilon_i, & k = 0, \end{cases}$$
$$S_0 = 0,$$

and symmetrically if $k < j$.

Remark 2.4. using Definition 2.3 we have :

$$S_k = \begin{cases} \sum_{1 \leq i \leq k} \epsilon_i, & k = 1, 2, \cdots, \\ \sum_{k \leq i \leq -1} \epsilon_i, & k = -1, -2, \cdots, \end{cases} \tag{2.10}$$

however, if we use 2.10 for the definition of (S_k, k), ϵ_0 does not appear in this definition and moreover it is not clear, when $j < 0 < k$, what the difference $S_k - S_j$ means (see figure 1).

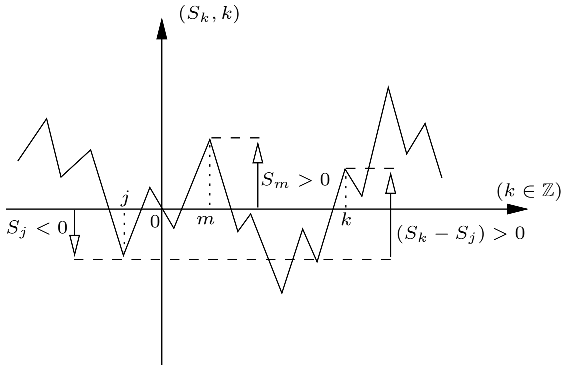

Figure 1: Trajectory of the random potential

Definition 2.5. We will say that the triplet $\{M', m, M''\}$ is a *valley* if

$$S_{M'} = \max_{M' \leq t \leq m} S_t, \tag{2.11}$$

$$S_{M''} = \max_{m \leq t \leq \tilde{M}''} S_t, \tag{2.12}$$

$$S_m = \min_{M' \leq t \leq M''} S_t \ . \tag{2.13}$$

If m is not unique we choose the one with the smallest absolute value.

Definition 2.6. We will call *depth of the valley* $\{M', m, M''\}$ and we will denote it $d([M', M''])$ the quantity

$$\min(S_{M'} - S_m, S_{M''} - S_m). \tag{2.14}$$

Now we define the operation of *refinement*

Definition 2.7. Let $\{M', m, M''\}$ be a valley and let M_1 and m_1 be such that $m \leq M_1 < m_1 \leq M''$ and

$$S_{M_1} - S_{m_1} = \max_{m \leq t' \leq t'' \leq M''} (S_{t'} - S_{t''}). \tag{2.15}$$

We say that the couple (m_1, M_1) is obtained by a *right refinement* of $\{M', m, M''\}$. If the couple (m_1, M_1) is not unique, we will take the one such that m_1 and M_1 have the smallest absolute value. In a similar way we define the *left refinement* operation.

We denote $\log_2 = \log\log$, in all this section we will suppose that n is large enough such that $\log_2 n$ is positive.

Definition 2.8. Let $n > 3$ and $\Gamma_n \equiv \log n + 12 \log_2 n$, we say that a valley $\{M', m, M''\}$ contains 0 and is of depth larger than Γ_n if and only if

1. $0 \in [M', M'']$,
2. $d([M', M'']) \geq \Gamma_n$,
3. if $m < 0$, $S_{M''} - \max_{m \leq t \leq 0} (S_t) \geq 12 \log_2 n$,
   if $m > 0$, $S_{M'} - \max_{0 \leq t \leq m} (S_t) \geq 12 \log_2 n$.

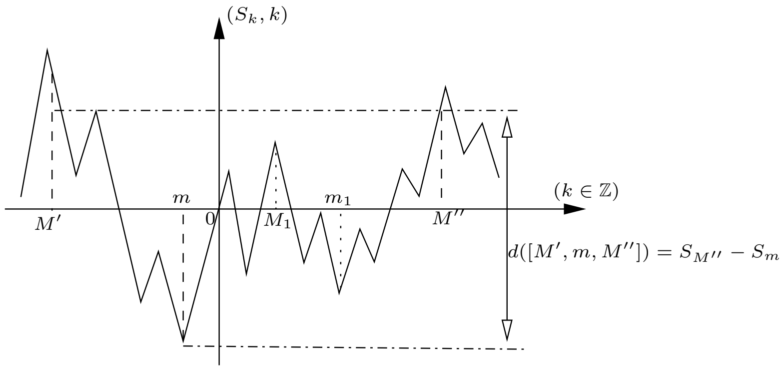

Figure 2: Depth of a valley and refinement operation

The basic valley $\{M_n{}', m_n, M_n\}$

We recall the notion of *basic valley* introduced by Sinai and denoted here $\{M_n{}', m_n, M_n\}$. The definition we give is inspired by the work of Kesten [1986]. First let $\{M', m_n, M''\}$ be the smallest *valley that contains* 0 *and of depth larger than* Γ_n. Here smallest means that if we construct, with the operation of refinement, other valleys in $\{M', m_n, M''\}$ such valleys will not satisfy one of the properties of Definition 2.8. $M_n{}'$ and M_n are defined from m_n in the following way: if $m_n > 0$

$$M_n{}' = \sup\left\{l \in \mathbb{Z}_-,\ l < m_n,\ S_l - S_{m_n} \geq \Gamma_n,\ S_l - \max_{0 \leq k \leq m_n} S_k \geq 12 \log_2 n\right\}, \tag{2.16}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+,\ l > m_n,\ S_l - S_{m_n} \geq \Gamma_n\right\}. \tag{2.17}$$

if $m_n < 0$

$$M_n{}' = \sup\left\{l \in \mathbb{Z}_-,\ l < m_n,\ S_l - S_{m_n} \geq \Gamma_n\right\}, \tag{2.18}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+,\ l > m_n,\ S_l - S_{m_n} \geq \Gamma_n,\ S_l - \max_{m_n \leq k \leq 0} S_k \geq 12 \log_2 n\right\}. \tag{2.19}$$

if $m_n = 0$

$$M_n{}' = \sup\left\{l \in \mathbb{Z}_-,\ l < 0,\ S_l - S_{m_n} \geq \Gamma_n\right\}, \tag{2.20}$$

$$M_n = \inf\left\{l \in \mathbb{Z}_+,\ l > 0,\ S_l - S_{m_n} \geq \Gamma_n\right\}. \tag{2.21}$$

$\{M_n{}', m_n, M_n\}$ exists with a Q probability as close to one as we need. In fact it is not difficult to prove the following lemma

Lemma 2.9. *Assume 1.6, 1.7 and 1.8 hold, for all* n *we have*

$$Q\left[\{M_n{}', m_n, M_n\} \neq \varnothing\right] = 1 - o(1). \tag{2.22}$$

we denote $o(1)$ *a positive function of* n *such that* $\lim_{n \to \infty} o(1) = 0$.

Proof.
One can find the proof of this Lemma in Section 5.2 of Andreoletti [2005]. ■
Let $x \in \mathbb{Z}$, define

$$T_x = \begin{cases} \inf\{k \in \mathbb{N}^*,\ X_k = x\} \\ +\infty, \text{ if such } k \text{ does not exist.} \end{cases} \tag{2.23}$$

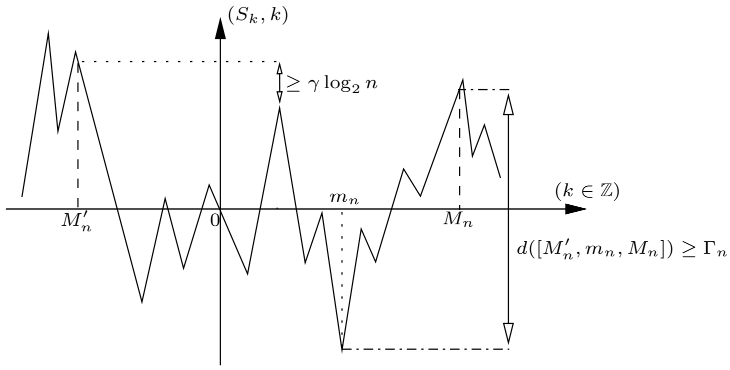

Figure 3: Basic valley, case $m_n > 0$

Definition 2.10. Let $c_0 > 0$, $c_3 > 0$, $0 \leq \beta < 1$ and $\omega \in \Omega_1$, we will say that $\alpha \equiv \alpha(\omega)$ is a *good environment* if there exists n_0 such that for all $n \geq n_0$ the sequence $(\alpha_i,\ i \in \mathbb{Z}) = (\alpha_i(\omega),\ i \in \mathbb{Z})$ satisfies the properties 2.24 to 2.26

(2.24) • $\{{M_n}', m_n, M_n\} \neq \varnothing$,

(2.25) • ${M_n}' \geq (\sigma^{-1} \log n)^2,\ M_n \leq (\sigma^{-1} \log n)^2$,

(2.26) • $\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}(\tilde{\Theta}(n, \beta), T_{m_n})\right] \leq \dfrac{2c_0}{\sqrt{\tilde{c}_3(\beta)}}$.

where $\tilde{\Theta}(n, \beta) = [M'_n, M'_n + 1, \cdots, m_n - \tilde{c}_3(\beta)] \cup [m_n + \tilde{c}_3(\beta), m_n + \tilde{c}_3(\beta) + 1, \cdots, M_n]$, recall that $\tilde{c}_3(\beta) = c_3(1 - \beta)^{-2}$.

Define the *set of good environments*

$$G_n \equiv G_n(c_0, c_3, \beta) = \{\omega \in \Omega_1,\ \alpha(\omega) \text{ is a } \textit{good environment}\}. \tag{2.27}$$

G_n depends on c_0 and n, however we do not make explicit its c_0 and β dependence.

Proposition 2.11. *There exists a numerical constant $d_2 > 0$ such that if 1.6, 1.7 and 1.8 hold, there exists $c_0 \equiv c_0(Q) > 0$ and n_0 such that for all $c_3 > 0$, $0 \leq \beta < 1$ and $n > n_0$*

$$Q\,[G_n] \geq 1/2. \tag{2.28}$$

Proof.
We have to prove that the three properties 2.24-2.26 are true with a Q probability strictly positive. In Andreoletti [2005] we prove that the two first properties are true with a probability close to one, so we only have to prove that the third one is true with a probability larger than 1/2, this is done in the Appendix B. ■

2.2.2 Argument for the walk (environment fixed $\alpha \in G_n$)

In this section we assume that n is sufficiently large such that Proposition 2.11 is true and we assume also that the random environment is fixed and belongs to G_n (denoted $\alpha \in G_n$).

Proposition 2.12. *For all $0 \leq \beta < 1$, n large enough and $\alpha \in G_n$ we have*

$$\mathbb{P}^{\alpha}_0\left[\max_x \mathcal{L}\left(\theta_\beta(x), n\right) \geq \beta n\right] > 1/2, \tag{2.29}$$

recall that $\theta_\beta(x) = [x - \tilde{c}_3(\beta), x + \tilde{c}_3(\beta)]$, $c_3 \equiv c_3(Q) > 0$.

Proof.
To get this result, it is enough to prove that,

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\theta_\beta(m_n),n\right)\geq \beta n\right] > 1/2, \tag{2.30}$$

we will prove the following equivalent fact

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\Theta(n,\beta),n\right)\geq (1-\beta) n\right] < 1/2, \tag{2.31}$$

where $\Theta(n,\beta)$ is the complementary of $\theta_\beta(m_n)$ in $\mathbb{Z}$.

First we recall the two following elementary results

Lemma 2.13. *For all n and $\alpha \in G_n$ we have*

$$\mathbb{P}_0^\alpha\left[\bigcup_{m=0}^{n}\left\{X_m \notin \left[M_n', M_n\right]\right\}\right] = o(1). \tag{2.32}$$

$$\mathbb{P}_0^\alpha\left[T_{m_n} > \frac{n}{(\log n)^4}\right] = o(1). \tag{2.33}$$

Recall that $\lim_{n\to\infty} o(1) = 0$.

Proof.
This is a basic result for Sinai's walk, it makes use Properties 2.24 and 2.25. One can find the details of this proof in Andreoletti [2005] : Proposition 4.7 and Lemma 4.8. ■

First we use 2.32 to reduce the set $\Theta(n,\beta)$ to $\tilde{\Theta}(n,\beta)$ defined just after 2.26, we get

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\Theta(n,\beta),n\right)\geq (1-\beta) n\right] \leq \mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}(n,\beta),n\right)\geq (1-\beta) n\right] + \mathrm{o}(1). \tag{2.34}$$

Now using 2.33 we get

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}(n,\beta),n\right)\geq (1-\beta) n\right] \quad\leq\quad \mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}(n,\beta),n\right)\geq (1-\beta) n,\ T_{m_n}\leq \frac{n}{(\log n)^4}\right] + \mathrm{o}(1). \tag{2.35}$$

Let us denote $N_0 \equiv \left[n(\log n)^{-4}\right]+1$ and $1-\beta_n \equiv 1-\beta-N_0/n$. By the Markov property and the homogeneity of the Markov chain we obtain

$$\mathbb{P}_0^\alpha\left[\mathcal{L}\left(\tilde{\Theta}(n,\beta),n\right)\geq (1-\beta) n,\ T_{m_n}\leq \frac{n}{(\log n)^4}\right] \quad\leq\quad \mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{n}\mathbb{I}_{\left\{X_k\in\tilde{\Theta}(n,\beta)\right\}}\geq (1-\beta_n)n\right]. \tag{2.36}$$

Let $j\geq 2$, define the following return times

$$T_{m_n,j} \equiv \left\{\begin{array}{l}\inf\{k > T_{m_n,j-1},\ X_k = m_n\},\\ +\infty, \text{ if such } k \text{ does not exist.}\end{array}\right.$$
$$T_{m_n,1}\equiv T_{m_n} \text{ (see 2.23)}.$$

Since by definition $T_{m_n,n} > n$, $\left\{\sum_{k=1}^{n}\mathbb{I}_{\left\{X_k\in\tilde{\Theta}(n,\beta)\right\}}\geq(1-\beta_n)n\right\} \subset \left\{\sum_{k=1}^{T_{m_n,n}}\mathbb{I}_{\left\{X_k\in\tilde{\Theta}(n,\beta)\right\}}\geq(1-\beta_n)n\right\}$, then using the definition of the local time and the Markov inequality we get

$$\mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{n}\mathbb{I}_{\left\{X_k\in\tilde{\Theta}(n,\beta)\right\}}\geq (1-\beta_n)n\right] \quad\leq\quad \mathbb{P}_{m_n}^\alpha\left[\sum_{k=1}^{T_{m_n},n}\mathbb{I}_{\left\{X_k\in\tilde{\Theta}(n,\beta)\right\}}\geq (1-\beta_n)n\right] \tag{2.37}$$

$$\leq\quad \mathbb{E}_{m_n}^\alpha\left[\mathcal{L}\left(\tilde{\Theta}(n,\beta),T_{m_n}\right)\right](1-\beta_n)^{-1}, \tag{2.38}$$

and we have used the fact that, by the strong Markov property, the random variables $\mathcal{L}\left(s, T_{m_n,i+1} - T_{m_n,i}\right)$ $(0 \leq i \leq n-1)$ are *i.d.*. Using the property 2.26, there exists c_0 such that

$$\mathbb{E}^{\alpha}_{m_n}\left[\mathcal{L}\left(\tilde{\Theta}(n,\beta), T_{m_n}\right)\right] \leq \frac{2c_0(1-\beta)}{(c_3)^{1/2}}. \tag{2.39}$$

Collecting what we did above, we finally get for n sufficiently large

$$\mathbb{P}^{\alpha}_0\left[\mathcal{L}\left(\Theta(n,\beta), n\right) \geq (1-\beta)n\right] \leq \frac{4c_0}{(c_3)^{1/2}}. \tag{2.40}$$

we get 2.31 choosing $c_3 = 64(c_0)^2$. ■

This ends the proof of Theorem 1.3, to get Theorem 1.2, we remark that

$$\begin{aligned}\{Y_n \leq c_1\} &\equiv \left\{\inf_{x\in\mathbb{Z}} \min\left\{k > 0 \ : \ \mathcal{L}([x-k, x+k], n) > n/2\right\} \leq c_1\right\} \\ &\subset \left\{\mathcal{L}^*(n) > \frac{n}{2c_1}\right\},\end{aligned}$$

we conclude with Theorem 1.1.

Remark 2.14. We have seen that Theorem 1.3 implies Theorem 1.2, moreover thanks to the result of Gantert and Shi [2002] extended to Sinai's walk we know that $\mathbb{P}\left[\limsup_n \frac{\mathcal{L}^*(n)}{n} = \text{const} \in]0,\infty]\right] = 1$, therefore there exists $c_2 > 0$ and $c_3 > 0$ such that for all $0 \leq \beta < 1$

$$\mathbb{P}\left[\limsup_n \left\{Y_{n,\beta} \leq c_3(1-\beta)^{-2},\ \mathcal{L}^*(n)/n > c_2\right\}\right] = 1. \tag{2.41}$$

3 Proof of Theorem 1.4

To prove this Theorem we use Remark 2.14. We begin with the following nice facts, define

$$\mathcal{A}_n = \max_{(x,y)\in\mathbb{F}_n^2} |x-y| \leq c_4, \tag{3.1}$$

$$\mathcal{B}_n = \max_x \mathcal{L}\left(\left[x - c_4/2, x + c_4/2\right], n\right) > n - \mathcal{L}^*(n), \tag{3.2}$$

recalling that $\mathcal{L}^*(n) = \max_x \mathcal{L}(x,n)$, c_4 is for the moment a free parameter that will be chosen latter.
Fact 1 We have $\mathcal{B}_n \subseteq \mathcal{A}_n$, indeed it is easy to check that

$$\bigcap_{x\in\mathbb{F}_n}\left\{\sum_{k=x-c_4/2}^{x+c_4/2} \mathcal{L}(k,n) > n - \mathcal{L}^*(n)\right\} \subseteq \mathcal{A}_n \tag{3.3}$$

moreover $\mathbb{F}_n \subset \mathbb{Z}$, so it is clear that

$$\bigcap_{x\in\mathbb{F}_n}\left\{\sum_{k=x-c_4/2}^{x+c_4/2} \mathcal{L}(k,n) > n - \mathcal{L}^*(n)\right\} \supseteq \bigcap_{x\in\mathbb{Z}}\left\{\sum_{k=x-c_4/2}^{x+c_4/2} \mathcal{L}(k,n) > n - \mathcal{L}^*(n)\right\} \equiv \mathcal{B}_n. \tag{3.4}$$

Fact 2 Using 2.41 with $\beta = 1 - c_2$ we have

$$\mathbb{P}\left[\limsup_n \left\{Y_{n,(1-c_2)} \leq c_3(c_2)^{-2} \text{ and } \mathcal{L}^*(n)/n \geq c_2\right\}\right] = 1 \tag{3.5}$$

Now, using the Definition of the "lim inf" and Fact 1 we get

$$\mathbb{P}\left[\liminf_n \max_{(x,y)\in\mathbb{F}_n^2} |x-y| \le c_4\right] \ge \mathbb{P}\left[\limsup_n \mathcal{B}_n\right]. \tag{3.6}$$

It is clear that

$$\mathbb{P}\left[\limsup_n \mathcal{B}_n\right] \quad \ge \quad \mathbb{P}\left[\limsup_n \left\{B_n \ , \frac{\mathcal{L}^*(n)}{n} \ge c_2\right\}\right], \tag{3.7}$$

moreover

$$\left\{B_n \ , \frac{\mathcal{L}^*(n)}{n} \ge c_2\right\} \supseteq \left\{\max_x \mathcal{L}\left([x-c_4/2, x+c_4/2], n\right) > n(1-c_2), \ \frac{\mathcal{L}^*(n)}{n} \ge c_2\right\}. \tag{3.8}$$

Therefore choosing $c_4 = c_3/(c_2^2)$, we finally get that :

$$\mathbb{P}\left[\limsup_n \mathcal{B}_n\right] \quad \ge \quad \mathbb{P}\left[\limsup_n \left\{\max_x \mathcal{L}\left(\left[x-c_3/c_2^2, x+c_3/c_2^2\right], n\right) > n(1-c_2), \ \frac{\mathcal{L}^*(n)}{n} \ge c_2\right\}\right] \tag{3.9}$$

$$\equiv \quad \mathbb{P}\left[\limsup_n \left\{Y_{n,(1-c_2)} \le c_3(c_2)^{-2}, \ \frac{\mathcal{L}^*(n)}{n} \ge c_2\right\}\right] \tag{3.10}$$

$$= \quad 1 \tag{3.11}$$

where the last equality comes from Fact 2. ■

4 Conclusion remarks

We have seen that using the method of Andreoletti [2005] and the Proposition 3.1 of Gantert and Shi [2002] we get easily annealed result for the concentration variable Y_n. We also point out that the result on the concentration variable implies both results on the maximum of the local time and on the favorite sites.

Here we only get the "lim inf" asymptotic of Y_n, what can we say about the "lim sup" ? We notice that if we have something like $\liminf \mathcal{L}^*(n)\phi(n)/n = cte > 0$ $\mathbb{P}.a.s$ then $\limsup Y_n/\phi(n) = cte \in]0+\infty]$, $\mathbb{P}.a.s$ but is $\phi(n)$ the good asymptotic for the "lim sup" of Y_n ? Notice that forthcoming work of Gantert shows that $\liminf \mathcal{L}^*(n)\log\log\log n/n = cte > 0$ $\mathbb{P}.a.s$ and forthcoming work of Z. Shi and O. Zindy implies that $\limsup Y_n/\log\log\log n = cte \in]0+\infty]$, $\mathbb{P}.a.s.$

Now, forgetting the hypothesis 1.6 and using the ones of Gantert and Shi [2002] (originally introduced by Kesten et al. [1975])

$$-\infty < \mathbb{E}_Q\left[\log\frac{1-\alpha_0}{\alpha_0}\right] < 0, \tag{4.1}$$

and that there is $0 < \kappa < 1$ such that

$$0 < \mathbb{E}_Q\left[\left(\frac{1-\alpha_0}{\alpha_0}\right)^\kappa\right] = 1. \tag{4.2}$$

Thanks to their work, it appears clearly that for small β one can find $c_1 \equiv c_1(\beta) > 0$ such that :

$$\liminf Y_{n,\beta} \le c_1, \ \mathbb{P}.a.s \tag{4.3}$$

a question that is maybe interesting is to understand how this β depends on κ, for example, can we find κ such that 4.3 is true for $\beta = 1/2$? We could say that Sinai's walk is concentrated uniformly for $0 < \beta < 1$ whereas Kesten et al. walk is uniformly concentrated for $0 < \beta < \beta_c \equiv \beta_c(\kappa)$. What can we say about β_c ?

A Basic results for birth and death processes

For completeness we recall some results of Chung [1967] and Révész [1989] on inhomogeneous discrete time birth and death processes.
Let x, a and b in $\mathbb{Z}$, assume $a < x < b$, the two following lemmata can be found in Chung [1967] (pages 73-76), the proof follows from the method of difference equations.

Lemma A.1. *Recalling 2.23, for all α we have*

$$\mathbb{P}_x^\alpha \left[T_a > T_b\right] = \frac{\sum_{i=a+1}^{x-1} \exp\left(S_i - S_a\right) + 1}{\sum_{i=a+1}^{b-1} \exp\left(S_i - S_a\right) + 1}, \tag{A.1}$$

$$\mathbb{P}_x^\alpha \left[T_a < T_b\right] = \frac{\sum_{i=x+1}^{b-1} \exp\left(S_i - S_b\right) + 1}{\sum_{i=a+1}^{b-1} \exp\left(S_i - S_b\right) + 1}. \tag{A.2}$$

Now we give some explicit expressions for the local times that can be found in Révész [1989] (page 279)

Lemma A.2. *For all α and $i \in \mathbb{Z}$, we have, if $x > i$*

$$\mathbb{E}_i^\alpha \left[\mathcal{L}(x, T_i)\right] = \frac{\alpha_i \mathbb{P}_{i+1}^\alpha \left[T_x < T_i\right]}{\beta_x \mathbb{P}_{x-1}^\alpha \left[T_x > T_i\right]}, \tag{A.3}$$

if $x < i$

$$\mathbb{E}_i^\alpha \left[\mathcal{L}(x, T_i)\right] = \frac{\beta_i \mathbb{P}_{i-1}^\alpha \left[T_x < T_i\right]}{\alpha_x \mathbb{P}_{x+1}^\alpha \left[T_x > T_i\right]}. \tag{A.4}$$

B Proof of the good properties for the environment

Here we give the main ideas for the proof of the Proposition 2.11, we begin with some

B.1 Elementary results for sum of i.i.d. random variables

We will always work on the right hand side of the origin, that means with $(S_m, m \in \mathbb{N})$, by symmetry we obtain the same result for $m \in \mathbb{Z}_-$.

We introduce the following stopping times, for $a > 0$,

$$V_a^+ \equiv V_a^+(S_j, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*,\ S_m \geq a\}, \\ +\infty, \text{ if such a } m \text{ does not exist.} \end{cases} \tag{B.1}$$

$$V_a^- \equiv V_a^-(S_j, j \in \mathbb{N}) = \begin{cases} \inf\{m \in \mathbb{N}^*,\ S_m \leq -a\}, \\ +\infty, \text{ if such a } m \text{ does not exist.} \end{cases} \tag{B.2}$$

The following lemma is an immediate consequence of the Wald equality (see Neveu [1972])

Lemma B.1. *Assume 1.6, 1.7 and 1.8, let $a > 0$, $d > 0$ we have*

$$Q\left[V_a^- < V_d^+\right] \leq \frac{d + I_{\eta_0}}{d + a + I_{\eta_0}}, \tag{B.3}$$

$$Q\left[V_a^- > V_d^+\right] \leq \frac{a + I_{\eta_0}}{d + a + I_{\eta_0}}, \tag{B.4}$$

with $I_{\eta_0} \equiv \log((1 - \eta_0)(\eta_0)^{-1})$.

The following lemma is a basic fact for sums of i.i.d. random variables

Lemma B.2. *Assume 1.6, 1.7 and 1.8 hold, there exists $b \equiv b(Q) > 0$ such that for all $r > 0$*

$$Q\left[V_0^- > r\right] \leq \frac{b}{\sqrt{r}}. \tag{B.5}$$

B.2 Proof of Proposition 2.11

It is in this part where the $(1-\beta)^{-2}$ dependance occuring in Theorem 1.3 will become clear. The main difficulty is to get an upper bound for the expectation $\mathbb{E}_Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}(n,\beta), T_{m_n})\right]\right]$.

B.2.1 Preliminaries

By linearity of the expectation we have :

$$\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}(n,\beta), T_{m_n})\right] \equiv \sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(j, T_{m_n})\right] + \sum_{j=M_n'}^{m_n-\tilde{c}_3(\beta)} \mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(j, T_{m_n})\right] + 1, \tag{B.6}$$

recall that $\tilde{c}_3(\beta) = c_3(1-\beta)^{-2}$ with $c_3 > 0$ and $0 \leq \beta < 1$. Now using Lemma A.1 and hypothesis 1.8 we easily get the following lemma

Lemma B.3. *Assume 1.8, for all ${M_n}' \leq k \leq M_n$, $k \neq m_n$*

$$\frac{\eta_0}{1-\eta_0}\frac{1}{e^{S_k - S_{m_n}}} \leq \mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(k, T_{m_n})\right] \leq \frac{1}{\eta_0}\frac{1}{e^{S_k - S_{m_n}}}, \tag{B.7}$$

with a Q probability equal to one.

The following lemma is easy to prove :

Lemma B.4. *For all $n > 3$, with a Q probability equal to one we have*

$$\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \frac{1}{e^{S_j - S_{m_n}}} \leq \sum_{i=1}^{N_n+1} \frac{1}{e^{a(i-1)}} \sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}, \tag{B.8}$$

$$\sum_{j=M_n'}^{m_n-\tilde{c}_3(\beta)} \frac{1}{e^{S_j - S_{m_n}}} \leq \sum_{i=1}^{N_n+1} \frac{1}{e^{a(i-1)}} \sum_{j=M_n'}^{m_n-\tilde{c}_3(\beta)} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}, \tag{B.9}$$

where $a = \frac{I_{\eta_0}}{4}$, $N_n = [(\Gamma_n + I_{\eta_0})/a]$, recall that $\mathbb{I}$ is the indicator function.

Using B.6, Lemma B.3 and B.4, we have for all $n > 3$

$$\begin{aligned}\mathbb{E}_Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}(n,\beta), T_{m_n})\right]\right] \leq 1 \quad &+ \quad \frac{1}{\eta_0}\sum_{i=1}^{N_n+1}\frac{1}{e^{a(i-1)}}\mathbb{E}_Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right] \\ &+ \quad \frac{1}{\eta_0}\sum_{i=1}^{N_n+1}\frac{1}{e^{a(i-1)}}\mathbb{E}_Q\left[\sum_{j=M_n'}^{m_n-\tilde{c}_3(\beta)} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right].\end{aligned} \tag{B.10}$$

The next step for the proof is to show that the two expectations $\mathbb{E}_Q[...]$ on the right hand side of B.10 are bounded by a constant depending only on the distribution Q times a polynomial in i times $1/\sqrt{\tilde{c}_3(\beta)}$:

Lemma B.5. *There exits a constant $c \equiv c(Q)$ such that for all n large enough :*

$$\mathbb{E}_Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[}\right] \le \frac{c\times i^3}{\sqrt{\tilde{c}_3(\beta)}}, \tag{B.11}$$

$$\mathbb{E}_Q\left[\sum_{j=M'_n}^{m_n-\tilde{c}_3(\beta)} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[}\right] \le \frac{c\times i^3}{\sqrt{\tilde{c}_3(\beta)}}. \tag{B.12}$$

B.2.2 Proof of Lemma B.5

Remark B.6. We give some details of the proof of Lemma B.5 mainly because it helps to understand the appearance of the $(1-\beta)^{-2}$ in Theorem 1.3, moreover it is based on a very nice cancellation that occurs between two $\Gamma_n \equiv \log n + \log_2 n$, see formulas B.18 and B.20. Similar cancellation is already present in Kesten [1986].

Let us define the following stopping times, let $i > 1$:

$$\begin{aligned} &u_0 = 0, \\ &u_1 \equiv V_0^- = \inf\{m > 0,\ S_m < 0\}, \\ &u_i = \inf\{m > u_{i-1},\ S_m < S_{u_{i-1}}\}. \end{aligned}$$

The following lemma give a way to characterize the point m_n, it is inspired by the work of Kesten [1986] and is just inspection

Lemma B.7. *Let $n > 3$ and $\gamma > 0$, recall $\Gamma_n = \log n + \gamma \log_2 n$, assume $m_n > 0$, for all $l \in \mathbb{N}^*$ we have*

$$m_n = u_l \quad \Rightarrow \quad \begin{cases} \bigcap_{i=0}^{l-1}\left\{\max_{u_i\le j\le u_{i+1}}(S_i) - S_{u_i} < \Gamma_n\right\} \text{ and} \\ \max_{u_l\le j\le u_{l+1}}(S_i) - S_{u_i} \ge \Gamma_n \text{ and} \\ M_n = V^+_{\Gamma_n,l} \end{cases} \tag{B.13}$$

where

$$V^+_{z,l} \equiv V^+_{z,l}(S_j, j\ge 1) = \inf\left(m > u_l,\ S_m - S_{u_l} \ge z\right). \tag{B.14}$$

A similar characterization of m_n if $m_n \le 0$ can be done (the case $m_n = 0$ is trivial). We will only prove B.11, we get B.12 symmetrically moreover we assume that $m_n > 0$, computations are similar for the case $m_n \le 0$. Thinking on the basic definition of the expectation, we need an upper bound for the probability :

$$Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[} = k.\right]$$

First we make a partition over the values of m_n and then we use Lemma B.7, we get :

$$\begin{aligned} Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[} = k.\right] &\equiv \sum_{l\ge 0} Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n}\in[a(i-1),ai[} = k,\ m_n = u_l\right] \\ &\le \sum_{l\ge 0} Q\left[\mathcal{A}^+_{\Gamma_n,l}, \max_{u_l\le j\le u_{l+1}}(S_j) - S_{u_l} \ge \Gamma_n, \mathcal{A}^-_{\Gamma_n,l}\right] \end{aligned} \tag{B.15}$$

where

$$\mathcal{A}^+_{\Gamma_n,l} = \sum_{s=u_l+\tilde{c}_3(\beta)}^{V^+_{\Gamma_n,l}} \mathbb{I}_{\{S_j-S_{u_l}\in[a(i-1),ai[\}} = k,$$
$$\mathcal{A}^-_{\Gamma_n,l} = \bigcap_{r=0}^{l-1}\left\{\max_{u_r\leq j\leq u_{r+1}}(S_r) - S_{u_r} < \Gamma_n\right\},\ \mathcal{A}^-_0 = \Omega_1.$$

for all $l \geq 0$. By the strong Markov property we have :

$$\text{(B.16)}\quad Q\left[\mathcal{A}^+_{\Gamma_n,l},\ \max_{u_l\leq j\leq u_{l+1}}(S_j) - S_{u_l} \geq \Gamma_n,\ \mathcal{A}^-_{\Gamma_n,l}\right] \leq Q\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right] Q\left[\mathcal{A}^-_{\Gamma_n,l}\right].$$

The strong Markov property gives also that the sequence $(\max_{u_r\leq j\leq u_{r+1}}(S_r) - S_{u_r} < \Gamma_n, r \geq 1)$ is i.i.d., therefore :

$$\text{(B.17)}\qquad Q\left[\mathcal{A}^-_{\Gamma_n,l}\right] \leq \left(Q\left[V^-_0 < V^+_{\Gamma_n}\right]\right)^{l-1}.$$

We notice that $Q\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right]$ does not depend on l, therefore, using B.15, B.16 and B.17 we get :

$$\text{(B.18)}Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j-S_{m_n}\in[a(i-1),ai[} = k.\right] \leq (1 + (Q\left[V^-_0 \geq V^+_{\Gamma_n}\right])^{-1})Q\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right].$$

Using the Markov property we obtain that

$$\text{(B.19)}\qquad Q\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right] \leq Q\left[V^-_0 > \tilde{c}_3(\beta)\right] \max_{0\leq x\leq \tilde{c}_3(\beta)/I_{\eta_0}} \left\{Q_x\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right]\right\}.$$

I_{η_0} is given just after B.4. To get an upper bound for $Q_x\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right]$, we introduce the following sequence of stopping times, let $k > 0$:

$$H_{ia,0} = 0,$$
$$H_{ia,k} = \inf\{m > H_{ia,k-1},\ S_m \in [(i-1)a, ia[\}.$$

Making a partition over the values of $H_{ia,k}$ and using the Markov property we get:

$$\begin{aligned}
& Q_x\left[\mathcal{A}^+_{\Gamma_n,0},\ V^-_0 > V^+_{\Gamma_n}\right] \\
\leq\ & \sum_{w\geq 0}\int_{(i-1)a}^{ia} Q_x\left[H_{ia,k} = w, S_w \in dy, \bigcap_{s=0}^{w}\{S_s > 0\}, \bigcap_{s=w+1}^{\inf\{l>w,S_l\geq\Gamma_n-x\}}\{S_s > 0\}\right] \\
\leq\ & Q_x\left[H_{ia,k} < V^-_0\right] \max_{(i-1)a\leq y\leq ia}\left\{Q_y\left[V^+_{\Gamma_n-y} < V^-_y\right]\right\} \\
\text{(B.20)}\qquad \equiv\ & Q_x\left[H_{ia,k} < V^-_0\right] Q_{ia}\left[V^+_{\Gamma_n-ia} < V^-_{ia}\right]
\end{aligned}$$

To finish we need an upper bound for $Q_x\left[H_{ia,k} < V^-_0\right]$, we do not want to give details of the computations for this because it is not difficult, however the reader can find these details in Andreoletti

[2003] pages 142-145. We have for all $i > 1$:

$$Q_x\left[H_{ia,k} < V_0^-\right] \leq Q_x\left[V_0^- > V_{(i-1)a}^+\right]\left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{2}\right] Q_{(i-1)a-\frac{I_{\eta_0}}{4}}\left[V_{(i-1)a}^+ \geq V_0^-\right]\right)^{k-1}$$

$$\text{(B.21)} \qquad \leq \left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{2}\right] Q_{(i-1)a-\frac{I_{\eta_0}}{4}}\left[V_{(i-1)a}^+ \geq V_0^-\right]\right)^{k-1},$$

and in the same way

$$\text{(B.22)} \qquad Q_x\left[H_{a,k} < V_0^-\right] \leq \left(1 - Q\left[\epsilon_0 < -\frac{I_{\eta_0}}{4}\right]\right)^{k-1} .$$

So using B.18-B.22, Lemmata B.1 and B.2 one can find a constant $c \equiv c(Q)$ that depends only on the distribution Q such that for all $i \geq 0$:

$$\mathbb{E}_Q\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[}\right] \equiv \sum_{k=1}^{+\infty} kQ\left[\sum_{j=m_n+\tilde{c}_3(\beta)}^{M_n} \mathbb{I}_{S_j - S_{m_n} \in [a(i-1), ai[} = k\right] \leq \frac{c \times i^3}{\sqrt{\tilde{c}_3(\beta)}},$$

which provide B.11. ■

Using both B.10 and Lemma B.5 we get that there exists $c_0 \equiv c_0(Q)$ such that

$$\text{(B.23)} \qquad \mathbb{E}_Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}(n,\beta), T_{m_n})\right]\right] \leq \frac{c_0}{\sqrt{\tilde{c}_3(\beta)}}.$$

Now, using the elementary Markov inequality, we get :

$$\begin{aligned} & Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}(n,\beta), T_{m_n})\right] \leq 2\frac{c_0}{\sqrt{\tilde{c}_3(\beta)}}\right] \\ \equiv\ & 1 - Q\left[\mathbb{E}_{m_n}^{\alpha}\left[\mathcal{L}(\tilde{\Theta}(n,\beta), T_{m_n})\right] > 2\frac{c_0}{\sqrt{\tilde{c}_3(\beta)}}\right] \\ \geq\ & 1/2. \end{aligned}$$

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