Sublattices of lattices of order-convex sets, II. Posets of finite length

Marina V. Semenova, Friedrich Wehrung

► To cite this version:

Marina V. Semenova, Friedrich Wehrung. Sublattices of lattices of order-convex sets, II. Posets of finite length. *International Journal of Algebra and Computation*, 2003, 13 (5), pp.543–564. 10.1142/S0218196703001547 . hal-00003979

HAL Id: hal-00003979

<https://hal.science/hal-00003979>

Submitted on 21 Jan 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SUBLATTICES OF LATTICES OF ORDER-CONVEX SETS, II. POSETS OF FINITE LENGTH

MARINA SEMENOVA AND FRIEDRICH WEHRUNG

ABSTRACT. For a positive integer n , we denote by **SUB** (resp., **SUB** $_n$) the class of all lattices that can be embedded into the lattice **Co**(P) of all order-convex subsets of a partially ordered set P (resp., P of length at most n). We prove the following results:

- (1) **SUB** $_n$ is a finitely based variety, for any $n \geq 1$.
- (2) **SUB** $_2$ is locally finite.
- (3) A finite atomistic lattice L without D -cycles belongs to **SUB** iff it belongs to **SUB** $_2$; this result does not extend to the nonatomistic case.
- (4) **SUB** $_n$ is not locally finite for $n \geq 3$.

1. INTRODUCTION

For a partially ordered set (from now on *poset*) $(P, \trianglelefteq)$, a subset X of P is *order-convex*, if $x \trianglelefteq z \trianglelefteq y$ and $\{x, y\} \subseteq X$ implies that $z \in X$, for all $x, y, z \in P$. The set **Co**(P) of all order-convex subsets of P forms a lattice under inclusion. It gives an important example of *convex geometry*, see K. V. Adaricheva, V. A. Gorbunov, and V. I. Tumanov [1]. In M. Semenova and F. Wehrung [10], the following result is proved:

Theorem. *The class **SUB** of all lattices that can be embedded into some **Co**(P) is a variety.*

This implies the nontrivial result that *every homomorphic image of a member of **SUB** belongs to **SUB***. It is in fact proved in [10] that the variety **SUB** is *finitely based*, it is defined by three identities that are denoted by (S), (U), and (B).

In the present paper, we extend this result to the class **SUB** $_n$ of all lattices that can be embedded into **Co**(P) for some poset P of length n , for a given positive integer n :

Theorem 6.4. *The class **SUB** $_n$ is a finitely based variety, for every positive integer n .*

Date: January 21, 2005.

2000 Mathematics Subject Classification. Primary: 06B05, 06B15, 06B23, 08C15. Secondary: 05B25, 05C05.

Key words and phrases. Lattice, embedding, poset, order-convex, variety, length, join-irreducible, join-seed.

The first author was supported by INTAS grant no. YSF: 2001/1-65, by RFBR grants no. 99-01-00485 and 01-01-06178, by GA UK grant no. 162/1999, and by GA CR grant no. 201/99. The second author was partially supported by the Fund of Mobility of the Charles University (Prague), by FRVS grant no. 2125, by institutional grant CEZ:J13/98:113200007, and by the Barrande program.

It is well-known that for $n = 1$, the class $\mathbf{SUB}_n$ is the variety of all *distributive* lattices. This fact is contained in G. Birkhoff and M. K. Bennett [2].

For $n = 2$, $\mathbf{SUB}_n = \mathbf{SUB}_2$ is much more interesting, it is the variety of all lattices that can be embedded into some $\mathbf{Co}(P)$ *without D -cycle on its atoms*. We find a simple finite set of identities characterizing $\mathbf{SUB}_2$, see Theorem 3.7. In addition, we prove the following results:

- The variety $\mathbf{SUB}_2$ is locally finite (see Theorem 4.10), and we provide an explicit upper bound for the cardinality of the free lattice on m generators in $\mathbf{SUB}_2$.
- A finite atomistic lattice without D -cycle belongs to $\mathbf{SUB}$ iff it belongs to $\mathbf{SUB}_2$ (see Proposition 3.9).

We also prove that $\mathbf{SUB}_n$ is not locally finite for $n \geq 3$ (see Theorem 7.1), and that $\mathbf{SUB}_n$ is a proper subvariety of $\mathbf{SUB}_{n+1}$ for every n (see Corollary 6.7).

2. BASIC CONCEPTS

We recall some of the definitions and concepts used in [10]. For elements a, b, c of a lattice L such that $a \leq b \vee c$, we say that the (formal) inequality $a \leq b \vee c$ is a *nontrivial join-cover*, if $a \not\leq b, c$. We say that it is *minimal in b* , if $a \not\leq x \vee c$ holds, for all $x < b$, and we say that it is a *minimal nontrivial join-cover*, if it is a nontrivial join-cover and it is minimal in both b and c .

The *join-dependency* relation $D = D_L$ (see R. Freese, J. Ježek, and J. B. Nation [4]) is defined on the set $J(L)$ of all join-irreducible elements of L by putting

$$p D q, \text{ if } p \neq q \text{ and } \exists x \text{ such that } p \leq q \vee x \text{ holds and is minimal in } q. \quad (2.1)$$

It is important to observe that $p D q$ implies that $p \not\leq q$, for all $p, q \in J(L)$. Furthermore, $p \not\leq x$ in (2.1).

We say that L is *finitely spatial* (resp., *spatial*) if every element of L is a join of join-irreducible (resp., completely join-irreducible) elements of L . It is well known that every dually algebraic lattice is lower continuous—see Lemma 2.3 in P. Crawley and R. P. Dilworth [3], and spatial (thus finitely spatial)—see Theorem I.4.22 in G. Gierz *et al.* [5] or Lemma 1.3.2 in V. A. Gorbunov [6].

A lattice L is *dually 2-distributive*, if it satisfies the identity

$$a \wedge (x \vee y \vee z) = (a \wedge (x \vee y)) \vee (a \wedge (x \vee z)) \vee (a \wedge (y \vee z)).$$

A stronger identity is the *Stirlitz identity* (S) introduced in [10]:

$$a \wedge (b' \vee c) = (a \wedge b') \vee \bigvee_{i < 2} \left(a \wedge (b_i \vee c) \wedge ((b' \wedge (a \vee b_i)) \vee c) \right),$$

where we put $b' = b \wedge (b_0 \vee b_1)$. Two other important identities are the *Udav identity* (U),

$$\begin{aligned} x \wedge (x_0 \vee x_1) \wedge (x_1 \vee x_2) \wedge (x_0 \vee x_2) \\ = (x \wedge x_0 \wedge (x_1 \vee x_2)) \vee (x \wedge x_1 \wedge (x_0 \vee x_2)) \vee (x \wedge x_2 \wedge (x_0 \vee x_1)), \end{aligned}$$

and the *Bond identity* (B),

$$\begin{aligned} x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) &= \bigvee_{i < 2} \left((x \wedge a_i \wedge (b_0 \vee b_1)) \vee (x \wedge b_i \wedge (a_0 \vee a_1)) \right) \\ &\quad \vee \bigvee_{i < 2} (x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge (a_0 \vee b_i) \wedge (a_1 \vee b_{1-i})). \end{aligned}$$

It is proved in [10] that a lattice L belongs to **SUB** iff it satisfies (S), (U), and (B). Although these identities are quite complicated, they have the following respective consequences, their so-called *join-irreducible interpretations*, that can be easily visualized on the poset P in case $L = \mathbf{Co}(P)$ for a poset P :

- (S_j): For all $a, b, b_0, b_1, c \in J(L)$, the inequalities $a \leq b \vee c$, $b \leq b_0 \vee b_1$, and $a \neq b$ imply that either $a \leq \bar{b} \vee c$ for some $\bar{b} < b$ or $b \leq a \vee b_i$ and $a \leq b_i \vee c$ for some $i < 2$.
- (U_j): For all $x, x_0, x_1, x_2 \in J(L)$, the inequalities $x \leq x_0 \vee x_1, x_0 \vee x_2, x_1 \vee x_2$ imply that either $x \leq x_0$ or $x \leq x_1$ or $x \leq x_2$.
- (B_j): For all $x, a_0, a_1, b_0, b_1 \in J(L)$, the inequalities $x \leq a_0 \vee a_1, b_0 \vee b_1$ imply that either $x \leq a_i$ or $x \leq b_i$ for some $i < 2$ or $x \leq a_0 \vee b_0, a_1 \vee b_1$ or $x \leq a_0 \vee b_1, a_1 \vee b_0$.

It is proved in [10] that (S) implies (S_j), (U) implies (U_j), and (B) implies (B_j).

A *Stirlitz track* of L is a pair $(\langle a_i \mid 0 \leq i \leq n \rangle, \langle a'_i \mid 1 \leq i \leq n \rangle)$, where the a_i -s and the a'_i -s are join-irreducible elements of L that satisfy the following relations:

- (i) the inequality $a_i \leq a_{i+1} \vee a'_{i+1}$ holds, for all $i \in \{0, \dots, n-1\}$, and it is a minimal nontrivial join-cover;
- (ii) the inequality $a_i \leq a'_i \vee a_{i+1}$ holds, for all $i \in \{1, \dots, n-1\}$.

For a poset P , the *length* of P , denoted by $\text{length } P$, is defined as the supremum of the numbers $|C| - 1$, where C ranges over the finite subchains of P . We say that P with predecessor relation $\prec$ is *tree-like*, if it has no infinite bounded chain and between any points a and b of P there exists at most one finite sequence $\langle x_i \mid 0 \leq i \leq n \rangle$ with distinct entries such that $x_0 = a$, $x_n = b$, and either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$, for all $i \in \{0, \dots, n-1\}$.

3. THE IDENTITY (L₂)

Let (L₂) be the following lattice-theoretical identity:

$$\begin{aligned} a \wedge \left((b \wedge (c \vee c')) \vee b' \right) &= \\ &= (a \wedge b \wedge (c \vee c')) \vee \left(a \wedge ((b \wedge c) \vee b') \right) \vee \left(a \wedge ((b \wedge c') \vee b') \right). \end{aligned}$$

Taking $b = c \vee c'$ implies immediately the following:

Lemma 3.1. *The identity (L₂) implies dual 2-distributivity.*

In order to find an alternative formulation for (L₂) and many other identities, it is convenient to introduce the following definition.

Definition 3.2. A subset Σ of a lattice L is a *join-seed*, if the following assertions hold:

- (i) $\Sigma \subseteq J(L)$;
- (ii) every element of L is a join of elements of Σ ;

- (iii) for all $p \in \Sigma$ and all $a, b \in L$ such that $p \leq a \vee b$ and $p \not\leq a, b$, there are $x \leq a$ and $y \leq b$ both in Σ such that $p \leq x \vee y$ is minimal in x and y .

Two important examples of join-seeds are provided by the following:

Lemma 3.3. *Any of the following assumptions implies that the subset Σ is a join-seed of the lattice L :*

- (i) $L = \mathbf{Co}(P)$ and $\Sigma = \{\{p\} \mid p \in P\}$, for some poset P .
- (ii) L is a dually 2-distributive, complete, lower continuous, finitely spatial lattice, and $\Sigma = J(L)$.

Proof. (i) is obvious, while (ii) follows immediately from [10, Lemma 3.2]. $\square$

Proposition 3.4. *Let L be a lattice, let $\Sigma \subseteq J(L)$. We consider the following statements on L, Σ :*

- (i) L satisfies (L_2) .
- (ii) There are no elements a, b, c of Σ such that $a D b D c$.

Then (i) implies (ii). Furthermore, if Σ is a join-seed of L , then (ii) implies (i).

Proof. (i) $\Rightarrow$ (ii) Suppose that there are $a, b, c \in \Sigma$ such that $a D b D c$. Let $b', c' \in L$ such that both inequalities $a \leq b \vee b'$ and $b \leq c \vee c'$ hold and are minimal, respectively, in b and in c . From the assumption that L satisfies (L_2) it follows that

$$a = (a \wedge b) \vee \left(a \wedge ((b \wedge c) \vee b') \right) \vee \left(a \wedge ((b \wedge c') \vee b') \right).$$

Since a is join-irreducible and $a \not\leq b$, there exists $x \in \{c, c'\}$ such that $a \leq (b \wedge x) \vee b'$. But $b \wedge x \leq b$, thus, by the minimality statement on b , $b \leq x$, a contradiction.

(ii) $\Rightarrow$ (i) under the additional assumption that Σ is a join-seed of L . Let $a, b, b', c, c' \in L$, denote by u (resp., v) the left hand side (resp., right hand side) of the identity (L_2) formed with these elements. It is clear that $v \leq u$. Conversely, let $x \leq u$ in Σ , we prove that $x \leq v$. If either $x \leq b \wedge (c \vee c')$ or $x \leq b'$ then this is clear. Suppose that $x \not\leq b \wedge (c \vee c'), b'$. Since $x \leq (b \wedge (c \vee c')) \vee b'$ and Σ is a join-seed of L , there are $y \leq b \wedge (c \vee c')$ and $y' \leq b'$ in Σ such that $x \leq y \vee y'$ is a minimal nontrivial join-cover. If either $y \leq c$ or $y \leq c'$ then either $x \leq a \wedge ((b \wedge c) \vee b')$ or $x \leq a \wedge ((b \wedge c') \vee b')$, in both cases $x \leq v$. Suppose that $y \not\leq c, c'$. Since $y \leq c \vee c'$ and Σ is a join-seed, there are $z \leq c$ and $z' \leq c'$ in Σ such that $y \leq z \vee z'$ is a minimal nontrivial join-cover. Hence $x D y D z$, a contradiction. Therefore, $x \leq v$. Since every element of L is a join of elements of Σ , $u \leq v$, whence $u = v$, which completes the proof that L satisfies (L_2) . $\square$

Corollary 3.5. *Let $(P, \trianglelefteq)$ be a poset. Then $\mathbf{Co}(P)$ satisfies (L_2) iff $\text{length } P \leq 2$.*

Proof. Put $\Sigma = \{\{p\} \mid p \in P\}$, the natural join-seed of $\mathbf{Co}(P)$. Suppose first that $\text{length } P > 2$, that is, P contains a four-element chain $o \triangleleft a \triangleleft b \triangleleft c$. Then $\{a\} D \{b\} D \{c\}$, thus, by Proposition 3.4, $\mathbf{Co}(P)$ does not satisfy (L_2) .

Conversely, suppose that $\mathbf{Co}(P)$ does not satisfy (L_2) . By Proposition 3.4, there are $a, b, c \in P$ such that $\{a\} D \{b\} D \{c\}$. Since $\{a\} D \{b\}$, there exists $b' \in P$ such that either $b \triangleleft a \triangleleft b'$ or $b' \triangleleft a \triangleleft b$, say, without loss of generality, $b' \triangleleft a \triangleleft b$. Since $\{b\} D \{c\}$, there are $u, v \in P$ such that $u \triangleleft b \triangleleft v$. Therefore, $b' \triangleleft a \triangleleft b \triangleleft v$ is a four-element chain in P . $\square$

In order to proceed, it is convenient to recall the following result from [10]:

Proposition 3.6. *Let L be a complete, lower continuous, dually 2-distributive lattice that satisfies (U) and (B). Then for every $p \in P$, there are subsets A and B of $[p]^D$ that satisfy the following properties:*

- (i) $[p]^D = A \cup B$ and $A \cap B = \emptyset$.
- (ii) For all $x, y \in [p]^D$, $p \leq x \vee y$ iff (x, y) belongs to $(A \times B) \cup (B \times A)$.

Moreover, the set $\{A, B\}$ is uniquely determined by these properties.

The set $\{A, B\}$ is called the *Udav-Bond partition* of $[p]^D$ associated with p .

We can now prove the following result:

Theorem 3.7. *Let L be a lattice. Then the following are equivalent:*

- (i) L belongs to **SUB**₂.
- (ii) L satisfies the identities (L₂), (U), and (B).
- (iii) There are a tree-like poset Γ of length at most 2 and a lattice embedding $\varphi: L \hookrightarrow \mathbf{Co}(\Gamma)$ that preserves the existing bounds. Furthermore, the following additional properties hold:
 - if L is finite, then Γ is finite;
 - if L is finite and subdirectly irreducible, then φ is atom-preserving.

Proof. (i) $\Rightarrow$ (ii) It has been already proved in [10] that every lattice in **SUB** (thus *a fortiori* in **SUB**₂) satisfies the identities (U) and (B). Furthermore, it follows from Corollary 3.5 that every lattice in **SUB**₂ satisfies (L₂).

(ii) $\Rightarrow$ (iii) Let L be a lattice satisfying (L₂), (U), and (B). We embed L into the lattice $\hat{L} = \text{Fil } L$ of all filters of L , partially ordered by reverse inclusion (see, e.g., G. Grätzer [7]); if L has no unit element, then we allow the empty set in $\hat{L}$, otherwise we require filters to be nonempty. This way, $\hat{L}$ is a dually algebraic lattice, satisfies the same identities as L , and the natural embedding $x \mapsto \uparrow x$ from L into $\hat{L}$ preserves the existing bounds.

Hence we have reduced the problem to the case where L is a dually algebraic lattice. In particular, L is complete, lower continuous, and finitely spatial (it is even spatial), and $\Sigma = J(L)$ is a join-seed of L (see Lemma 3.3). Since L satisfies the identity (L₂) and by Lemma 3.1, L is dually 2-distributive. Hence, by Proposition 3.6, every $p \in J(L)$ has a unique Udav-Bond partition $\{A_p, B_p\}$.

Our poset Γ is defined in a similar fashion as in [10, Section 7]. The underlying set of Γ is the set of all nonempty finite sequences $\alpha = \langle a_0, \dots, a_n \rangle$ of elements of $J(L)$ such that a_0 is D -minimal in $J(L)$ (this condition is added) and $a_i D a_{i+1}$, for all $i \in \{0, \dots, n-1\}$; as in [10], we call n the *length* of α and we put $e(\alpha) = a_n$. Since L satisfies (L₂) and by Proposition 3.4, the elements of Γ are of length either 1 or 2. Hence the partial ordering $\leq$ on Γ takes the following very simple form. The nontrivial coverings in Γ are those of the form $\langle p, a \rangle \triangleleft \langle p \rangle \triangleleft \langle p, b \rangle$, where $p \in J(L)$ and $(a, b) \in A_p \times B_p$. Since the elements of length 1 of Γ are either maximal or minimal, Γ has indeed length at most 2. The proof that Γ is tree-like proceeds *mutatis mutandis* as in [10, Proposition 7.3].

As in [10], we define a map φ from L to the powerset of Γ by the rule

$$\varphi(x) = \{\alpha \in \Gamma \mid e(\alpha) \leq x\}, \quad \text{for all } x \in L.$$

If $\langle p, a \rangle \triangleleft \langle p \rangle \triangleleft \langle p, b \rangle$ in Γ , then $p \leq a \vee b$; hence, for $x \in L$, if both $\langle p, a \rangle$ and $\langle p, b \rangle$ belong to $\varphi(x)$, then $\langle p \rangle \in \varphi(x)$; whence $\varphi(x) \in \mathbf{Co}(\Gamma)$.

It is clear that φ is a meet-homomorphism, and that it preserves the existing bounds. Let $x, y \in L$ such that $x \not\leq y$. Since L is finitely spatial, there exists

$a \in J(L)$ such that $a \leq x$ and $a \not\leq y$. If a is D -minimal in $J(L)$, then $\langle a \rangle$ belongs to $\varphi(x) \setminus \varphi(y)$. If a is not D -minimal in $J(L)$, then there exists $p \in J(L)$ such that $p D a$. Since there are no D -chains with three elements in $J(L)$, p is D -minimal, thus $\langle p, a \rangle$ belongs to $\varphi(a) \setminus \varphi(b)$. Therefore, φ is a meet-embedding from L into $\mathbf{Co}(\Gamma)$.

We now prove that φ is a join-homomorphism. It suffices to prove that $\varphi(x \vee y) \subseteq \varphi(x) \vee \varphi(y)$, for all $x, y \in L$. Let $\alpha \in \varphi(x \vee y)$, we prove that $\alpha \in \varphi(x) \vee \varphi(y)$. This is obvious if $\alpha \in \varphi(x) \cup \varphi(y)$, so suppose that $\alpha \notin \varphi(x) \cup \varphi(y)$. Put $p = e(\alpha)$. So $p \not\leq x, y$ while $p \leq x \vee y$, thus there are $u \leq x$ and $v \leq y$ in $J(L)$ such that $p \leq u \vee v$ is a minimal nontrivial join-cover. In particular, $p D u$ and $p D v$, thus $\alpha = \langle p \rangle$ and both $\langle p, u \rangle$ and $\langle p, v \rangle$ belong to Γ . It follows from $p \leq u \vee v$ that (u, v) belongs to $(A_p \times B_p) \cup (B_p \times A_p)$, thus either $\langle p, u \rangle \triangleleft \langle p \rangle \triangleleft \langle p, v \rangle$ or $\langle p, v \rangle \triangleleft \langle p \rangle \triangleleft \langle p, u \rangle$, in both cases $\alpha \in \varphi(x) \vee \varphi(y)$. This completes the proof that φ is a lattice embedding.

Of course, if L is finite, then Γ is finite. Now suppose that L is finite and subdirectly irreducible. Since there are no D -sequences of length three in $J(L)$, there are *a fortiori* no D -cycles, thus, since L is subdirectly irreducible, $J(L)$ has a unique D -minimal element p (see R. Freese, J. Ježek, and J.B. Nation [4, Chapter 3]). Hence, if x is an atom of L , then $\varphi(x)$ is equal to $\{\langle p \rangle\}$ if $x = p$ and to $\{\langle p, x \rangle\}$ otherwise, in both cases, $\varphi(x)$ is an atom of $\mathbf{Co}(\Gamma)$.

Finally, (iii) $\Rightarrow$ (i) is trivial. $\square$

Remark 3.8. It follows from [10, Example 8.1] that there exists a (non subdirectly irreducible) finite lattice L without D -cycle in $\mathbf{SUB}_2$ that cannot be embedded atom-preservingly into any lattice of the form $\mathbf{Co}(P)$.

Proposition 3.9. *Let L be a finite atomistic lattice without any D -cycle of the form $a D b D a$. Then L belongs to $\mathbf{SUB}$ iff L belongs to $\mathbf{SUB}_2$. In particular, L has no D -cycle.*

Proof. Suppose that L belongs to $\mathbf{SUB}$. For $a, b, c \in J(L)$ such that $a D b D c$, it follows from Lemma 3.3 that there are elements b' and c' in $J(L)$ such that both inequalities $a \leq b \vee b'$ and $b \leq c \vee c'$ hold and are minimal nontrivial join-covers. Since L satisfies (S_j) , there exists $x \in \{c, c'\}$ such that $b \leq a \vee x$ and $a \leq b' \vee x$. But $a \neq b$ and $b \neq x$ (because $a D b D x$), thus, since a, b , and x are atoms, the first inequality witnesses that $b D a$. Hence $a D b D a$, a contradiction. It follows from Proposition 3.4 that L satisfies (L_2) , and then it follows from Theorem 3.7 that L belongs to $\mathbf{SUB}_2$, in fact, there exists a finite poset Γ of length at most 2 such that L embeds into $\mathbf{Co}(\Gamma)$. It follows from Proposition 3.4 and Corollary 3.5 that $\mathbf{Co}(\Gamma)$ has no D -cycle (a direct proof is also very easy), thus neither has L . $\square$

As the following example shows, Proposition 3.9 does not extend to the nonatomistic case.

Example 3.10. *A finite subdirectly irreducible lattice without D -cycle that belongs to $\mathbf{SUB}_3 \setminus \mathbf{SUB}_2$.*

Proof. Let $P = \{\dot{a}, \dot{a}', \dot{b}, \dot{c}, \dot{u}, \dot{v}\}$ be the poset diagrammed on Figure 1.

Let L be the sublattice of $\mathbf{Co}(P)$ that consists of those subsets X such that

$$\begin{aligned} (\dot{a} \in X \Rightarrow \dot{a}' \in X) \text{ and } (\{\dot{b}, \dot{c}\} \subseteq X \Rightarrow \dot{a} \in X) \text{ and } (\{\dot{u}, \dot{v}\} \subseteq X \Rightarrow \dot{b} \in X) \\ \text{and } (\{\dot{a}', \dot{u}\} \subseteq X \Rightarrow \dot{b} \in X) \text{ and } (\{\dot{u}, \dot{c}\} \subseteq X \Rightarrow \dot{a} \in X). \end{aligned}$$

FIGURE 1. A finite poset of length 3

Then $J(L) = \{a, a', b, c, u, v\}$, where $a = \{\dot{a}, \dot{a}'\}$, $a' = \{\dot{a}'\}$, $b = \{\dot{b}\}$, $c = \{\dot{c}\}$, $u = \{\dot{u}\}$, $v = \{\dot{v}\}$. Hence L is the $\langle \vee, 0 \rangle$ -semilattice defined by the generators a, a', b, c, u, v , and the relations

$$a' \leq a; a \leq b \vee c; b \leq u \vee v; b \leq a' \vee u; a \leq u \vee c.$$

In particular, L has no D -cycle and it is subdirectly irreducible. Furthermore, L is a sublattice of $\mathbf{Co}(P)$, hence it belongs to $\mathbf{SUB}_3$. However, L has the three-element D -sequence $a D b D u$, thus it does not belong to $\mathbf{SUB}_2$. $\square$

4. LOCAL FINITENESS OF $\mathbf{SUB}_2$

We begin with a few elementary observations on complete congruences of lattices of the form $\mathbf{Co}(P)$. We recall that a congruence θ of a complete lattice L is *complete*, if $x \equiv y \pmod{\theta}$, for all $y \in Y$ implies $x \equiv \bigvee Y \pmod{\theta}$ and $x \equiv \bigwedge Y \pmod{\theta}$, for all $x \in L$ and all nonempty $Y \subseteq L$. We say that L is *completely subdirectly irreducible*, if it has a least nonzero complete congruence.

Definition 4.1. We say that a subset U of a poset $(P, \trianglelefteq)$ is *D-closed*, if $x \triangleleft p \triangleleft y$ and either $x \in U$ or $y \in U$ implies that $p \in U$, for all $x, y, p \in P$.

Equivalently, $\{p\} D \{x\}$ (in $\mathbf{Co}(P)$) and $x \in U$ implies that $p \in U$, for all $p, x \in P$. Observe in particular that every D -closed subset of P is convex. We leave to the reader the straightforward proof of the following lemma:

Lemma 4.2. Let P be a poset, let U be a D -closed subset of P . Then the binary relation θ_U on $\mathbf{Co}(P)$ defined by

$$X \equiv Y \pmod{\theta_U} \Leftrightarrow X \cup U = Y \cup U, \quad \text{for all } X, Y \in \mathbf{Co}(P)$$

is a complete lattice congruence on $\mathbf{Co}(P)$, and one can define a surjective homomorphism $h_U: \mathbf{Co}(P) \rightarrow \mathbf{Co}(P \setminus U)$ with kernel θ_U by the rule $h_U(X) = X \setminus U$, for all $X \in \mathbf{Co}(P)$. Furthermore, every complete lattice congruence θ of $\mathbf{Co}(P)$ has the form θ_U , with associated D -closed set $U = \{p \in P \mid \{p\} \equiv \emptyset \pmod{\theta}\}$.

We shall denote by $\mathcal{D}(P)$ the lattice of all D -closed subsets of a poset P under inclusion. It follows from Lemma 4.2 that $\mathcal{D}(P)$ is isomorphic to the lattice of all complete congruences of $\mathbf{Co}(P)$.

Lemma 4.3. The lattice $\mathcal{D}(P)$ is algebraic, for every poset P .

Proof. Evidently, $\mathcal{D}(P)$ is an algebraic subset of the powerset lattice $\mathcal{P}(P)$ of P , that is, a complete meet-subsemilattice closed under nonempty directed unions (see [6]). Since $\mathcal{P}(P)$ is algebraic, so is $\mathcal{D}(P)$. $\square$

We observe that Lemma 4.3 cannot be extended to complete congruences of arbitrary complete lattices: by G. Grätzer and H. Lakser [8], every complete lattice L is isomorphic to the lattice of complete congruences of some complete lattice K . By G. Grätzer and E. T. Schmidt [9], K can be taken distributive.

Corollary 4.4. *For a poset P , the lattice $\mathbf{Co}(P)$ is completely subdirectly irreducible iff there exists a least (for the inclusion) nonempty D -closed subset of P .*

The analogue of Birkhoff's subdirect decomposition theorem runs as follows:

Lemma 4.5. *Let P be a poset. Then there exists a family $\langle U_i \mid i \in I \rangle$ of D -closed subsets of P such that the diagonal map from $\mathbf{Co}(P)$ to $\prod_{i \in I} \mathbf{Co}(P \setminus U_i)$ is a lattice embedding, and all the $\mathbf{Co}(P \setminus U_i)$ are completely subdirectly irreducible.*

Proof. Let $\{U_i \mid i \in I\}$ denote the set of all completely meet-irreducible elements of $\mathcal{D}(P)$. It follows from Lemma 4.3 that $\mathcal{D}(P)$ is dually spatial, that is, every element of $\mathcal{D}(P)$ is a meet of some of the U_i -s. By applying this to the empty set, we obtain that the U_i -s have empty intersection, which concludes the proof. $\square$

Notation 4.6. For every positive integer n , we denote by $\mathbb{P}_n$ the class of all posets P of length at most n such that $\mathbf{Co}(P)$ is completely subdirectly irreducible (i.e., P has a least nonempty D -closed subset).

For every pair (I, J) of nonempty disjoint sets, set $P_{I,J} = I \cup J \cup \{p\}$, where p is some outside element, with nontrivial coverings $x \triangleleft p$ for $x \in I$ and $p \triangleleft y$ for $y \in J$.

Lemma 4.7. *The class $\mathbb{P}_2$ consists of the one-element poset and all posets of the form $P_{I,J}$, where I and J are nonempty disjoint sets.*

Proof. It is straightforward to verify that the one-element poset and the posets $P_{I,J}$ all belong to $\mathbb{P}_2$ (the monolith of $\mathbf{Co}(P_{I,J})$ is the congruence $\Theta(\emptyset, \{p\})$). Conversely, let P be a poset in $\mathbb{P}_2$. If $\text{length } P \leq 1$, then $\mathbf{Co}(P)$ is the powerset of P , thus it is distributive. Furthermore, every subset of P is D -closed, thus, since P is completely subdirectly irreducible, P is a singleton.

Suppose now that P has length 2. Thus there exists a three-element chain $a \triangleleft p \triangleleft b$ in P . Since P has length 2, a is minimal, b is maximal, and $\{p\}$ is D -closed. The latter applies to every element of height 1 instead of p , hence, by assumption on P , p is the only element of height 1 of P . Let x be a minimal element of P . If $x \not\triangleleft p$, then $\{x\}$ is D -closed, thus $x = p$, a contradiction; whence $x \triangleleft p$. Similarly, $p \triangleleft y$ for every maximal element y of P . Therefore, $P \cong P_{I,J}$, where I (resp., J) is the set of all minimal (resp., maximal) elements of P . $\square$

Notation 4.8. For a positive integer m , let $\mathbf{SUB}_{2,m}$ denote the class of all lattices that can be embedded into a product of lattices of the form $\mathbf{Co}(P_{I,J})$, where $|I| + |J| \leq m$.

Lemma 4.9. *Let L be a finitely generated lattice, let $m \geq 2$, let $a_0, \dots, a_{m-1}$ be generators of L . Let I and J be disjoint sets, let $f: L \rightarrow \mathbf{Co}(P_{I,J})$ be a lattice homomorphism. Then there are finite sets $I' \subseteq I$ and $J' \subseteq J$ such that, if*

$\pi: \mathbf{Co}(P_{I,J}) \rightarrow \mathbf{Co}(P_{I',J'})$, $X \mapsto X \cap P_{I',J'}$ is the canonical map, the following assertions hold:

- (i) $|I'| + |J'| \leq 2^m - 1$;
- (ii) $\pi \circ f$ is a lattice homomorphism;
- (iii) $\ker(f) = \ker(\pi \circ f)$.

Proof. Let $\mathbb{D}$ be the sublattice of the powerset lattice $\mathcal{P}(I \cup J)$ generated by the subset $\{f(a_i) \setminus \{p\} \mid i < m\}$. We observe that $\mathbb{D}$ is a finite distributive lattice. Moreover, every join-irreducible element of $\mathbb{D}$ has the form $\bigwedge_{i \in X} f(a_i)$, where X is a proper subset of $\{0, 1, \dots, m-1\}$, hence $|\mathbb{D}| \leq 2^m - 2$.

Claim 1. *The set $\mathbb{D}^* = \left(\mathbb{D} \cap (\mathcal{P}(I) \cup \mathcal{P}(J))\right) \cup \{X \cup \{p\} \mid X \in \mathbb{D}\}$ is a sublattice of $\mathbf{Co}(P_{I,J})$, and it contains the range of f .*

Proof of Claim. It is easy to verify that $\mathbb{D}^*$ is a sublattice of $\mathbf{Co}(P_{I,J})$. It contains all elements of the form $f(a_i)$, thus it contains the range of f . $\square$ Claim 1.

For all $A \in \mathbb{D}$, let $A^\dagger$ denote the largest element X of $\mathbb{D}$ such that $A \not\subseteq X$. Observe that $A^\dagger$ is meet-irreducible in $\mathbb{D}$. For every $A \in \mathbb{D}$, we pick $k_A \in A \setminus A^\dagger$. Furthermore, if the zero $0_{\mathbb{D}}$ of $\mathbb{D}$ is nonempty, we pick an element l of $0_{\mathbb{D}}$. We define $K_0 = \{k_A \mid A \in \mathbb{D}\}$, and we put $K = K_0$ if $0_{\mathbb{D}} = \emptyset$, $K = K_0 \cup \{l\}$ otherwise. Observe that K is a subset of $I \cup J$ and $|K| \leq 2^m - 1$. Finally, we put $I' = I \cap K$ and $J' = J \cap K$, and we let $\pi: \mathbf{Co}(P_{I,J}) \rightarrow \mathbf{Co}(P_{I',J'})$ be the canonical map.

Claim 2. *The following assertions hold:*

- (i) $X \not\subseteq Y$ implies that $X \cap K \not\subseteq Y \cap K$, for all $X, Y \in \mathbb{D}$.
- (ii) $X \neq \emptyset$ implies that $X \cap K \neq \emptyset$, for all $X \in \mathbb{D}$.

Proof of Claim. (i) There exists $A \in \mathbb{D}$ such that $A \subseteq X$ while $A \not\subseteq Y$. Hence $k_A \in A \setminus A^\dagger \subseteq X \setminus Y$.

(ii) If $0_{\mathbb{D}} = \emptyset$, then X contains an atom A of $\mathbb{D}$; hence $k_A \in A \subseteq X$. If $0_{\mathbb{D}} \neq \emptyset$, then $l \in 0_{\mathbb{D}} \subseteq X$. $\square$ Claim 2.

Now we can prove that $\pi \circ f$ is a lattice homomorphism. It is clearly a meet-homomorphism. To prove that it is a join-homomorphism, it suffices to prove the containment

$$(f(x) \vee f(y)) \cap P_{I',J'} \subseteq (f(x) \cap P_{I',J'}) \vee (f(y) \cap P_{I',J'}), \quad (4.1)$$

for all $x, y \in L$. Suppose otherwise. Since p is the only element of $P_{I,J}$ that is neither maximal nor minimal, it belongs to the left hand side of (4.1) but not to its right hand side. In particular, $p \notin f(x) \cup f(y)$, whence, say, $f(x) \subseteq I$ and $f(y) \subseteq J$. By Claim 1, $f(x), f(y) \in \mathbb{D}^*$, thus $f(x), f(y) \in \mathbb{D}$. Furthermore, $p \in f(x) \vee f(y)$ with $f(x) \subseteq I$ and $f(y) \subseteq J$, whence $f(x), f(y)$ are nonempty. By Claim 2(ii), both $f(x)$ and $f(y)$ meet K , whence $p \in (f(x) \cap I') \vee (f(y) \cap J')$, a contradiction. Therefore, $\pi \circ f$ is indeed a lattice homomorphism.

In order to conclude the proof of Lemma 4.9, it suffices to prove that $\ker(\pi \circ f)$ is contained in $\ker(f)$. So let $x, y \in L$ such that $f(x) \not\subseteq f(y)$. By Claim 1, both $f(x)$ and $f(y)$ belong to $\mathbb{D}^*$. If $f(x) \setminus \{p\} \subseteq f(y)$, then $p \in f(x)$, hence

$$p \in (f(x) \cap P_{I',J'}) \setminus (f(y) \cap P_{I',J'}) = (\pi \circ f(x)) \setminus (\pi \circ f(y)).$$

If $f(x) \setminus \{p\} \not\subseteq f(y)$, then, by Claim 2(i), there exists $k \in K$ with $k \in (f(x) \setminus \{p\}) \setminus (f(y) \setminus \{p\})$, whence $k \in (\pi \circ f(x)) \setminus (\pi \circ f(y))$. In both cases, $\pi \circ f(x) \not\subseteq \pi \circ f(y)$. $\square$

We can now prove the main result of this section:

Theorem 4.10. *Let $m \geq 2$ be an integer. Then every m -generated member of $\mathbf{SUB}_2$ belongs to $\mathbf{SUB}_{2,2^m-1}$. In particular, the variety $\mathbf{SUB}_2$ is locally finite.*

Proof. Let L be a m -generated member of $\mathbf{SUB}_2$. By Lemma 4.5, there exists a family $\langle (I_l, J_l) \mid l \in \Omega \rangle$ of pairs of nonempty disjoint sets, together with an embedding $f: L \hookrightarrow \prod_{l \in \Omega} \mathbf{Co}(P_{I_l, J_l})$. For all $l \in \Omega$, denote by $f_l: L \rightarrow \mathbf{Co}(P_{I_l, J_l})$ the l -th component of f . By Lemma 4.9, there are finite subsets $I'_l \subseteq I_l$ and $J'_l \subseteq J_l$ such that $|I'_l| + |J'_l| \leq 2^m - 1$, $\pi_l \circ f_l$ is a lattice homomorphism, and $\ker(f_l) = \ker(\pi_l \circ f_l)$, where $\pi_l: \mathbf{Co}(P_{I_l, J_l}) \rightarrow \mathbf{Co}(P_{I'_l, J'_l})$ is the canonical map. Therefore, the map

$$g: L \rightarrow \prod_{l \in \Omega} \mathbf{Co}(P_{I'_l, J'_l}), \quad x \mapsto \langle \pi_l \circ f_l(x) \mid l \in \Omega \rangle$$

is a lattice embedding of L into a member of $\mathbf{SUB}_{2,2^m-1}$. $\square$

The above argument gives a very rough upper bound for the cardinality of the free lattice F_m in $\mathbf{SUB}_2$ on m generators, namely, $e(m)^{e(m)^m}$, where $e(m) = 2^{2^m} + 2^{2^{m+1}-2} - 1$. Indeed, by Theorem 4.10, F_m embeds into A^{A^m} , where $A = P_{2^m-1, 2^m-1}$, and $|A| = e(m)$.

5. THE IDENTITIES (H_n)

Definition 5.1. For a positive integer n , we define inductively lattice polynomials $U_{i,n}$ (for $0 \leq i \leq n$), $V_{i,j,n}$ (for $0 \leq j \leq i \leq n-1$), $W_{i,j,n}$ (for $0 \leq j \leq i \leq n-2$), with variables $x_0, \dots, x_n, x'_1, \dots, x'_n$, as follows:

$$\begin{aligned} U_{n,n} &= x_n; \\ U_{i,n} &= x_i \wedge (U_{i+1,n} \vee x'_{i+1}) && \text{for } 0 \leq i \leq n-1; \\ V_{i,i,n} &= (x_i \wedge U_{i+1,n}) \vee (x_i \wedge x'_{i+1}) && \text{for } 0 \leq i \leq n-1; \\ V_{i,j,n} &= x_j \wedge (V_{i,j+1,n} \vee x'_{j+1}) && \text{for } 0 \leq j < i \leq n-1; \\ W_{i,i,n} &= x_i \wedge (x'_{i+1} \vee x'_{i+2}) \wedge ((U_{i+1,n} \wedge (x_i \vee x'_{i+2})) \vee x'_{i+1}) && \text{for } 0 \leq i \leq n-2; \\ W_{i,j,n} &= x_j \wedge (W_{i,j+1,n} \vee x'_{j+1}) && \text{for } 0 \leq j < i \leq n-2. \end{aligned}$$

Furthermore, we put

$$\begin{aligned} U_n &= U_{0,n}, \\ V_{i,n} &= V_{i,0,n} && \text{for } 0 \leq i \leq n-1; \\ W_{i,n} &= W_{i,0,n} && \text{for } 0 \leq i \leq n-2. \end{aligned}$$

Lemma 5.2. *Let n be a positive integer. The following inequalities hold in every lattice:*

- (i) $V_{i,j,n} \leq U_{j,n}$ for $0 \leq j \leq i \leq n-1$;
- (ii) $W_{i,j,n} \leq U_{j,n}$ for $0 \leq j \leq i \leq n-2$;
- (iii) $V_{i,n} \leq U_n$ for $0 \leq i \leq n-1$;
- (iv) $W_{i,n} \leq U_n$ for $0 \leq i \leq n-2$.

Proof. Items (i) and (ii) are easily established by downward induction on j . Items (iii) and (iv) follow immediately. $\square$

As in the following lemma, we shall often use the convenient notation

$$\vec{a} = \langle a_0, a_1, \dots, a_n \rangle, \quad \vec{a}' = \langle a'_1, \dots, a'_n \rangle.$$

Lemma 5.3. *Let n be a positive integer, let L be a lattice, let $a_0, \dots, a_n \in J(L)$ and $a'_1, \dots, a'_n \in L$ such that $a_i \leq a_{i+1} \vee a'_{i+1}$ is a nontrivial join-cover, for all $i \in \{0, \dots, n-1\}$, minimal in a_{i+1} for $i \leq n-2$. If the equality*

$$a_0 = \bigvee_{0 \leq i \leq n-1} V_{i,n}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}(\vec{a}, \vec{a}') \quad (5.1)$$

holds, then there exists $i \in \{0, \dots, n-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$.

Note. Of course, the meaning of the right hand side of the equation (5.1) for $n = 1$ is simply $V_{0,1}(\vec{a}, \vec{a}')$.

Proof. We first observe that the assumptions imply the following:

$$U_{i,n}(\vec{a}, \vec{a}') = a_i, \text{ for all } i \in \{0, \dots, n\}. \quad (5.2)$$

Now we put $c_{i,j} = V_{i,j,n}(\vec{a}, \vec{a}')$ and $c_i = c_{i,0}$ for $0 \leq j \leq i \leq n-1$, and $d_{i,j} = W_{i,j,n}(\vec{a}, \vec{a}')$ and $d_i = d_{i,0}$ for $0 \leq j \leq i \leq n-2$. We deduce from the assumption that one of the two following cases occurs:

Case 1. $a_0 = c_i$ for some $i \in \{0, \dots, n-1\}$. This can also be written $c_{i,0} = a_0$. Suppose that $c_{i,j} = a_j$, for $0 \leq j < i$. So $a_j \leq c_{i,j+1} \vee a'_{j+1}$ with $c_{i,j+1} \leq a_{j+1}$, thus, by the minimality assumption on a_{j+1} , we obtain that $c_{i,j+1} = a_{j+1}$. Hence $c_{i,j} = a_j$, for all $j \in \{0, \dots, i\}$, in particular, by (5.2),

$$a_i = c_{i,i} = (a_i \wedge a_{i+1}) \vee (a_i \wedge a'_{i+1}),$$

whence, by the join-irreducibility of a_i , either $a_i \leq a_{i+1}$ or $a_i \leq a'_{i+1}$, which contradicts the assumption. Thus, Case 1 cannot occur.

Case 2. $a_0 = d_i$ for some $i \in \{0, \dots, n-2\}$ (thus $n \geq 2$). As in Case 1, $d_{i,j} = a_j$, for all $j \in \{0, \dots, i\}$, whence, for $j = i$ and by (5.2),

$$a_i \leq (a'_{i+1} \vee a'_{i+2}) \wedge ((a_{i+1} \wedge (a_i \vee a'_{i+2})) \vee a'_{i+1})$$

Set $x = a_{i+1} \wedge (a_i \vee a'_{i+2})$, so $x \leq a_{i+1}$. Observe that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_i \leq x \vee a'_{i+1}$, whence, by the minimality assumption on a_{i+1} , we obtain that $x = a_{i+1}$, that is, $a_{i+1} \leq a_i \vee a'_{i+2}$.

This concludes the proof. $\square$

Lemma 5.4. *Let L be a lattice satisfying the Stirlitz identity (S), let Σ be a join-seed of L , let $x \in \Sigma$, let n be a positive integer, and let $a_0, \dots, a_n, a'_1, \dots, a'_n \in L$. If $x \leq U_n(\vec{a}, \vec{a}')$, then one of the following three cases occurs:*

- (i) *there exists $i \in \{0, \dots, n-1\}$ such that $x \leq V_{i,n}(\vec{a}, \vec{a}')$;*
- (ii) *there exists $i \in \{0, \dots, n-2\}$ such that $x \leq W_{i,n}(\vec{a}, \vec{a}')$;*
- (iii) *there are elements $x_i \leq U_{i,n}(\vec{a}, \vec{a}')$ ($0 \leq i \leq n$) and $x'_i \leq a'_i$ ($1 \leq i \leq n$) of Σ such that the pair $(\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$ is a Stirlitz track.*

Proof. We put $a_i^* = U_{i,n}(\vec{a}, \vec{a}')$ for $0 \leq i \leq n$, $c_{i,j} = V_{i,j,n}(\vec{a}, \vec{a}')$ for $0 \leq j \leq i \leq n-1$ and $d_{i,j} = W_{i,j,n}(\vec{a}, \vec{a}')$ for $0 \leq j \leq i \leq n-2$, then $c_i = c_{i,0}$ for $0 \leq i \leq n-1$ and $d_i = d_{i,0}$ for $0 \leq i \leq n-2$. We observe that $x \leq U_{0,n}(\vec{a}, \vec{a}') = a_0^*$.

Suppose that $x \not\leq c_i$, for all $i \in \{0, \dots, n-1\}$. Put $x_0 = x$. Suppose we have constructed $x_j \leq a_j^*$ in Σ , with $0 \leq j < n$, such that $x_j \not\leq c_{i,j}$, for all $i \in \{j, \dots, n-1\}$. If either $x_j \leq a_{j+1}^*$ or $x_j \leq a'_{j+1}$, then, since $x_j \leq a_j$, we obtain that $x_j \leq c_{j,j}$, a contradiction; whence $x_j \not\leq a_{j+1}^*, a'_{j+1}$. On the other hand, $x_j \leq a_j^* \leq a_{j+1}^* \vee a'_{j+1}$, thus, since $x_j \in \Sigma$ and Σ is a join-seed of L , there

are $x_{j+1} \leq a_{j+1}^*$ and $x'_{j+1} \leq a'_{j+1}$ in Σ such that $x_j \leq x_{j+1} \vee x'_{j+1}$ is a minimal nontrivial join-cover. Suppose that $x_{j+1} \leq c_{i,j+1}$ for some $i \in \{j+1, \dots, n-1\}$. Then

$$x_j \leq a_j \wedge (x_{j+1} \vee x'_{j+1}) \leq a_j \wedge (c_{i,j+1} \vee a'_{j+1}) = c_{i,j},$$

a contradiction. Hence $x_{j+1} \not\leq c_{i,j+1}$, for all $i \in \{j+1, \dots, n-1\}$, which completes the induction step.

Therefore, we have constructed elements $x_0 \leq a_0^*, \dots, x_n \leq a_n^*, x'_1 \leq a'_1, \dots, x'_n \leq a'_n$ of Σ such that $x_0 = x$ and $x_i \leq x_{i+1} \vee x'_{i+1}$ is a minimal nontrivial join-cover, for all $i \in \{0, \dots, n-1\}$. Suppose that $(\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$ is not a Stirlitz track. Then, since all the x_i -s and the x'_i -s are join-irreducible and L satisfies the axiom (S_j) (see [10, Proposition 4.4]), there exists $i \in \{0, \dots, n-2\}$ such that

$$x_{i+1} \leq x_i \vee x'_{i+2} \text{ and } x_i \leq x'_{i+1} \vee x'_{i+2}. \quad (5.3)$$

It follows from this that $x_{i+1} \leq a_{i+1}^* \wedge (a_i \vee a'_{i+2})$, whence

$$x_i \leq a_i \wedge (a'_{i+1} \vee a'_{i+2}) \wedge ((a_{i+1}^* \wedge (a_i \vee a'_{i+2})) \vee a'_{i+1}) = d_{i,i}.$$

For $0 \leq j < i$, suppose we have proved that $x_{j+1} \leq d_{i,j+1}$. Since $x_j \leq x_{j+1} \vee x'_{j+1}$, we obtain that $x_j \leq a_j \wedge (d_{i,j+1} \vee a'_{j+1}) = d_{i,j}$. Hence we have proved that $x_j \leq d_{i,j}$, for all $j \in \{0, \dots, i\}$. In particular, $x = x_0 \leq d_{i,0} = d_i = W_{i,n}(\vec{a}, \vec{a}')$, which concludes the proof. $\square$

For a positive integer n , let (H_n) be the following lattice identity:

$$U_n = \bigvee_{0 \leq i \leq n-1} V_{i,n} \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}.$$

It is not hard to verify directly that (H_1) is equivalent to distributivity.

Proposition 5.5. *Let n be a positive integer, let L be a lattice satisfying (S) and (U), let Σ be a subset of $J(L)$. We consider the following statements on L, Σ :*

- (i) *L satisfies (H_n) .*
- (ii) *For all elements $a_0, \dots, a_n, a'_1, \dots, a'_n$ of Σ , if $a_i \leq a_{i+1} \vee a'_{i+1}$ is a nontrivial join-cover, for all $i \in \{0, \dots, n-1\}$, minimal in a_{i+1} for $i \neq n-1$, then there exists $i \in \{0, \dots, n-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$.*
- (iii) *There is no Stirlitz track of length n with entries in Σ .*

Then (i) implies (ii) implies (iii). Furthermore, if Σ is a join-seed of L , then (iii) implies (i).

Proof. (i) $\Rightarrow$ (ii) Let $a_0, \dots, a_n, a'_1, \dots, a'_n \in \Sigma$ satisfy the assumption of (ii). Observe that $U_{i,n}(\vec{a}, \vec{a}') = a_i$ for $0 \leq i \leq n$, in particular, $U_n(\vec{a}, \vec{a}') = a_0$. From the assumption that L satisfies (H_n) it follows that

$$a_0 = \bigvee_{0 \leq i \leq n-1} V_{i,n}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}(\vec{a}, \vec{a}').$$

The conclusion of (ii) follows from Lemma 5.3.

(ii) $\Rightarrow$ (iii) Let $\sigma = (\langle a_i \mid 0 \leq i \leq n \rangle, \langle a'_i \mid 1 \leq i \leq n \rangle)$ be a Stirlitz track of L with entries in Σ . From (ii) it follows that there exists $i \in \{0, \dots, n-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$, whence $a_{i+1} \leq a'_{i+1} \vee a'_{i+2}$. Since σ is a Stirlitz track, the inequality $a_{i+1} \leq a'_{i+1} \vee a_{i+2}$ also holds, whence, since

$a_{i+1} \leq a_{i+2} \vee a'_{i+2}$ and by (U_j), either $a_{i+1} \leq a'_{i+1}$ or $a_{i+1} \leq a_{i+2}$ or $a_{i+1} \leq a'_{i+2}$, a contradiction.

(iii)⇒(i) under the additional assumption that Σ is a join-seed of L . Let $a_0, \dots, a_n, a'_1, \dots, a'_n \in L$, define $c, d \in L$ by

$$c = U_n(\vec{a}, \vec{a}'), \quad d = \bigvee_{0 \leq i \leq n-1} V_{i,n}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq n-2} W_{i,n}(\vec{a}, \vec{a}').$$

It follows from Lemma 5.2 that $d \leq c$. Conversely, let $x \in \Sigma$ such that $x \leq c$, we prove that $x \leq d$. Otherwise, $x \not\leq V_{i,n}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, n-1\}$ and $x \not\leq W_{i,n}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, n-2\}$, thus, by Lemma 5.4, there are elements $x_0 = x, x_1, \dots, x_n, x'_1, \dots, x'_n$ of Σ such that the pair

$$(\langle x_i \mid 0 \leq i \leq n \rangle, \langle x'_i \mid 1 \leq i \leq n \rangle)$$

is a Stirlitz track of L , a contradiction. Since every element of L is a join of elements of Σ , it follows that $c \leq d$. Therefore, $c = d$, so L satisfies (H_n). $\square$

Corollary 5.6. *Let $(P, \trianglelefteq)$ be a poset, let n be a positive integer. Then $\mathbf{Co}(P)$ satisfies (H_n) iff length $P \leq n$.*

Proof. It follows from [10, Section 4] that $\mathbf{Co}(P)$ satisfies (S) and (U). Furthermore, $\Sigma = \{\{p\} \mid p \in P\}$ is a join-seed of $\mathbf{Co}(P)$.

Suppose first that length $P \geq n+1$, that is, P contains a $n+2$ -element chain, say, $y \triangleleft x_0 \triangleleft \dots \triangleleft x_n$. Then the pair

$$(\langle \{x_i\} \mid 0 \leq i \leq n \rangle, \langle \{y\} \mid 1 \leq i \leq n \rangle)$$

is a Stirlitz track of length n in $\mathbf{Co}(P)$, thus, by Proposition 5.5, $\mathbf{Co}(P)$ does not satisfy (H_n).

Conversely, suppose that P does not contain any $n+2$ -element chain. By Proposition 5.5, in order to prove that $\mathbf{Co}(P)$ satisfies (H_n), it suffices to prove that $\mathbf{Co}(P)$ has no Stirlitz track of length n with entries in Σ . Suppose that there exists such a Stirlitz track, say,

$$(\langle \{x_i\} \mid 0 \leq i \leq n \rangle, \langle \{x'_i\} \mid 1 \leq i \leq n \rangle).$$

Since $\{x_0\} \leq \{x_1\} \vee \{x'_1\}$ is a nontrivial join-cover, either $x_1 \triangleleft x_0 \triangleleft x'_1$ or $x'_1 \triangleleft x_0 \triangleleft x_1$, say, $x'_1 \triangleleft x_0 \triangleleft x_1$. Similarly, for all $i \in \{0, \dots, n-1\}$, either $x_{i+1} \triangleleft x_i \triangleleft x'_{i+1}$ or $x'_{i+1} \triangleleft x_i \triangleleft x_{i+1}$. Suppose that the first possibility occurs, and take i minimum such. Thus $i > 0$ and $x'_i \triangleleft x_{i-1} \triangleleft x_i \triangleleft x'_{i+1}$ and $x_{i+1} \triangleleft x_i$ while $\{x_i\} \leq \{x'_i\} \vee \{x_{i+1}\}$, a contradiction. Thus $x'_{i+1} \triangleleft x_i \triangleleft x_{i+1}$. It follows that

$$x'_1 \triangleleft x_0 \triangleleft \dots \triangleleft x_n$$

is a $n+2$ -element chain in P , a contradiction. $\square$

6. THE IDENTITIES (H_{m,n})

Definition 6.1. For positive integers m and n and a lattice L , a *bi-Stirlitz track of index (m, n)* is a pair (σ, τ) , where

$$\begin{aligned} \sigma &= (\langle a_i \mid 0 \leq i \leq m \rangle, \langle a'_i \mid 1 \leq i \leq m \rangle), \\ \tau &= (\langle b_j \mid 0 \leq j \leq n \rangle, \langle b'_j \mid 1 \leq j \leq n \rangle) \end{aligned}$$

are Stirlitz tracks with the same base $a_0 = b_0 \leq a_1 \vee b_1$.

For positive integers m and n , we define the identity $(H_{m,n})$, with variable symbols t, x_i, x'_i ($1 \leq i \leq m$), y_j, y'_j ($1 \leq j \leq n$) as follows, where we put $x_0 = y_0 = t$:

$$\begin{aligned} U_m(\vec{x}, \vec{x}') \wedge U_n(\vec{y}, \vec{y}') = & \bigvee_{0 \leq i \leq m-1} (V_{i,m}(\vec{x}, \vec{x}') \wedge U_n(\vec{y}, \vec{y}')) \\ & \vee \bigvee_{0 \leq i \leq m-2} (W_{i,m}(\vec{x}, \vec{x}') \wedge U_n(\vec{y}, \vec{y}')) \\ & \vee \bigvee_{0 \leq j \leq n-1} (U_m(\vec{x}, \vec{x}') \wedge V_{j,n}(\vec{y}, \vec{y}')) \\ & \vee \bigvee_{0 \leq j \leq n-2} (U_m(\vec{x}, \vec{x}') \wedge W_{j,n}(\vec{y}, \vec{y}')) \\ & \vee (U_m(\vec{x}, \vec{x}') \wedge U_n(\vec{y}, \vec{y}') \wedge (x_1 \vee y'_1) \wedge (x'_1 \vee y_1)). \end{aligned}$$

The analogue of Proposition 5.5 for the identity $(H_{m,n})$ is the following:

Proposition 6.2. *Let m and n be positive integers, let L be a lattice satisfying (S), (U), and (B), let Σ be a subset of $J(L)$. We consider the following statements on L, Σ :*

- (i) L satisfies $(H_{m,n})$.
- (ii) *For all elements $a_0, \dots, a_m, a'_1, \dots, a'_m, b_0, \dots, b_n, b'_1, \dots, b'_n$ of Σ with $a_0 = b_0$, if $a_i \leq a_{i+1} \vee a'_{i+1}$ is a nontrivial join-cover, for all $i \in \{0, \dots, m-1\}$, minimal in a_{i+1} for $i \neq m-1$ and if $b_j \leq b_{j+1} \vee b'_{j+1}$ is a nontrivial join-cover, for all $j \in \{0, \dots, n-1\}$, minimal in b_{j+1} for $j \neq n-1$, then one of the following occurs:*
 - (a) *there exists $i \in \{0, \dots, m-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$;*
 - (b) *there exists $j \in \{0, \dots, n-2\}$ such that $b_j \leq b'_{j+1} \vee b'_{j+2}$ and $b_{j+1} \leq b_j \vee b'_{j+2}$;*
 - (c) $a_0 \leq (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)$.
- (iii) *There is no bi-Stirlitz track of index (m, n) with entries in Σ .*

Then (i) implies (ii) implies (iii). Furthermore, if Σ is a join-seed of L , then (iii) implies (i).

Proof. (i) $\Rightarrow$ (ii) Let $a_0, \dots, a_m, a'_1, \dots, a'_m, b_0, \dots, b_n, b'_1, \dots, b'_n \in \Sigma$ satisfy the assumption of (ii). Observe that $U_{m,i}(\vec{a}, \vec{a}') = a_i$ for $0 \leq i \leq m$ and $U_{n,j}(\vec{b}, \vec{b}') = b_j$ for $0 \leq j \leq n$. Put $p = a_0 = b_0$. From the assumption that L satisfies $(H_{m,n})$ it follows that

$$\begin{aligned} p = & \bigvee_{0 \leq i \leq m-1} (V_{i,m}(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}')) \vee \bigvee_{0 \leq i \leq m-2} (W_{i,m}(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}')) \\ & \vee \bigvee_{0 \leq j \leq n-1} (U_m(\vec{a}, \vec{a}') \wedge V_{j,n}(\vec{b}, \vec{b}')) \vee \bigvee_{0 \leq j \leq n-2} (U_m(\vec{a}, \vec{a}') \wedge W_{j,n}(\vec{b}, \vec{b}')) \quad (6.1) \\ & \vee (U_m(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}') \wedge (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)). \end{aligned}$$

Since p is join-irreducible, three cases can occur:

Case 1. $p = \bigvee_{0 \leq i \leq m-1} (V_{i,m}(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}')) \vee \bigvee_{0 \leq i \leq m-2} (W_{i,m}(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}'))$.

From Lemma 5.2 it follows that the equality

$$p = \bigvee_{0 \leq i \leq m-1} V_{i,m}(\vec{a}, \vec{a}') \vee \bigvee_{0 \leq i \leq m-2} W_{i,m}(\vec{a}, \vec{a}')$$

also holds. By Lemma 5.3, there exists $i \in \{0, \dots, m-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$.

Case 2. $p = \bigvee_{0 \leq j \leq n-1} (U_m(\vec{a}, \vec{a}') \wedge V_{j,n}(\vec{b}, \vec{b}')) \vee \bigvee_{0 \leq j \leq n-2} (U_m(\vec{a}, \vec{a}') \wedge W_{j,n}(\vec{b}, \vec{b}'))$.

As in Case 1, we obtain $j \in \{0, \dots, n-2\}$ such that $b_j \leq b'_{j+1} \vee b'_{j+2}$ and $b_{j+1} \leq b_j \vee b'_{j+2}$.

Case 3. $p \leq (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)$.

In all three cases above, the conclusion of (ii) holds.

(ii) $\Rightarrow$ (iii) Let (σ, τ) be a bi-Stirlitz track as in Definition 6.1. Put $p = a_0 = b_0$. It follows from the assumption (ii) that either there exists $i \in \{0, \dots, m-2\}$ such that $a_i \leq a'_{i+1} \vee a'_{i+2}$ and $a_{i+1} \leq a_i \vee a'_{i+2}$, or there exists $j \in \{0, \dots, n-2\}$ such that $b_j \leq b'_{j+1} \vee b'_{j+2}$ and $b_{j+1} \leq b_j \vee b'_{j+2}$, or $p \leq (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)$. In the first case, $a_{i+1} \leq a'_{i+1} \vee a'_{i+2}$, but σ is a Stirlitz track, thus also $a_{i+1} \leq a'_{i+1} \vee a_{i+2}$, a contradiction since $a_{i+1} \leq a_{i+2} \vee a'_{i+2}$ and by (U_j) . The second case leads to a similar contradiction. In the third case, $p \leq a_1 \vee b'_1$, a contradiction by (U_j) since $p \leq a_1 \vee b_1$ and $p \leq a_1 \vee a'_1$.

(iii) $\Rightarrow$ (i) under the additional assumption that Σ is a join-seed of L . Let $a_0 = b_0$, $a_1, \dots, a_m, a'_1, \dots, a'_m, b_1, \dots, b_n, b'_1, \dots, b'_n \in L$, put $c = U_m(\vec{a}, \vec{a}') \wedge U_n(\vec{b}, \vec{b}')$ and define $d \in L$ as the right hand side of (6.1). Further, put $a_i^* = U_{i,m}(\vec{a}, \vec{a}')$ for $0 \leq i \leq m$ and $b_j^* = U_{j,n}(\vec{b}, \vec{b}')$ for $0 \leq j \leq n$. It follows from Lemma 5.2 that $d \leq c$. Conversely, let $z \in \Sigma$ such that $z \leq c$, we prove that $z \leq d$. Otherwise, $z \not\leq V_{i,m}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, m-1\}$, and $z \not\leq W_{i,m}(\vec{a}, \vec{a}')$, for all $i \in \{0, \dots, m-2\}$, and $z \not\leq V_{j,n}(\vec{b}, \vec{b}')$, for all $j \in \{0, \dots, n-1\}$, and $z \not\leq W_{j,n}(\vec{b}, \vec{b}')$, for all $j \in \{0, \dots, n-2\}$, and $z \not\leq (a_1 \vee b'_1) \wedge (a'_1 \vee b_1)$, say, $z \not\leq a_1 \vee b'_1$. By Lemma 5.4, there are $x_1 \leq a_1^*, \dots, x_m \leq a_m^*, x'_1 \leq a'_1, \dots, x'_m \leq a'_m, y_1 \leq b_1^*, \dots, y_n \leq b_n^*, y'_1 \leq b'_1, \dots, y'_n \leq b'_n$ in Σ such that, putting $x_0 = y_0 = z$, both pairs

$$\begin{aligned} \sigma &= (\langle x_i \mid 0 \leq i \leq m \rangle, \langle x'_i \mid 1 \leq i \leq m \rangle), \\ \tau &= (\langle y_j \mid 0 \leq j \leq n \rangle, \langle y'_j \mid 1 \leq j \leq n \rangle) \end{aligned}$$

are Stirlitz tracks. By assumption, the pair (σ, τ) is not a bi-Stirlitz track, whence $z \not\leq x_1 \vee y_1$. Furthermore, from $z \not\leq a_1 \vee b'_1$ it follows that $z \not\leq x_1 \vee y'_1$ (observe that $x_1 \leq a_1^* \leq a_1$). However, from the fact that $z \leq x_1 \vee x'_1, y_1 \vee y'_1$ are nontrivial join-covers and (B_j) it follows that either $z \leq x_1 \vee y_1$ or $z \leq x_1 \vee y'_1$, a contradiction. $\square$

Corollary 6.3. *Let m and n be positive integers, let P be a poset. Then $\mathbf{Co}(P)$ satisfies $(H_{m,n})$ iff $\text{length } P \leq m + n - 1$.*

Proof. Suppose first that P contains a $m + n + 1$ -element chain, say,

$$x_m \triangleleft \dots \triangleleft x_1 \triangleleft x_0 = y_0 \triangleleft y_1 \triangleleft \dots \triangleleft y_n.$$

Then both pairs σ and τ defined as

$$\begin{aligned} \sigma &= (\langle \{x_i\} \mid 0 \leq i \leq m \rangle, \langle \{y_1\} \mid 1 \leq i \leq m \rangle) \\ \tau &= (\langle \{y_j\} \mid 0 \leq j \leq n \rangle, \langle \{x_1\} \mid 1 \leq j \leq n \rangle) \end{aligned}$$

are Stirlitz tracks with the same base $\{x_0\} = \{y_0\} \leq \{x_1\} \vee \{y_1\}$, hence (σ, τ) is a bi-Stirlitz track of index (m, n) . By Proposition 6.2, $\mathbf{Co}(P)$ does not satisfy $(H_{m,n})$.

Conversely, suppose that P does not contain any $m + n + 1$ -element chain. By Proposition 6.2, in order to prove that $\mathbf{Co}(P)$ satisfies $(H_{m,n})$, it suffices to prove that it has no bi-Stirlitz track of index (m, n) with entries in $\Sigma = \{\{p\} \mid p \in P\}$. Let

$$\begin{aligned}\sigma &= (\langle \{x_i\} \mid 0 \leq i \leq m \rangle, \langle \{x'_i\} \mid 1 \leq i \leq m \rangle) \\ \tau &= (\langle \{y_j\} \mid 0 \leq j \leq n \rangle, \langle \{y'_j\} \mid 1 \leq j \leq n \rangle)\end{aligned}$$

be pairs such that (σ, τ) is such a bi-Stirlitz track. By an argument similar as the one used in the proof of Corollary 5.6, since σ is a Stirlitz track, either $x'_1 \triangleleft x_0 \triangleleft \cdots \triangleleft x_m$ or $x_m \triangleleft \cdots \triangleleft x_0 \triangleleft x'_1$; without loss of generality, the second possibility occurs. Similarly, since τ is a Stirlitz track, either $y'_1 \triangleleft y_0 \triangleleft \cdots \triangleleft y_n$ or $y_n \triangleleft \cdots \triangleleft y_0 \triangleleft y'_1$. If the second possibility occurs, then $y_1 \triangleleft y_0 = x_0$ and $x_1 \triangleleft x_0$ while $\{x_0\} \leq \{x_1\} \vee \{y_1\}$, a contradiction. Therefore, the first possibility occurs, hence

$$x_m \triangleleft \cdots \triangleleft x_1 \triangleleft x_0 = y_0 \triangleleft y_1 \triangleleft \cdots \triangleleft y_n$$

is a $m + n + 1$ -element chain in P , a contradiction. $\square$

Now let us recall some results of [10]. In case L belongs to the variety **SUB**, so does the lattice $\widehat{L} = \text{Fil } L$ of all filters of L partially ordered by reverse inclusion (see Section 3), and $\text{J}(\widehat{L})$ is a join-seed of $\widehat{L}$. Furthermore, one can construct two posets R and Γ with the following properties:

- (i) There are natural embeddings $\varphi: L \hookrightarrow \mathbf{Co}(R)$ and $\psi: L \hookrightarrow \mathbf{Co}(\Gamma)$, and they preserve the existing bounds.
- (ii) R is finite in case L is finite.
- (iii) Γ is tree-like (as defined in Section 2, see also [10]).
- (iv) There exists a natural map $\pi: \Gamma \rightarrow R$ such that $\alpha \prec \beta$ in Γ implies that $\pi(\alpha) \prec \pi(\beta)$ in R . In particular, π is order-preserving.
- (v) $\psi(x) = \pi^{-1}[\varphi(x)]$, for all $x \in L$.

The main theorem of this section is the following:

Theorem 6.4. *Let n be a positive integer, let L be a lattice that belongs to the variety **SUB**. Consider the posets R and Γ constructed in [10] from $\widehat{L}$. Then the following are equivalent:*

- (i) $\text{length } R \leq n$;
- (ii) $\text{length } \Gamma \leq n$;
- (iii) *there exists a poset P such that $\text{length } P \leq n$ and L embeds into $\mathbf{Co}(P)$;*
- (iv) *L satisfies the identities (H_n) and $(H_{k,n+1-k})$ for $1 < k < n$;*
- (v) *L satisfies the identities (H_n) and $(H_{k,n+1-k})$ for $1 \leq k \leq n$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\text{length } R \leq n$, we prove that $\text{length } \Gamma \leq n$. Otherwise, there exists a $n + 2$ -element chain $\alpha_0 \prec \cdots \prec \alpha_{n+1}$ in Γ , thus, applying the map π , we obtain a $n + 2$ -element chain $\pi(\alpha_0) \prec \cdots \prec \pi(\alpha_{n+1})$ in R , a contradiction.

(ii) $\Rightarrow$ (iii) Since L embeds into $\mathbf{Co}(\Gamma)$, it suffices to take $P = \Gamma$.

(iii) $\Rightarrow$ (iv) follows immediately from Corollaries 5.6 and 6.3.

(iv) $\Rightarrow$ (v) Suppose that L satisfies the identities (H_n) and $(H_{k,n+1-k})$ for $1 < k < n$; then so does the filter lattice $\widehat{L}$ of L . Since $\widehat{L}$ satisfies (H_n) , it has no Stirlitz track of length n (see Proposition 5.5), thus, *a fortiori*, it has no bi-Stirlitz track of index

either $(n, 1)$ or $(1, n)$. Since $J(\widehat{L})$ is a join-seed of $\widehat{L}$, it follows from Proposition 6.2 that $\widehat{L}$ satisfies both $(H_{n,1})$ and $(H_{1,n})$.

(v) $\Rightarrow$ (i) Suppose that L satisfies the identities (H_n) and $(H_{k,n+1-k})$ for $1 \leq k \leq n$; then so does the filter lattice $\widehat{L}$ of L . We prove that $\text{length } R \leq n$. Otherwise, R has an oriented path $\mathbf{r} = \langle r_0, \dots, r_{n+1} \rangle$ of length $n+2$, that is, $r_i \prec r_{i+1}$, for all $i \in \{0, \dots, n\}$. By [10, Lemma 6.4], we can assume that $\mathbf{r}$ is ‘reduced’. If there are n successive values of the r_i that are of the form $\langle a_i, b_i, \varepsilon \rangle$ for a constant $\varepsilon \in \{+, -\}$, then, by [10, Lemma 6.1], there exists a Stirlitz track of length n in $\widehat{L}$ (with entries in $J(\widehat{L})$), which contradicts the assumption that $\widehat{L}$ satisfies (H_n) and Proposition 5.5. Therefore, $\mathbf{r}$ has the form

$$\langle \langle a_{k-1}, a_k, - \rangle, \dots, \langle a_0, a_1, - \rangle, \langle p \rangle, \langle b_0, b_1, + \rangle, \dots, \langle b_{l-1}, b_l, + \rangle \rangle$$

for some positive integers k and l and elements $a_0, \dots, a_k, b_0, \dots, b_l$ of $J(\widehat{L})$. By [10, Lemma 6.1], there are Stirlitz tracks of the form

$$\begin{aligned} \sigma &= (\langle a_i \mid 0 \leq i \leq k \rangle, \langle a'_i \mid 1 \leq i \leq k \rangle), \\ \tau &= (\langle b_j \mid 0 \leq j \leq l \rangle, \langle b'_j \mid 1 \leq j \leq l \rangle) \end{aligned}$$

for elements $a'_1, \dots, a'_k, b'_1, \dots, b'_l$ of $J(\widehat{L})$. Observe that $p = a_0 = b_0$. Furthermore, from $\langle a_0, a_1, - \rangle \prec \langle p \rangle \prec \langle b_0, b_1, + \rangle$ and the definition of $\prec$ on R it follows that $p \leq a_1 \vee b_1$. Therefore, (σ, τ) is a bi-Stirlitz track of index (k, l) with $k+l = n+1$ in $\widehat{L}$, which contradicts the assumption that $\widehat{L}$ satisfies $(H_{k,l})$ and Proposition 6.2. $\square$

The main result of [10] is that **SUB** is a finitely based variety of lattices. We thus obtain the following:

Corollary 6.5. *Let n be a positive integer. The class $\mathbf{SUB}_n$ of all lattices L that can be embedded into $\mathbf{Co}(P)$ for a poset P of length at most n is a finitely based variety, defined by the identities (S), (U), (B), (H_n) , and $(H_{k,n+1-k})$ for $1 < k < n$.*

Since finiteness of L implies finiteness of R , we also obtain the following:

Corollary 6.6. *Let n be a positive integer. A finite lattice L belongs to $\mathbf{SUB}_n$ iff it can be embedded into $\mathbf{Co}(P)$ for some finite poset P of length at most n .*

For a positive integer m , denote by $\mathbf{m}$ the m -element chain. As a consequence of Corollaries 5.6 and 6.3 and of Theorem 6.4, we obtain immediately the following:

Corollary 6.7. *For positive integers m and n , $\mathbf{Co}(\mathbf{m})$ belongs to $\mathbf{SUB}_n$ iff $m \leq n+1$. In particular, $\mathbf{SUB}_n$ is a proper subvariety of $\mathbf{SUB}_{n+1}$, for every positive integer n .*

7. NON-LOCAL FINITENESS OF $\mathbf{SUB}_3$

We have seen in Section 4 that the variety $\mathbf{SUB}_2$ is locally finite. In contrast with this, we shall now prove the following:

Theorem 7.1. *There exists an infinite, three-generated lattice in $\mathbf{SUB}_3$. Hence $\mathbf{SUB}_n$ is not locally finite for $n \geq 3$.*

Proof. Let P be the poset diagrammed on Figure 2.

We observe that the length of P is 3. We define order-convex subsets A, B, C of P as follows:

$$A = \{a_n \mid n < \omega\}, \quad B = \{d_0\} \cup \{b_n \mid n < \omega\}, \quad C = \{c_n \mid n < \omega\} \cup \{d_n \mid n < \omega\}.$$

FIGURE 2. An infinite poset of length 3

We put $A_0 = A$, $B_0 = B$, $A_{n+1} = A \vee (B_n \cap C)$, and $B_{n+1} = B \vee (A_n \cap C)$, for all $n < \omega$. A straightforward computation yields that both c_n and d_n belong to $A_{2n+1} \setminus A_{2n}$, for all $n < \omega$. Hence the sublattice of $\mathbf{Co}(P)$ generated by $\{A, B, C\}$ is infinite. $\square$

8. OPEN PROBLEMS

So far we have studied the following $(\omega + 1)$ -chain of varieties:

$$\mathbf{D} = \mathbf{SUB}_1 \subset \mathbf{SUB}_2 \subset \mathbf{SUB}_3 \subset \cdots \subset \mathbf{SUB}_n \subset \cdots \subset \mathbf{SUB}. \quad (8.1)$$

We do not know the answer to the following simple question, see also Problem 1 in [10]:

Problem 1. Is $\mathbf{SUB}$ the quasivariety join of all the $\mathbf{SUB}_n$, for $n > 0$?

Every variety from the chain (8.1) is the variety $\mathbf{SUB}(\mathcal{K})$ generated by all $\mathbf{Co}(P)$, where $P \in \mathcal{K}$, for some class $\mathcal{K}$ of posets.

Problem 2. Can one classify all the varieties of the form $\mathbf{SUB}(\mathcal{K})$? In particular, are there only countably many such varieties?

Problem 3. What are the *complete* sublattices of the lattices of the form $\mathbf{Co}(P)$ for some poset P ?

Problem 4. Give an estimate for the cardinality of the free lattice in $\mathbf{SUB}_2$ on m generators, for a positive integer m .

Problem 5. Classify all the subvarieties of $\mathbf{SUB}_2$.

9. ACKNOWLEDGMENTS

This work was partially completed while both authors were visiting the Charles University during the fall of 2001. Excellent conditions provided by the Department of Algebra are highly appreciated. Special thanks are due to Jiří Tůma and Vaclav Slavík.

REFERENCES

- [1] K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov, *Join-semidistributive lattices and convex geometries*, Adv. Math. **173** (2003), 1–49.
- [2] G. Birkhoff and M.K. Bennett, *The convexity lattice of a poset*, Order **2** (1985), 223–242.
- [3] P. Crawley and R.P. Dilworth, “Algebraic Theory of Lattices”, Prentice-Hall, New Jersey, 1973. vi+201 p.

- [4] R. Freese, J. Ježek, and J. B. Nation, “Free Lattices”, Mathematical Surveys and Monographs, **42**, Amer. Math. Soc., Providence, 1995. viii+293 p.
- [5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, “A Compendium of Continuous Lattices”, Springer-Verlag, Berlin, New York, 1980, xx+371 p.
- [6] V. A. Gorbunov, “Algebraic theory of quasivarieties”, (Algebraichesкая teoriya kvazimnogobrazij) (Russian) Sibirskaya Shkola Algebry i Logiki. 5. Novosibirsk: Nauchnaya Kniga, 1999. xii+368 p. English translation by Plenum, New York, 1998. xii+298 p.
- [7] G. Grätzer, “General Lattice Theory. Second edition”, new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäuser Verlag, Basel, 1998. xx+663 p.
- [8] G. Grätzer and H. Lakser, *On complete congruence lattices of complete lattices*, Trans. Amer. Math. Soc. **327** (1991), 385–405.
- [9] G. Grätzer and E. T. Schmidt, *Complete congruence lattices of complete distributive lattices*, J. Algebra **170** (1995), 204–229.
- [10] M. Semenova and F. Wehrung, *Sublattices of lattices of order-convex sets, I. The main representation theorem*, preprint 2002.

(M. Semenova) INSTITUTE OF MATHEMATICS OF THE SIBERIAN BRANCH OF RAS, ACAD. KOP-
TYUG PROSP. 4, 630090 NOVOSIBIRSK, RUSSIA
E-mail address: semenova@math.nsc.ru

(F. Wehrung) CNRS, UMR 6139, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE CAEN,
14032 CAEN CEDEX, FRANCE
E-mail address: wehrung@math.unicaen.fr
URL: <http://www.math.unicaen.fr/~wehrung>