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Chantal Berline, Antonino Salibra

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C. Berline [a], A. Salibra [b,1]

[a] *CNRS et Université Paris 7, Equipe PPS (case 7014), 2 place Jussieu, 72251 Paris Cedex 05, France*

[b] *Dipartimento di Informatica, Università Ca'Foscari di Venezia, Via Torino 155, 30172 Venezia, Italy*

Abstract

We generalize Baeten and Boerboom's method of forcing to show that there is a fixed sequence $(u_k)_{k\in\omega}$ of closed (untyped) λ-terms satisfying the following properties:

a) For any countable sequence $(g_k)_{k\in\omega}$ of Scott continuous functions (of arbitrary arity) on the power set of an arbitrary countable set, there is a graph model such that $(\lambda x.xx)(\lambda x.xx)u_k$ represents g_k in the model.

b) For any countable sequence $(t_k)_{k\in\omega}$ of closed λ-terms there is a graph model that satisfies $(\lambda x.xx)(\lambda x.xx)u_k = t_k$ for all k.

We apply these two results, which are corollaries of a unique theorem, to prove the existence of

(1) a finitely axiomatized λ-theory $\mathcal{L}$ such that the interval lattice constituted by the λ-theories extending $\mathcal{L}$ is distributive;

(2) a continuum of pairwise inconsistent graph theories (= λ-theories that can be realized as theories of graph models);

(3) a congruence distributive variety of combinatory algebras (lambda abstraction algebras, respectively).

Key words: Untyped λ-calculus, graph models, easy terms, Scott's continuous semantics, lattice of λ-theories, webbed models, combinatory algebras, lambda abstraction algebras.
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1 Introduction

Lambda theories are equational extensions of the untyped λ-calculus that are closed under derivation. They arise by syntactic and semantical considerations: a lambda theory may correspond to an operational semantics of the lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function (see e.g. [6,14]). The set of lambda-theories ordered by inclusion is naturally equipped with a structure of complete lattice (see Chapter 4 in [6]), where the meet of a family of lambda theories is their intersection, and the join is the least equivalence relation containing their union. The bottom element of this lattice is the minimal λ-theory λ_β, while the top element is the inconsistent λ-theory. The lattice of lambda theories, hereafter denoted by λT, has a continuum of elements (Barendregt's thesis, 1971, see [6, Ch. 6.2]). Since researchers have mainly focused their interest on a limited number of λ-theories, very little is known about the structure and equational theory of λT (see [38,45]).

Since syntactic techniques are usually difficult to use in the study of λ-theories, then semantical methods have been extensively investigated. Topology is at the center of the known approaches to giving models of the untyped lambda calculus; in particular, the first non syntactic model was found by Scott in 1969 in the category of complete lattices and Scott continuous functions. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1,6,14,43]. Scott's continuous semantics [48] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics introduced by Berry [15] and the strongly stable semantics introduced by Bucciarelli-Ehrhard [16] are a strengthening of the continuous semantics, introduced to capture the sequential features of lambda calculus. All these semantics are structurally and equationally rich in the sense that each of them is able to represent 2^ω distinct λ-theories [31,32,35], where *a semantics (or a class of models) represents a λ-theory T* if it contains a model $\mathcal{M}$ whose equational theory is exactly T.

Nevertheless, each of the above denotational semantics is *equationally incomplete*, in the sense that it is possible to produce λ-theories which are not represented in it. The problem of the equational incompleteness was positively solved by Honsell and Ronchi della Rocca [25] for the continuous semantics (who even produced a λ-theory induced by an operational semantics as a

Email addresses: berline@pps.jussieu.fr (C. Berline),
salibra@dsi.unive.it (A. Salibra).

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counter-example), by Bastonero and Gouy [24,10,11] for the stable semantics, and by Salibra [46,47] for the strongly stable semantics. As for λT, results on the structure of the set of λ-theories induced by a semantics are still rare, and there exist several longstanding very basic open questions (see [14] for a survey). In particular it is still open to know whether λ_β, the least λ-theory, could be the theory of a non-syntactic model in Scott's continuous semantics.

In this paper we concentrate on the semantics $\mathcal{G}$ of lambda calculus given in terms of graph models, *graph semantics* for short. These models, isolated in the seventies by Plotkin, Scott and Engeler [37] within the continuous semantics, have proved useful for giving proofs of consistency of extensions of lambda calculus and for studying operational features of lambda calculus (see [14]). For example, the simplest graph model, namely the Engeler and Plokin's model, has been used by Berline [14] to give concise proofs of the head-normalization theorem and of the left-normalization theorem of lambda calculus. Bucciarelli and Salibra [17,18] have recently proved that the set $\mathcal{G}T$, consisting of all the graph theories (= λ-theories that can be represented as theories of graph models), admits a least element, which is strictly greater than λ_β; in particular λ_β cannot be the theory of a graph model. These authors have also proved in [18] results about the "smaller" class $\mathcal{G}_sT$ of all sensible graph theories (a theory is *sensible* if all the unsolvable (or non-headnormalizable) terms are congruent). Smaller here only means that $\mathcal{G}_sT$ is strictly included in $\mathcal{G}T$, since from Kerth [33] [36] and David [21] it follows that $\mathcal{G}_sT$ also contains 2^ω λ-theories (however, the result is much harder to prove than for $\mathcal{G}T$).

Graph models are "webbed models" in the sense of [14]. Roughly speaking, a model of lambda calculus is a webbed model if it can be generated from a simpler structure, called its *web*. The web has a carrier set D and λ-terms are interpreted as (possibly special) subsets of D.

The reasons to concentrate on $\mathcal{G}$ are the following. First, $\mathcal{G}$ is, by far, the simplest class of models, in the sense that the webs of graph models are the simplest existing webs. Second, $\mathcal{G}T$ nevertheless contains a continuum of elements [31], so it is a rich class, in the sense that its cardinality is the maximal possible one, but it contains no extensional theories. Third, it is quite clear that the techniques and results for $\mathcal{G}$ and $\mathcal{G}T$ can often be transferred to other classes of webbed models, whether more general ones or belonging to other semantics.

It is a well known result by Jacopini [27] that Ω can be consistently equated to any closed term t of the (untyped) λ-calculus, where Ω is the paradigmatic unsolvable term $(\lambda x.xx)\ \lambda x.xx$ (this is called *the easiness of* Ω). Baeten and Boerboom gave in [5] the first semantical proof of this result by showing that for all closed terms t one can build a graph model satisfying the equation $\Omega = t$. This semantical result extends to other classes of models and to some other terms which share with Ω enough of its good will (cf. [14] for a survey of such results).

We recall that *a graph model* is, by definition, a reflexive Scott domain, which is generated by a web of the form (D, p), where D is an infinite set and $p : D^* \times D \to D$ is a total injection, D^* being the set of finite subsets of D (see Section 2.2). For brevity, we shall confuse graph models and their webs, but one should keep present in mind that the underlying domain of the model (D, p) is the full powerset $\mathcal{P}(D)$ ordered by inclusion, which is therefore independent of p. Starting from the set $D = N$ of natural numbers, Baeten and Boerboom build p by a method of "forcing", which, although much simpler than the forcing techniques used in set theory, is somewhat in the same spirit. In the Baeten and Boerboom setting, a *forcing condition* is a partial injection $q : D^* \times D \rightharpoonup D$ and "q forces $\alpha \in t$", abbreviated by $q \Vdash \alpha \in t$, means that for all total injections $p \supseteq q$ we have that α is in the interpretation of t in the model (D, p). The game is to build p as an increasing union of forcing conditions which successively put in the interpretation of Ω all the elements which are forced to be in the interpretation of t and exclude all the other ones.

In this paper we address the question of the "easiness" of sequences of λ-terms and of the λ-representability of sequences of continuous functions on $\mathcal{P}(D)$, where D is any countable infinite set. Given two sequences $\bar{t}$ and $\bar{v}$ of the same length, we denote by $\bar{t} = \bar{v}$ the set consisting of all the equations $t_k = v_k$. We say that a (possibly infinite) sequence $\bar{t}$ of closed λ-terms is

(1) *easy* if, for every other sequence $\bar{v}$ (of same length) of closed λ-terms, the set $\bar{t} = \bar{v}$ is consistent.
(2) *graph easy* if, for every other sequence $\bar{v}$ (of same length) of closed λ-terms, there is a graph model satisfying $\bar{t} = \bar{v}$. (Of course, "graph easy" implies "easy").
(3) *graph easy for functionals* if, for every sequence $\bar{f}$ (of same length) of Scott continuous functions on $\mathcal{P}(D)$, there exists a graph model (D, p) such that t_k represents f_k in the model for every k.

We generalize Baeten and Boerboom's method of forcing, and apply it to show that there is a sequence $(u_k)_{k\in\omega}$ of closed λ-terms satisfying the conditions expressed in the following two theorems.

Theorem 1. The sequence $(\Omega u_k : k < \omega)$ is graph easy.

Theorem 2. The sequence $(\Omega u_k : k < \omega)$ is graph easy for functionals.

The above theorems have clear incidence on our knowledge of λT and on all the subsets $\mathcal{C}T$ of λT, where $\mathcal{C}$ is any interesting class of models of λ-calculus in the continuous semantics which contains the class $\mathcal{G}$ of all graph-models, and $\mathcal{C}T$ is the set consisting of the λ-theories of the models in $\mathcal{C}$. For example, Theorem 1 implies the existence of 2^ω pairwise inconsistent graph theories (see Corollary 40), and hence it shows that $\mathcal{G}T$, and all the $\mathcal{C}T$ are as "wide" as possible.

The question of the λ-representability of (sequences of) continuous functions has not yet been addressed, as far as we know. Related works are only the very recent papers by Alessi et al. [3] and Dezani-Lusin [22], where the authors use intersection type systems (see [3,7,20]) for synthesizing filter models of lambda calculus in which the interpretation of a simple easy term can be any filter described by a continuous predicate. The notion of simple easiness was introduced by Alessi-Lusin [4] as a semantical tool to prove easiness. In fact, simple easiness implies easiness, while it is an open question whether easiness implies simple easiness. We should like to point out here that the main result in [3] (that the interpretation of a simple easy term can be any filter described by a continuous predicate) can be also interpreted as a generalization of Baeten and Boerboom's method of forcing via the use of intersection type systems. However, the framework we have developed in this paper is more direct and general than the one used in [3]. We illustrate this with two examples, concerning the λ-representability of the minimal fixed point operator and of the pair union/intersection.

One application of Theorem 2 that we develop here, concerns the lattice λT of all λ-theories ordered by inclusion. In particular, by instantiating Theorem 2 we get the distributivity of the interval sub-lattice $[\mathcal{L}) = \{S \in \lambda T : \mathcal{L} \subseteq S\}$ for a suitable finitely axiomatized λ-theory $\mathcal{L}$. The existence of a distributive interval sub-lattice of λT was an open question, which arises naturally since Salibra [45] proved that the lattice λT *does not satisfy the modularity law* (which is a weak form of distributivity), and since Lusin and Salibra [38] have shown, among other results on λT, the existence of an interval sub-lattice satisfying a restricted form of distributivity (called meet semi-distributivity) expressed in the form of a quasi-identity. The interest for interval sub-lattices of λT rather than arbitrary sub-lattices of λT is explained in Section 6.

Another application that we develop here concerns the variety (i.e., equational class) of lambda abstraction algebras (**LAA**'s) and the variety of combinatory algebras (**CA**'s). **LAA**'s were introduced by Pigozzi and Salibra in [40,41] as a purely algebraic theory of the untyped lambda calculus which nevertheless, and in contrast to Combinatory Logic, keeps all the functional intuitions. There is a close relationship between the lattice λT of lambda theories and the variety **LAA**. In [44] Salibra has shown that, for every variety of **LAA**'s, there exists exactly one lambda theory whose term algebra generates the variety. Thus, the properties of an arbitrary lambda theory can be studied by means of the variety of **LAA**'s generated by its term algebra. Many longstanding open problems of lambda calculus can be restated in terms of algebraic properties of varieties of **LAA**'s. For example, the open problem of the order-incompleteness of lambda calculus [49,47] asks for the existence of a lambda theory not arising as the equational theory of a non-trivially partially ordered model of lambda calculus. The order-incompleteness of lambda calculus is equivalent to the existence of an n-permutable variety of **LAA**'s for some natural number $n \geq 2$ (see the remark after Theorem 3.4 in [49]; the definition of n -permutability can be found in [39]). As a consequence of Theorem 2, we show that there exist a congruence distributive variety of **LAA**'s and a congruence distributive variety of **CA**'s. The existence of varieties of **LAA**'s or **CA**'s satisfying strong algebraic properties, such as n-permutability or congruence distributivity, was an open problem since Salibra [45] proved that the variety **LAA** is not congruence modular. The existence of a congruence distributive variety of **LAA**'s shows, against a common belief, that the lambda calculus satisfies strong algebraic properties. We express hope to positively solve in the future the order-incompleteness problem by showing the existence of an n-permutable variety of **LAA**'s.

The paper is organized as follows. Section 2 is a preliminary section containing the definition of a graph model and recalling the two possible ways of building graph models out of partial webs, namely "*canonical completion*" and "*completion by forcing*". This section also surveys the most recent results about the lambda theories represented by graph models. In Section 3 we introduce the generalized terms, which allow continuous functions of arbitrary arity as first-order function symbols, and we extend the classic notion of easiness of Ω to sequences of generalized terms. In Section 4 we show that Baeten and Boerboom's method works not only for forcing but more generally for weakly continuous operators, and also for generalized terms. This allows for the (optional) use of the (continuous) notion of partial interpretation as an alternative to forcing. We provide sequences of lambda terms of arbitrary finite length that are functionally graph easy. In Section 5 we introduce the technical notions of flattening and of an orthogonal system of representatives (osr); then we give examples of infinite sequences of terms that admit an osr. These technicalities are applied to get infinite sequences of terms that are functionally graph easy. In Section 6 it is shown that there exist a distributive interval sub-lattice of the lattice of lambda theories, a congruence distributive variety of lambda abstraction algebras, and a congruence distributive variety of combinatory algebras. Section 7 is devoted to conclusions and future work.

2 Preliminaries

2.1 Basic notations and conventions

2.1.1 λ-calculus

In this paper λ-*calculus* will always mean *untyped* λ-calculus, and we adopt the notations of [6]. In particular Λ and Λ° are, respectively, the set of λ-terms and of closed λ-terms. A λ-*theory* is a congruence on Λ (with respect to the operators of abstraction and application), which contains (α)- and (β)-conversion. There is a smallest λ-theory, denoted here by λ_β, which is nothing else than (α)- and (β)-conversion itself. λ-theories can of course also be seen as (specific) sets of equations between λ-terms. A λ-theory is *sensible* if all the unsolvable terms are congruent, and *semi-sensible* if no solvable term is equated to an unsolvable term (it is well known and easy to prove that sensible theories are semi-sensible). The smallest sensible λ-theory is traditionally denoted by $\mathcal{H}$.

2.1.2 The lattice of λ-theories

The set of lambda-theories ordered by inclusion is naturally equipped with a structure of complete lattice (see Chapter 4 in [6]), where the meet of a family of λ-theories is their intersection, and the join is the least equivalence relation containing their union (and hence a congruence too). The bottom element of this lattice is the minimal λ-theory λ_β, while the top element is the inconsistent λ-theory. The lattice of λ-theories will be denoted by λT. The sets of semi-sensible λ-theories and of sensible λ-theories constitute sub-lattices of λT.

2.1.3 Lattice identities

In the context of lattices an identity in the binary symbols $\{+, \cdot\}$ is called a *lattice identity*. ("+" is intended for *sup* and "·" for *inf*). A lattice identity is trivial if it holds in every lattice and nontrivial otherwise.

Given the lattice λT of λ-theories, we interpret the variables of a lattice identity as λ-theories, and for arbitrary λ-theories T and S we interpret $T + S$ as the lambda theory generated by the union of the two relations, and $T \cdot S$ as the intersection (as usual, we shall write TS for $T \cdot S$).

A *quasi-identity* is an implication with an equational conclusion and a finite number of equational premises. A quasi-identity in the language of lattices is satisfied by the lattice of lambda-theories if the conclusion of the quasi-identity is satisfied by all the lambda theories that satisfy the premises.

2.1.4 Sets

For every set S, S^* is the set of all finite subsets of S, while $\mathcal{P}(S)$ is the powerset of S and $S^{<\omega}$ (resp. S^ω, $S^{\leq\omega}$) is the set of all finite (resp. infinite, resp. finite or infinite) sequences of elements of S; $l(\bar{s})$ denotes the length of the sequence $\bar{s}$. When writing $g(\bar{x})$, where g is a function and $\bar{x}$ a sequence of elements of the domain of g, we shall of course always understand that $l(\bar{x})$ is the arity of g. Finally, for any function $f : S \to S'$ we shall define $f^+ : \mathcal{P}(S) \to \mathcal{P}(S')$ by $f^+(X) = \{ f(x) : x \in X \}$.

2.1.5 Scott's semantics

Cpos (complete partial orders) and *(Scott-) continuous functions* between cpos are defined in [6, Chapter I.2]. Given a set S, and an element $\bot$ not in S, *the flat cpo* $S_\bot$ is the order $(S \cup \{\bot\}, \leq)$ where $x \leq y$ if and only if $x = \bot$ or $x = y$. If C, C' are cpos then $[C \to C']$ denotes the cpo of all the continuous functions from C into C'. A *reflexive cpo* is a triple (C, A, λ) such that $\lambda \in [[C \to C] \to C]$ and $A \in [C \to [C \to C]]$ and $A \circ \lambda = id$. Reflexive cpos are models of λ-calculus in a way which is recalled in Section 2.2 (for more details see [6, Chapter V.5]). We are mainly (but not always) interested in cpos of the form $(\mathcal{P}(D), \subseteq)$, for some infinite countable set D. In this case $\subseteq$ will be understood as set inclusion. By "*a continuous function g of arity n on $\mathcal{P}(D)$*" we mean: $g \in [\mathcal{P}(D)^n \to \mathcal{P}(D)]$.

2.1.6 Further conventions

Greek letters $\alpha, \beta, ..$ will always denote elements of a set D specified by the context (from Section 3 on, D will be any fixed countable infinite set). Small Latin letters a, b, c will denote elements of D^*, and $\bar{a}, \bar{b}, \bar{c}...$ elements of $(D^*)^{<\omega}$. Also, (a, α) is the usual set-theoretical pair, and $(\bar{a}, \alpha)$ is defined by induction as follows: $(b\bar{c}, \alpha) =_{def} (b, (\bar{c}, \alpha))$.

2.1.7 Traces of continuous functions

A continuous function g on $\mathcal{P}(D)$, of any arity, is completely determined by its *trace*, which is defined by:

$$tr(g) =_{def} \{ (\bar{a}, \alpha) \ : \alpha \in g(\bar{a}) \} \tag{1}$$

The trace is, essentially, the relevant part of $graph(g)$, the graph of g; "essentially" refers to the fact that, if g is unary, say, then $tr(g) \subseteq D^* \times D \subseteq \mathcal{P}(D) \times D$, while $graph(g) \subseteq \mathcal{P}(D) \times \mathcal{P}(D)$.

2.2 Graph models

The class of graph models belongs to Scott's continuous semantics. Graph models owe their name to the fact that continuous functions are encoded in them via (a sufficient fragment of) their graphs, namely their traces.

As mentioned in the introduction, a *graph model* is a model of the untyped λ-calculus that is generated from a web (D, p) in a way that will be recalled below. Historically, the first graph model was Plotkin and Scott's P_ω (see e.g. [6]), which is also known in the literature as "the graph model". The simplest graph model, $\mathcal{E}$, was introduced soon afterwards, and independently, by Engeler [23] and Plotkin [42]. More examples can be found in [14].

For brevity we shall confuse the model and its web and so we define:

Definition 1 *A* graph model *is a pair (D, p), where D is an infinite set and $p : D^* \times D \to D$ is an injective total function.*

Such a pair will also be called a *total pair*. A total pair (D, p) generates a reflexive cpo $(\mathcal{P}(D), \lambda_p, A_p)$, and hence a model of λ-calculus. The continuous function $\lambda_p \in [[\mathcal{P}(D) \to \mathcal{P}(D)] \to \mathcal{P}(D)]$ is defined by $\lambda_p = p^+ \circ tr$, where tr is defined in (1) above, and p^+ is the straightforward extension of p to $\mathcal{P}(D^* \times D)$. This definition extends to continuous functions of arbitrary arity on $\mathcal{P}(D)$; in other words, for any such function g, we have:

$$\lambda_p(g) = \{\, p(\bar{a}, \alpha) \ : \ \alpha \in g(\bar{a}) \,\} \tag{2}$$

The left inverse $A_p \in [\mathcal{P}(D) \to [\mathcal{P}(D) \to \mathcal{P}(D)]]$ of λ_p (that allows one to interpret application in the model) is defined by:

$$A_p(X)(Y) = \{\alpha \in D : (\exists a \subseteq Y)\ p(a, \alpha) \in X\}.$$

where X, Y are arbitrary subsets of D. When no ambiguity will occur we write XY instead of $A_p(X)(Y)$. More generally, for $\bar{Y} = (Y_1, ..., Y_n)$, $X\bar{Y}$ is defined as $(..((XY_1)...)Y_n)$.

Let Env_D be the set of D-environments ρ mapping the set of the variables of λ-calculus into $\mathcal{P}(D)$. For $\rho \in Env_D$ and $X \in \mathcal{P}(D)$ let $\rho[x : X]$ be the environment which takes value X on x and coincides with ρ on all other variables. The interpretation $t^p : Env_D \to \mathcal{P}(D)$ of a λ-term t that is relative to (D, p) is defined by induction as follows:

- $x^p_\rho = \rho(x)$
- $(tu)^p_\rho = A_p(t^p_\rho)(u^p_\rho) = \{\alpha : (\exists a \subseteq u^p_\rho) \;\; p(a,\alpha) \in t^p_\rho\}$
- $(\lambda x.t)^p_\rho = \lambda_p(X \in \mathcal{P}(D) \mapsto t^p_{\rho[x:X]}) = \{\, p(a,\alpha) : \alpha \in t^p_{\rho[x:a]}\}$

Since t^p_ρ only depends on the value of ρ on the free variables of t, we just write t^p if t is closed. The following trivial example will be used in the Appendix.

Example 2 $(\lambda x.x)^p = \{\, p(a,\alpha) \,:\, \alpha \in a\,\}$

We turn now to the interpretation of $\Omega = \delta\delta$ in graph models, where $\delta = \lambda x.xx$. It is easy to check that the interpretation of Ω in P_ω and $\mathcal{E}$ is $\emptyset$; but, fortunately, this is not always the case. The following lemma gives a necessary condition and a sufficient condition for $\alpha \in D$ to be in the interpretation of Ω in (D,p), but, first, two remarks on the interpretation of δ are in order.

Remark 3 *(i)* $p(a,\alpha) \in \delta^p \Longleftrightarrow \alpha \in a\,a$.

(ii) $(\alpha \in XX$ *and* $X \subseteq \delta^p) \Longrightarrow \exists a \subseteq X\,(\,p(a,\alpha) \in X$ *and* $\alpha \in aa\,)$

Lemma 4 *[5] Let (D,p) be a graph model and $\alpha \in D$, then:*

(i) If $\alpha \in \Omega^p$, then there exists a such that $p(a,\alpha) \in a$.

(ii) If there exists $\beta \in D$ such that $p(\{\beta\},\alpha) = \beta$, then $\alpha \in \Omega^p$.

Proof. (i) If $\alpha \in \Omega^p = \delta^p\delta^p$ then:

$\exists a_1 \subseteq \delta^p\,(\,p(a_1,\alpha) \in \delta$ and $\alpha \in a_1a_1\,)$ (Remark 3 (ii) with $X = \delta^p$)

$\exists a_2 \subseteq a_1\,(\,p(a_2,\alpha) \in a_1$ and $\alpha \in a_2a_2\,)$ (Remark 3 (ii) with $X = a_1$)

...

$\exists a_{n+1} \subseteq a_n\,(\,p(a_{n+1},\alpha) \in a_n$ and $\alpha \in a_{n+1}a_{n+1}\,)$ (Rem. 3 (ii) with $X = a_n$)

Now, since a_1 is a finite set and the sequence a_n is decreasing, there is an n such that $a_n = a_{n+1}$; hence $p(a_n,\alpha) \in a_n$.

(ii) By definition of application, $p(\{\beta\},\alpha) = \beta$ implies $\alpha \in \{\beta\}\{\beta\}$, hence $p(\{\beta\},\alpha) \in \delta^p$ (Remark 3 (i)); hence $\beta \in \delta^p$ and $\alpha \in \delta^p\delta^p = \Omega^p$, since application is monotone with respect to inclusion. ■

A graph model (D,p) *satisfies* $t = u$, written $(D,p) \models t = u$, if $t^p = u^p$, or, equivalently, if $t^p_\rho = u^p_\rho$ for all environments ρ. The λ-theory $Th(D,p)$ induced by (D,p) is defined as

$$Th(D,p) = \{t = u \,:\, t,u \in \Lambda \text{ and } t^p = u^p\}.$$

A λ-theory induced by a graph model will be called a *graph theory*. A graph model is called *sensible* (rep. *semi-sensible*) if its theory is.

Notation 5 *$\mathcal{G}$ and $\mathcal{G}_s$ are the classes of graph models and sensible graph models respectively, while $\mathcal{G}T$, $\mathcal{G}_sT$ are respectively the classes of graph theories, and of sensible graph theories.*

2.3 Building graph models from partial pairs

There are other classes of models that can be generated from webs, but graph models are the models with the simplest (=less structured) webs, and the most easily feasible to deal with the interpretation of terms. Some of these classes belong to the continuous semantics and include $\mathcal{G}$, others belong to other semantics (for example the Berry/Girard stable semantics). These classes of webbed models, as well as the techniques for studying these models and their λ-theories are surveyed in [14].

For proving the consistency of extensions of λ-calculus, or more generally for studying the lattice λT of λ-theories one is interested in building models subject to specified equational or /and inequational constraints. The class of graph models offers a great wealth of models that are furthermore feasible. For this reason this is the first class of models to experiment with.

There are two known methods for building graph models, namely: by *forcing* or by *canonical completion*. Both methods can be extended to the other classes of webbed models (with more or less ease!), both methods consist in completing a partial pair into a total one, i.e. into a graph model.

In the setting of graph models, the *general definition of a partial pair* (see [14]), which allows one to cover both methods, is the following: A *partial pair* is a pair (A, q) where A is any set and q is a partial (possibly total) injection from $A^* \times A$ to A, written $q : A^* \times A \rightharpoonup A$. Examples of partial pairs are: all the graph models, and the empty pair $(\emptyset, \emptyset)$. For dealing only with the forcing method, a more restricted definition is sufficient, which we shall introduce later on.

The canonical completion method was, de facto, introduced by Plotkin and Engeler, since their model $\mathcal{E}$ is nothing else than the canonical completion of $(\emptyset, \emptyset)$. It was systematized by Longo for graph models [37], who proved in particular that the graph model P_ω is the canonical completion of $(\,\{0\}, \{(\emptyset, 0), 0\}\,)$, up to isomorphism. It was then used on a larger scale by Kerth, who used it, for example, to prove the existence of 2^ω distinct graph theories, and also transferred it to other semantics [33,32,35], and by Bucciarelli-Salibra in [17,18]. *Canonical* here refers to the fact that the graph model (D, p) is built in an inductive (and "canonical") way from the partial pair (A, q) we start with, and is completely determined by it. Furthermore, if the partial web is *positive* (in the sense of [14]) then (D, p) is sensible. Finally if one can apply the strong approximation theorem in the spirit of Hyland [26] and Wadsworth [51], which is the case for P_ω and $\mathcal{E}$, then $Th(D, p)$ is completely known: (D, p) equates two terms if and only if they have the same Böhm tree. For more details, and for the extension of the method to other classes of webbed models see [14].

The forcing method that we shall present below, originates in Baeten and Boerboom [5]. In the simpler presentation proposed by Zylberajch [52], it starts from a partial pair (D, p_0) [2], where D is an infinite countable set, and builds by induction a total $p : D^* \times D \to D$, hence a graph model (D, p). The inductive construction depends here not only on p_0 but also on the consistency problem we are interested in, and it heavily exploits the fact that the interpretation of Ω can be quite freely constrained. The method was generalized to other classes of webbed models in Jiang [29,30], Kerth [33,34], and such a generalization was used by Bastonero to build an extensional model of the continuous semantics, whose theory could be realized neither by a model in the stable semantics nor by a hypercoherence model (such models belong to the strongly stable semantics) [8,9]. It was also generalized to families of terms having a similar behavior as Ω by Zylberajch [52]. Note that, although $(D, \emptyset)$ is a positive web, no model built by completing $(D, \emptyset)$ by forcing will be sensible, and furthermore most of them will be clearly non-semi-sensible.

A last difference between both methods is that if we start with a recursive partial web, the canonical completion will build a recursive total web (hence a graph model that can be viewed as a reasonable intersection type system), whilst nontrivial forcing always creates a nonrecursive web.

2.3.1 The partial interpretation method

In this paper we highlight the fact that the key reason why constructions by forcing are possible is that forcing induces a family of "weakly continuous functions" (see Definition 10). We also introduce the notion of a partial interpretation of a term and note that it induces a family of Scott-continuous functions. Hence partial interpretations can be used as an alternative to forcing to build models by using a similar method; in particular, all the results proved in this paper can be obtained in both ways. The two notions are distinct (forcing is not continuous, as proved in the Appendix), but their use is essentially equivalent; in most cases it is a matter of taste, even if sometimes one or the other may appear to be more direct.

[2] As a matter of fact $p_0 = \emptyset$ in [5] and in all the other authors quoted, but here we shall need this more general setting.

2.3.2 Convention

From now on we shall only deal with the forcing-like methods, and hence we shall work with some fixed countable infinite set D.

3 Generalized terms and easy sequences: basic definitions

3.1 Generalized terms

In the next section we shall extend the classic notion of easiness of Ω to a more general class of terms, which allows continuous functions of arbitrary arity as first-order function symbols. All the results proved in the remaining sections, could be proved by working with pure λ-terms only (we first did it that way), but with more sophisticated tools. The interest of putting continuous functions in the language is that it allows for cleaner statements, simpler and more straightforward proofs, and, finally, that all the applications are evident corollaries. One may also wonder why adding genuine functions and not only elements of $\mathcal{P}(D)$ is necessary, since after all every function f is coded in each (D,p) by $\lambda_p(f) \in \mathcal{P}(D)$; once more, the answer is that it is much simpler to do it that way.

Definition 6 *The set* Λ_D *of the* generalized λ-terms *(relatively to D), or* gen–terms *is defined as the smallest set such that:*

(i) $V \subseteq \Lambda_D$, *where* V *is the set of variables of* Λ

(ii) $\mathcal{P}(D) \subseteq \Lambda_D$

(iii) if $t, u \in \Lambda_D$, *then* tu *is in* Λ_D

(iv) if $t \in \Lambda_D$ *and* $x \in V$ *then* $\lambda x.t \in \Lambda_D$

(v) if $f \in [\mathcal{P}(D)^n \to \mathcal{P}(D)]$, $1 \leq n$, *and* $\bar{t} \in \Lambda_D^n$, *then* $f(\bar{t}) \in \Lambda_D$.

Λ_D° *is defined as the set of closed gen-terms.*

Recall that Λ is the set of terms obtained by removing (ii) and (v) from the above definition. Hereafter the elements of Λ will be called *pure terms*.

Thus, f is not a gen-term, while $\lambda\bar{x}.f(\bar{x})$ is ($\lambda\bar{x}$ should be understood as $\lambda x_1 ... \lambda x_n$ if $\bar{x} = (x_1, ..., x_n)$). To be more formal we should have introduced one new symbol for each element of $\mathcal{P}(D) \cup \cup_{n\in\omega} [\mathcal{P}(D)^n \to \mathcal{P}(D)]$. A *redex* is a gen-term of the form $(\lambda x.t)u$, where t, u are gen-terms, and its reduct is defined as usual. We extend β-equivalence to gen-terms in a straightforward way: we just add to the usual rules the fact that it should be a congruence also with respect to the first-order functions, in other words $t_1 =_\beta t'_1, ., t_n =_\beta t'_n$ should imply $f(t_1, ..., t_n) =_\beta f(t'_1, ..., t'_n)$; in particular, no rule taking the evaluation of functions into account is given at the syntactic level. *The interpretation* t^p of the gen-term t in the graph model (D, p) is once more defined by induction on t. Cases (i), (iii) and (iv) are as in Section 2.2, while the interpretations of $X \subseteq D$ and $f(t_1, ..., t_n)$ are the obvious ones:

$$X^p_\rho = X; \quad f(t_1, ..., t_n)^p_\rho =_{def} f((t_1)^p_\rho, ..., (t_n)^p_\rho).$$

It is clear that this interpretation coincides with that of Section 2.2 for pure λ-terms. Satisfaction in (D, p) of an equation $t = t'$, for $t, t' \in \Lambda_D$ is defined as usual by $t^p_\rho = t'^p_\rho$ for all ρ. It is then clear that any graph model equates β-equivalent gen-terms and respects the behavior of the added functions, that is, if f is an n-ary continuous function which takes value $Y \in \mathcal{P}(D)$ on $X_1, ..., X_n \in \mathcal{P}(D)$, then all graph models on D will satisfy $f(X_1, ..., X_n) = Y$. Furthermore, it is easy to check that

$$(D, p) \vDash \lambda \bar{x}.f(\bar{x}) = \lambda_p(f),$$

where $\lambda_p(f)$ is the code of f in (D, p).

3.2 Partial interpretations

We extend the notion of interpretation of a gen-term from total pairs to partial pairs. In the sequel we shall always have the choice of using either total interpretations plus forcing, or partial interpretations (and no forcing).

Definition 7 *Let (D, q) be a partial pair. Given $t \in \Lambda_D$ we define t^q by induction on t :*

(i) $x^q_\rho = \rho(x)$

(ii) $X^q_\rho = X$

(iii) $(tu)^q_\rho = \{\, \alpha \in D : (\exists a \subseteq u^q_\rho)\ [(a, \alpha) \in dom(q) \wedge q(a, \alpha) \in t^q_\rho]\}$

(iv) $(\lambda x.t)^q_\rho = \{\, q(c, \gamma) \in D : (c, \gamma) \in dom(q) \wedge \gamma \in t^q_{\rho[x:c]}\,\}$

(v) $(f(t_1, ..., t_n))^q_\rho = f((t_1)^q_\rho, ..., (t_n)^q_\rho)$

We write t^q for t^q_ρ if $t \in \Lambda^\circ_D$ is a closed gen-term.

3.3 Easy sequences of terms

We now define easy sequences of terms.

Given two sequences $\bar{t}$ and $\bar{t}'$ of the same length, we denote by $\bar{t} = \bar{t}'$ the set consisting of all the equations $t_k = t'_k$.

Definition 8 *Let $\bar{s}$ be a (possibly infinite) sequence of closed pure λ-terms, then:*

(i) $\bar{s}$ is easy *if for all sequences $\bar{t} \in (\Lambda^\circ)^{l(\bar{s})}$ the set $\bar{t} = \bar{s}$ is consistent.*

(ii) $\bar{s}$ is graph easy *if for all sequences $\bar{t} \in (\Lambda^\circ)^{l(\bar{s})}$ there is a graph model satisfying $\bar{t} = \bar{s}$.*

(iii) $\bar{s}$ is functionally graph easy *if for all countable sets D and all sequences $\bar{t} \in (\Lambda^\circ_D)^{l(\bar{s})}$ there is a graph model of web D satisfying $\bar{t} = \bar{s}$.*

Of course $(iii) \Longrightarrow (ii) \Longrightarrow (i)$.

4 Baeten and Boerboom's proof revisited

4.1 Weakly continuous operators are the point

We observe here that Baeten and Boerboom's proof, in Zylberajch's style, works for any weakly continuous operator (instead of forcing) and that easiness with respect to all closed gen-terms holds.

Notation 9 *Q is the cpo of partial (including total) injections $q : D^* \times D \rightharpoonup D$, partially ordered by inclusion of their graphs.*

By "*a total p*" we shall always mean "an element of Q which is total" (equivalently: which is maximal). The domain and range of $q \in Q$ are denoted by $dom(q)$ and $range(q)$, we shall also confuse the partial injections and their graphs.

Given any set S and any function $H : Q \to \mathcal{P}(S)$, we shall use H_q for $H(q)$ when more convenient.

Definition 10 *A function $H : Q \to \mathcal{P}(S)$, where S is any countable infinite set, is* weakly continuous *if it is monotone with respect to inclusion and if furthermore, for all total $p \in Q$ and $\alpha \in H(p)$, there is a finite $q \subseteq p$ such that $\alpha \in H(q)$.*

Since we are working with a countable infinite D, the difference with continuity comes of course from the fact that there exist infinite elements of Q which are not total.

Theorem 11 *Given any weakly continuous function $H : Q \to \mathcal{P}(D)$, there is a total p such that $(D, p) \models \Omega = H_p$.*

Proof. We are going to build an increasing sequence of partial injective maps p_n, starting from p_0, and a sequence of elements $\alpha_n \in D \cup \{v\}$, where v is some new element, such that: $p =_{def} \cup p_n$ is a total injection (in fact a bijection), and $(D, p) \models \Omega = A = H_p$, where $A =_{def} \{\, \alpha_n \,:\, n \in \omega \,\} \cap D$.

We fix an enumeration of D, and an enumeration of $D^* \times D$.

We start from $p_0 = \emptyset$.

Assume that p_n and $\alpha_0, ..., \alpha_{n-1}$ have been built.

Let α_n be the first element of $H_{p_n} - \{\alpha_0, ..., \alpha_{n-1}\}$ if this set is non-empty, and v otherwise.

Let (b_n, δ_n) be the first element in $D^* \times D \;-\; dom(p_n)$ and γ_n be the first element in $D \;-\; (range(p_n) \cup b_n)$.

Case 1. $\alpha_n = v$ we let

$$p_{n+1} = p_n \cup \{\, ((b_n, \delta_n), \gamma_n) \,\}$$

Case 2. $\alpha_n \in D$ we let :

$$p_{n+1} = p_n \cup \{\, ((b_n, \delta_n), \gamma_n),\; ((\{\beta_n\}, \alpha_n)\,, \beta_n) \,\}$$

where β_n is the first element of D such that :

$$\begin{aligned} (\{\beta_n\}, \alpha_n) &\in D^* \times D \;-\; (dom(p_n) \cup \{(b_n, \delta_n)\}) \text{ and} \\ \beta_n &\in D \;-\; (range(p_n) \cup \{\gamma_n\}) \end{aligned}$$

It is clear that p_n is a strictly increasing sequence of well-defined partial injective maps and that $p = \cup p_n$ is total. It is also surjective since there are infinitely many elements of $D^* \times D$ of the form $(\emptyset, \delta)$, $\delta \in D$: these elements are successively introduced at steps, say, n_k (where the n_k form a strictly increasing sequence of integers), and are then given as image the first element in $D - range(p_{n_k})$, hence the k-th element of D will necessarily belong to $range(p_{n_k+1})$.

There remains to see that $(D, p) \models \Omega = A = H_p$.

$A \subseteq H_p$ follows from the definition of α_n and from the fact that $H_{p_n} \subseteq H_p$.

$H_p \subseteq A$: suppose $\gamma \in H_p$; then, since H is weakly continuous, $\gamma \in H_{p_m}$ for some m (and for all the larger ones). If $\gamma \notin A$ then, for all $n \geq m$, $\alpha_n \in D$ has smaller rank than γ in the enumeration of D, contradicting the fact that there is only a finite number of such elements.

$A \subseteq \Omega^p$: $\alpha_n \in \Omega^p$ follows immediately from the fact that $((\{\beta_n\}, \alpha_n), \beta_n) \in p_{n+1} \subseteq p$ and from Lemma 4 (ii).

$\Omega^p \subseteq A$: if $\varepsilon \in \Omega^p$ then there is an $a \in D^*$ such that $p(a, \varepsilon) \in a$ (by Lemma 4 (i)). Since $p = \cup p_n$ and because of the choices of the γ_n, this may only occur if ε is one of the α_n. ■

For showing the existence of infinite graph easy sequences we shall need to have available the following slight extension of Theorem 11.

Definition 12 *$p_0 \in Q$ is* free for Ω *if:*

(i) $D^* \times D - dom(p_0)$ and $D - range(p_0)$ are infinite, and

(ii) $(a, \alpha) \in dom(p_0)$ implies $p_0(a, \alpha) \notin a$.

Theorem 13 *If $H : Q \to \mathcal{P}(D)$ is weakly continuous and $p_0 \in Q$ is free for Ω, then there is a total $p \supseteq p_0$ such that $(D, p) \models \Omega = H_p$.*

Proof. Indeed, the proof of Theorem 11 only used that $\emptyset$ was free for Ω. ■

We now show that Theorem 11 and Theorem 13 can be applied to two different classes of functions $H : Q \to \mathcal{P}(D)$, respectively arising from forcing (as defined below) and partial interpretation (cf. Definition 7).

Definition 14 *(Forcing) For $t \in \Lambda_D^\circ$, $q \in Q$ and $\alpha \in D$, the abbreviation $q \Vdash \alpha \in t$ means that for all total injections $p \supseteq q$ we have that $(D, p) \models \alpha \in t^p$. Furthermore $q \Vdash X \subseteq t$ means that $q \Vdash \alpha \in t$ for all $\alpha \in X$.*

Thus, for p total, $p \Vdash \alpha \in t$ if and only if $\alpha \in t^p$. Moreover if $q_i \Vdash \alpha_i \in t$ for all $i \in I$ then $\cup q_i \Vdash \{\alpha_i : i \in I\} \subseteq t$.

Lemma 15 *For all $t \in \Lambda_D^\circ$ the function $F_t : Q \to \mathcal{P}(D)$ defined by $F_t(q) = \{\alpha \in D : q \Vdash \alpha \in t\}$ is weakly continuous, and we have $F_t(p) = t^p$ for each total p.*

Proof. The proof of the weak continuity of F_t is a straightforward induction on the complexity of the closed gen-term t; we detail it anyway.

If t is an element X of $\mathcal{P}(D)$ then F_t is the constant function with value X.

Let now $p \in Q$ be total.

If $t = uv$ and $\alpha \in t^p$, then there exists $a \subseteq v^p$ such that $p(a, \alpha) \in u^p$. Choose such an a and let $\gamma = p(a, \alpha)$. By induction hypothesis there is a finite $q \subseteq p$ such that $q \Vdash a \subseteq v$ and a finite $r \subseteq p$ such that $r \Vdash \gamma \in u$; then it is clear that $q \cup r \cup \{((a, \alpha), \gamma)\} \Vdash \alpha \in t$.

If $t = \lambda x.u$ and $\alpha \in t^p$ then there is a unique pair (b, β) such that $\alpha = p(b, \beta)$ and $\beta \in u[x : b]^p$. By induction hypothesis there is a finite $q \subseteq p$ such that $q \Vdash \beta \in u[x : b]$; then it is clear that $q \cup \{((b, \beta), \alpha)\} \Vdash \alpha \in t$.

If $t = f(t_1, ..., t_n)$ and $\alpha \in f(t_1, ..., t_n)^p = f(t_1^p, ..., t_n^p)$, then from the continuity of f it follows the existence of finite $b_1 \subseteq t_1^p$ ... $b_n \subseteq t_n^p$ such that $\alpha \in f(b_1, ..., b_n)$. Since the b_i's are finite and the t_i's are of lower complexity than t, there are finite $q_1, ..., q_n \subseteq p$ such that $q_i \Vdash b_i \subseteq t_i$ for all i; then we clearly have $q \Vdash b_i \subseteq t_i$ for all i, where $q =_{def} \cup\{ q_i : i \leq n \}$. The conclusion $q \Vdash \alpha \in f(t_1, ..., t_n)$ follows from $\alpha \in f(b_1, ..., b_n)$, $q \Vdash b_i \subseteq t_i$ for all i , and the monotonicity of f. ■

We note that the function F_t defined in the above lemma is not continuous as shown in Appendix.

Lemma 16 *For all $t \in \Lambda_D^\circ$, the function $I_t : Q \to \mathcal{P}(D)$ defined by $I_t(q) = t^q$ is continuous (where t^q is the interpretation of the gen-term t in the partial pair (D, q)).*

Proof. The proof of the continuity of I_t is a straightforward induction on the complexity of the closed gen-term t. ■

4.2 Easy terms

In this section we show that the λ-term Ω is functionally easy. Then every continuous function on $\mathcal{P}(D)$ is λ-represented by Ω.

Theorem 17 *Ω is functionally graph easy, that is, for all closed gen-terms $t \in \Lambda_D^\circ$ there is a p such that $(D, p) \models \Omega = t$.*

Proof. It is enough to apply Theorem 11 either to the weakly continuous function F_t defined in Lemma 15 or to the continuous function I_t defined in

Lemma 16. ■

Let us give now a few applications of this result.

The following is the classic result by Baeten and Boerboom.

Corollary 18 *[5] Ω is graph easy, that is, for all closed pure terms $t \in \Lambda^\circ$ there is a graph model (D, p) such that $(D, p) \models \Omega = t$.*

Definition 19*(i) A continuous function f on $\mathcal{P}(D)$ is* λ-represented *by a pure term $t \in \Lambda^\circ$ in a graph model (D, p) if $(D, p) \models t = \lambda\bar{x}.f(\bar{x})$.*

(ii) A (possibly infinite) sequence $\bar{f}$ of continuous functions is λ-represented by $\bar{t} \in (\Lambda^\circ)^{l(\bar{f})}$ in (D, p) if (D, p) satisfies $t_k = \lambda\bar{x}.f_k(\bar{x})$ for all k.

The preceding definition would of course trivialize if the term t in (i) and the sequence $\bar{t}$ of terms in (ii) were asked to be gen-terms.

Corollary 20 *Each continuous function f on $\mathcal{P}(D)$ is λ-represented by Ω in some graph model.*

Proof. From Theorem 17 there is a graph model satisfying $\Omega = \lambda\bar{x}.f(\bar{x})$, which implies clearly that Ω represents f in this graph model. ■

The least fixed point operator on a cpo $\mathcal{C}$ is the continuous function $L \in [[\mathcal{C} \to \mathcal{C}] \to \mathcal{C}]$ defined by $L(f) = \cup_{n\in\omega} f^n(\bot)$, where $\bot$ is the least element of $\mathcal{C}$. Using the formalism of intersection type systems and filter models, Alessi and al. [3] proved that there exists a reflexive cpo where L is represented by Ω, in the sense that the least fixed point operator of the underlying cpo is the interpretation of Ω in the model. It is hence interesting to note that we can get this result in a more economical way, and with a simpler model.

Corollary 21 *There is a graph model where Ω represents L.*

Proof. The smallest element of the cpo $\mathcal{P}(D)$ is $\bot = \emptyset$. By Corollary 20 there is a graph model (D, p) where Ω represents the unary continuous function defined on $\mathcal{P}(D)$ by: $h(X) = \cup_{n\in\omega} X^n\emptyset$ (where e.g. $X^2\emptyset$ means $X(X\emptyset)$). But, then, for all unary continuous functions f, we automatically have: $[\Omega(\lambda x.f(x))]^p = \cup_{n\in\omega} f^n(\emptyset)$. Thus Ω represents L in (D, p). ■

We now look for easy sequences of terms.

The existence of finite easy sequences of pure terms could be proved without using generalized terms (using variations of the tools built in Section 5), and it will also show up as a particular case of a result proved in Section 5.3. But in the present setting, which allows one to use the standard tricks in λ-calculus, it appears as a direct corollary of Theorem 17.

Theorem 22 *For each $n \in \omega$ there is a sequence $\bar{u} \in (\Lambda^\circ)^n$ of pure terms such that $(\Omega u_k)_{k\leq n}$ is a functionally graph easy sequence.*

Proof. We only treat the case $n = 2$, and claim that the two projections $T = \lambda x.\lambda y.x$ and $F = \lambda x.\lambda y.y$ work. Using Theorem 17, for all closed gen-terms t_1 and t_2, we get a graph model satisfying $\Omega = \lambda z.zt_1t_2$. Then it is clear that in the same graph model we have that $\Omega T = t_1$ and $\Omega F = t_2$. ■

Proposition 23 *For each $n \in \omega$ there is a sequence $\bar{u} \in (\Lambda^\circ)^n$ of pure terms such that each sequence $\bar{f}$ of continuous functions on $\mathcal{P}(D)$ is λ -represented by $(\Omega u_k)_{k\leq n}$ in some graph model over D.*

Proof. By the above theorem. ■

Corollary 24 *The pair $(\cup, \cap)$ consisting of union and intersection on $\mathcal{P}(D)$, is λ-represented by ΩT and ΩF.*

Proof. Immediate consequence of the preceding corollary since $\cup, \cap$ are continuous functions (continuity of $\cap$ follows from the fact that $\mathcal{P}(D)$ is a distributive lattice). ■

Interesting applications of this result to the structure of the lattice of lambda theories are shown in Section 6.1.

In [22] Dezani and Lusin have shown the existence of a filter model of lambda calculus representing union, and the existence of a filter model representing intersection could be derived along the same way; but the existence of a unique model for both, as we have shown in Corollary 24, was left open.

5 Infinite easy sequences of terms

In Section 4.3 we have proved the existence of functionally graph easy sequences of every finite length (Theorem 22). In this section we introduce the two technical notions of *flattening* and *osr,* which give us another way to obtain easy sequences and, in particular, to prove the existence of *infinite* easy sequences.

5.1 *Flattenings*

Notation 25 *Let* $E =_{def} \bigcup_{n\geq 0} E_n$ *where the* E_n *are defined by:* $E_0 =_{def} D$ *and* $E_{n+1} =_{def} (E_n^* \times E_n) \cup E_n$.

It is easy to check that $E_{n+1} = (E_n^* \times E_n) \cup D$, and also that $tr(g) \subseteq E_{n+1}$ for every continuous function $g : D^n \to D$.

It is also easy to check that each partial injection $q \in Q$ extends to a partial function $f_q : E \to D$, satisfying the following properties:

(i) $f_q(x) =_{def} x$ if $x \in D$;
(ii) $f_q(e, \varepsilon) =_{def} q(f_q^+(e), f_q(\varepsilon))$ if $e \cup \{\varepsilon\} \subseteq dom(f_q)$ and $(f_q^+(e), f_q(\varepsilon)) \in dom(q)$, undefined otherwise, where: $f_q^+(e) =_{def} \{f_q(x) : x \in e\}$.

Thus $f_q(x) \neq \perp$ if and only if q is hereditarily defined on all the internal components of x. We now define a total function $f_q^\bullet : \mathcal{P}(E) \to \mathcal{P}(D)$.

Notation 26 $f_q^\bullet(G) =_{def} \{ f_q(x) \ : / \, x \in G \cap dom(f_q) \}$, *for any* $G \subseteq E$.

Definition 27 *For* $x \in E$ *and* $G \subseteq E$ *we shall respectively call* $f_q(x)$ *and* $f_q^\bullet(G)$ the q-flattening of x *and* G.

Example 28 *For all* $q \in Q$ *and* $G \subseteq D$ *we have* $f_q^\bullet(G) = G$.

In particular, for all $t \in \Lambda_D^\circ$ *we have* $f_q^\bullet(t^q) = t^q$.

We see more sophisticated examples below (but these ones are relevant for our purpose).

Lemma 29 *The function* $f : E \times Q \to D_\perp$, *defined by* $f(x, q) = f_q(x)$ *if* $x \in dom(f_q)$ *and* $\perp$ *otherwise, is continuous with respect to* q.

Proof. Since $D_\perp$ is flat, continuity is here equivalent to saying that:

(i) If $q \subseteq q'$ and $f_q(x)$ is defined then $f_{q'}(x)$ is defined and $f_{q'}(x) = f_q(x)$.

(ii) If q is the union of an increasing sequence $(q_n)_{n\in\omega}$ then there exists n such that $f_q(x) = f_{q_n}(x)$.

The first point is trivial, and the second easily follows from the fact that the computation of $f_q(x)$ only requires a finite part of the graph of q. ■

Lemma 30 *The function $f^\bullet : \mathcal{P}(E) \times Q \to \mathcal{P}(D)$, defined by $f^\bullet(G,q) = f^\bullet_q(G)$, is continuous. It is even additive with respect to the first component G (i.e. commutes with all unions).*

Proof. A binary function is continuous iff it is continuous componentwise. Continuity with respect to q easily follows from the preceding lemma, and additivity with respect to G is trivial. ■

Lemma 31 *Let $G : Q \to \mathcal{P}(E)$ and let $H : Q \to \mathcal{P}(D)$ be defined by $H_q = f^\bullet_q(G_q)$ for all $q \in Q$, then:*

(i) If G is continuous then H is also continuous,

(ii) If G is weakly continuous then H is weakly continuous.

Proof. Since $H =_{def} f^\bullet \circ (G \times id)$ the monotonicity of H follows from the monotonicity of G, and similarly for continuity, using the continuity of $f^\bullet$ (previous Lemma). Suppose now that G is only weakly continuous and suppose $\alpha \in H_p$, where p is a total injection. By definition of H there is an $x \in G(p) \subseteq E$ such that $\alpha = f_p(x)$. By Lemma 29 there is a finite $s \subseteq p$ such that $\alpha = f_s(x)$; furthermore, since G is weakly continuous there is a finite $r \subseteq p$ such that $x \in G(r)$. Then, if $q = r \cup s$ we have $\alpha \in H_q = f^\bullet_q(G_q)$, by monotonicity of $f^\bullet$ and G. ■

5.2 Orthogonal system of representatives

Definition 32 *Let $\bar{u} \in (\Lambda^\circ)^{\leq\omega}$ be a sequence of closed pure terms and $p_0 \in Q$ be free for Ω. The sequence $\bar{\varepsilon} \in E^{\leq\omega}$ is an* orthogonal system of representatives *(*an osr*, for short) for $\bar{u}$* modulo p_0 *if: $l(\bar{\varepsilon}) = l(\bar{u})$ and, for all $j,k \leq l(\bar{u})$ and all total $p \supseteq p_0$, we have $f_p(\varepsilon_k) \in u^p_j$ iff $k = j$.*

Definition 33 *The sequence $\bar{u}$* admits an osr *if there exist $\bar{\varepsilon}$, p_0 such that $\bar{\varepsilon}$ is an osr for $\bar{u}$ modulo p_0.*

Of course not all sequences of pure terms admit an *osr*. Simple examples of finite and infinite sequences of terms admitting an osr will be given in Lemma 36 below. It is clear that any subsequence or permutation of a sequence admitting an osr also admits an osr. The interest of the notion of osr comes from the fact that, for all $\bar{u} \in \Lambda^{\leq\omega}$ admitting an osr, the sequence $(\Omega u_k)_{k\leq l(\bar{u})}$ is functionally graph easy (Theorem 37 in the next section).

Notation 34 *$\pi_{n,k} =_{def} \lambda x_1...\lambda x_n.x_k \in \Lambda$, for $1 \leq k \leq n$.*

$\pi'_k =_{def} \pi_{k+1,k+1}$, for $k \in \omega$.

Notation 35 *Let α be some fixed element of D.*

$\varepsilon_{n,k} =_{def} (\emptyset^{k-1}\{\alpha\}\emptyset^{n-k}, \alpha) \in E$, for $1 \leq k \leq n$.

$\varepsilon'_k =_{def} \varepsilon_{k+1,k+1} = (\emptyset^k\{\alpha\}, \alpha) \in E$, for $k \in \omega$.

(where $(\bar{a}_1\bar{a}_2...\bar{a}_n, \alpha)$ is defined as $(\bar{a}, \alpha)$ where $\bar{a}$ is the concatenation of $\bar{a}_1, \bar{a}_2, ..., \bar{a}_n$).

Lemma 36 *(i) $(\varepsilon_{n,k})_{k\leq n}$ is an osr for $(\pi_{n,k})_{k\leq n}$ modulo $\emptyset$.*

(ii) $(\varepsilon'_k)_{k\in\omega}$ is an osr for $(\pi'_k)_{k\in\omega}$ modulo $p_0 = \{((\emptyset, \alpha), \alpha)\}$.

Proof. (i) is clear, by definition of $\pi^p_{n,k}$.

(ii) Suppose that p is total and $p(\emptyset, \alpha) = \alpha$. Then it is easy to check successively that (D, p) satisfies:

(1) $\alpha \notin (\lambda x.x)^p$.
(2) $\{\alpha\}\emptyset^n = \{\alpha\}$ for all $n \geqslant 0$.
(3) $\{\alpha\} = \{\, p(\{\alpha\}, \alpha)\,\}\{\alpha\}$.
(4) $\forall n \geqslant 0\ (\alpha \notin \pi'_n)$
(this follows from 1,2, and the monotonicity of application).
(5) $\forall n \geqslant 1\ (\ p(\{\alpha\}, \alpha) \notin \pi'_n\ )$
(this follows from 1,3, and the monotonicity of application).
(6) $f_p(\emptyset^m, \{\alpha\}, \alpha) \in \pi'_n$ iff $m = n$.
(the case $n > m$ is excluded by 5 and the monotonicity of application, and $m > n$ contradicts 4).

■

5.3 Infinite easy sequences of terms

Theorem 37 *For all $\bar{u} \in (\Lambda^\circ)^{\leq\omega}$ admitting an osr, the sequence $(\Omega u_k)_{k\leq l(\bar{u})}$ is functionally graph easy (and then easy).*

Proof. Let $\bar{\varepsilon}$, p_0 be such that $\bar{\varepsilon}$ is an osr for $\bar{u}$ modulo p_0, and let $\bar{t} \in \Lambda_D^{l(\bar{u})}$. For all $q \in Q$, let $G_q = \{\, (\{\varepsilon_k\}, \alpha) \,/\, 1 \leq k \leq l(\bar{t}),\ \alpha \in t^q_k\} \subseteq E$. Since G_q is essentially the disjoint union of the subsets t^q_k of D, which are continuous wrt q, the function $G : Q \to \mathcal{P}(E)$ is continuous. From Lemma 31 the function F defined by $F(q) = f^\bullet_q(G_q)$ is also continuous. From Theorem 13 there is a total $p \supseteq p_0$ such that $\Omega^p = f_p^\bullet(G_p)$. Now, since $\bar{\varepsilon}$ is an osr for $\bar{u}$ relatively to p_0 we have that $f_p^\bullet(G_p).u_k^p = f_p^\bullet(G_p).\{f_p(\varepsilon_k)\} = f_p^\bullet(t_k^p) = t_k^p$ (by definition of application in (D,p)), thus $(\Omega u_k)^p = t_k^p$, and $(D,p) \models \Omega u_k = t_k$ for all k.

The alternative proof using forcing works in a similar way, using case (ii) of Lemma 31. ■

Recall that the pure λ-terms π'_k are defined in Notation 34.

Corollary 38 *The infinite sequence $(\Omega\pi'_k)_{k\geq 0}$ is functionally graph easy.*

Corollary 39 *For all infinite sequences $\bar{g}$ of continuous functions on $\mathcal{P}(D)$, there is a graph model (D,p) such that for all k we have: $(D,p) \vDash \Omega\pi'_k = \lambda\bar{x}.g_k(\bar{x})$, where $l(\bar{x})$ is the arity of g_k.*

In the next corollary we show that there exist 2^ω-pairwise inconsistent graph theories, so that $\mathcal{G}T$ is as "wide" as possible. This improves Kerth's result [31] stating the existence of 2^ω-graph theories.

Before stating the corollary, it is worth noting that from Kerth's proof one can already derive the existence of 2^ω-pairwise *incomparable* graph-theories (recall that two λ-theories T and S are incomparable if neither $T \subseteq S$ nor $S \subseteq T$). Indeed Kerth produces families of graph models $(G_W)_{W\in\mathcal{P}(S)}$ and of sets of equations $(R_W)_{W\in\mathcal{P}(S)}$, where S is an infinite countable set, such that $R_W \subseteq R_{W'}$ if and only if $W \subseteq W'$ and G_W satisfies all the equations of R_W and no equation of $R_{W'} - R_W$. From the fact that $(\mathcal{P}(S), \subseteq)$ contains 2^ω pairwise incomparable sets (this is easy to prove), we deduce immediately that there are 2^ω pairwise incomparable graph theories. Note that the G_W are built as canonical completions of partial pairs, and that Kerth's proofs (see [31] and [33]), even if not difficult, required some nontrivial observations, and some computations, which is not the case here (once generalized forcing is established).

Corollary 40 *There exist 2^ω pairwise inconsistent graph theories.*

Proof. Let $\bar{s}$ be an infinite graph easy sequence and let $\bar{t}$ be the sequence of Church integers. For any permutation σ on usual integers let p_σ be such that $(D,p_\sigma) \vDash s_k = t_{\sigma(k)}$ for all k. It is clear that the graph models (D,p_σ) are non equationally equivalent, and that their theories are pairwise inconsistent. ■

Kerth and David's result which asserts the existence of 2^ω sensible graph theories, mentioned in the introduction, is out of the scope of our techniques.

6 Applications

In this section we show that there exist

(1) a finitely axiomatized λ-theory $\mathcal{L}$ whose interval sub-lattice $[\mathcal{L}) = \{S \in \lambda T : \mathcal{L} \subseteq S\}$ has a continuum of elements and is a distributive sub-lattice of the lattice of λ-theories;
(2) a congruence distributive variety of lambda abstraction algebras;
(3) a congruence distributive variety of combinatory algebras.

6.1 The Lattice of λ-Theories

The set of the λ-theories ordered by inclusion is naturally equipped with a structure of complete lattice (see Section 2). The lattice λT of λ-theories has a very rich and complex structure. For example, Visser [50] has shown in first eighties that every countable partially ordered set embeds into λT by an order-preserving map, and that every interval of λT, whose bounds are recursively enumerable lambda theories, has a continuum of elements.

Lusin-Salibra [38] and Salibra [45] have employed results and techniques from universal algebra, in particular commutator theory and the theory of Mal'cev conditions, to obtain some results characterizing the structure and the equational theory of the lattice of lambda theories. Very little had previously been known about the equational theory of this lattice.

We briefly outline the approach developed in [38]. Consider the absolutely free algebra of pure terms:

$$\mathbf{\Lambda} := (\Lambda, \cdot^{\mathbf{\Lambda}}, \lambda x^{\mathbf{\Lambda}}, x^{\mathbf{\Lambda}})_{x \in Va}, \tag{3}$$

where Λ is the set of pure terms over an infinite set Va of variables and, for all $M, N \in \Lambda$,

$$M \cdot^{\mathbf{\Lambda}} N = (MN); \quad \lambda x^{\mathbf{\Lambda}}(M) = (\lambda x.M); \quad x^{\mathbf{\Lambda}} = x.$$

An equivalence relation T over the set Λ of pure terms is a lambda theory if, and only if, it is a congruence over $\mathbf{\Lambda}$ including (α) and (β)-conversion. For every lambda theory T, the congruence lattice of the term algebra $\mathbf{\Lambda}^T$, the quotient of $\mathbf{\Lambda}$ by T, is isomorphic to the interval sub-lattice $[T) = \{S : T \subseteq S\}$ of the lattice of the lambda theories. In particular, the isomorphism between the lattice λT and the congruence lattice of $\mathbf{\Lambda}^{\lambda_\beta}$ is the starting point for studying the structure of λT by universal algebraic methods.

It was shown by Salibra [45] that the lattice λT is not modular, i.e., it does not satisfy the following modular law (that is a weakened form of distributivity):

$$T(S+TR)=TS+TR,$$

while Lusin and Salibra [38] have shown that λT satisfies interesting quasi-identities in the language of bounded lattices. For example, the following quasi-identity holds in the lattice λT:

$$S+T=1,\ SG=TG\ \rightarrow\ G=GS=GT,$$

where 1 is the inconsistent lambda theory.

The same authors have shown in [38] that there exists a λ-theory $\mathcal{J}$, whose interval sub-lattice $[\mathcal{J})=\{\mathcal{S}\in\lambda\mathcal{T}:\mathcal{J}\subseteq\mathcal{S}\}$ satisfies the following restricted form of distributivity (called meet semidistributivity)

$$TR=TS\ \rightarrow\ TR=T(R+S),$$

and a nontrivial identity in the language of lattices enriched by the composition of binary relations.

In [38] it was conjectured that the lattice λT does not satisfy any nontrivial lattice identity. To support this conjecture the authors have shown in [38] that, for every nontrivial lattice identity e, there exists a natural number n such that e fails in the lattice of lambda theories in a language of λ-terms with n constants. We can relax the above conjecture by asking whether there exists an infinite interval sub-lattice of λT satisfying interesting lattice identities. In this section we show that there exists an infinite distributive interval sub-lattice of λT.

There are strong motivations to be interested in interval sub-lattices of λT rather than arbitrary sub-lattices of λT. The study of interval sub-lattices allows us to apply algebraic methods to lambda calculus. In the remaining part of this section we provide an interesting example of this connection. First we introduce a finitely axiomatized λ-theory $\mathcal{L}$, whose consistency is obtained by using the methods introduced in the previous sections. The equations defining $\mathcal{L}$, which make the lambda calculus consistent with the lattice operations of join and meet, are used to define lattice term operations on the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$, the quotient of $\mathbf{\Lambda}$ by the congruence $\mathcal{L}$. Since every algebra admitting lattice term operations is congruence distributive, then we immediately get that the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ is congruence distributive. The conclusion, that the interval sub-lattice $[\mathcal{L})$ is distributive, follows because $[\mathcal{L})$ is isomorphic to the congruence lattice of the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$. As it will be pointed out in the next section, algebraic properties of interval sub-lattices of λT are related in many cases to the existence of varieties of lambda abstraction algebras (combinatory algebras, respectively) satisfying strong algebraic properties.

Recall that $T =_{def} \lambda xy.x$ and $F =_{def} \lambda xy.y$.

Lemma 41 *The lambda theory $\mathcal{L}$, axiomatized by*

(1) $\Omega Txx = x; \quad \Omega Fxx = x.$
(2) $\Omega Txy = \Omega Tyx; \quad \Omega Fxy = \Omega Fyx.$
(3) $\Omega Tx(\Omega Tyz) = \Omega T(\Omega Txy)z; \quad \Omega Fx(\Omega Fyz) = \Omega F(\Omega Fxy)z.$
(4) $\Omega Tx(\Omega Fxy) = x; \quad \Omega Fx(\Omega Txy) = x.$
(5) $\Omega Tx(\Omega Fyz) = \Omega F(\Omega Txy)(\Omega Txz); \quad \Omega Fx(\Omega Tyz) = \Omega T(\Omega Fxy)(\Omega Fxz).$

is consistent.

Proof. From Corollary 24 it follows that there exists a graph model (D, p), where the set-theoretical union and intersection are λ-represented by ΩT and ΩF. Since $(\mathcal{P}(D), \cup, \cap)$ is a distributive lattice, then the λ-theory $\mathcal{L}$ is contained in the theory $Th(D, p)$ of (D, p). ■

Lemma 42 *The congruence lattice of the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ is isomorphic to the interval sub-lattice $[\mathcal{L}) = \{T : \mathcal{L} \subseteq T\}$ of the lattice of lambda theories.*

Proof. A λ-theory T satisfying the condition $\mathcal{L} \subseteq T$ can be interpreted as a congruence $\equiv_T$ on the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ (see [38]): for every $\tau, \sigma \in \mathbf{\Lambda}^{\mathcal{L}}$, $\tau \equiv_T \sigma$ if, and only if, there exist pure terms $t \in \tau$ and $u \in \sigma$ such that $T \vdash t = u$ (recall that τ, σ are equivalence classes of pure terms). ■

Lemma 43 *Let* $\mathbf{A}$ *be any algebra. If* $\mathbf{A}$ *admits two binary term operations satisfying the axioms of a distributive lattice, then the congruence lattice of* $\mathbf{A}$ *is distributive.*

Proof. Let A be the universe of the algebra $\mathbf{A}$, and $+, \cdot$ be the binary term operations of $\mathbf{A}$ satisfying the axioms of a distributive lattice. It is well known that the congruence lattice of every distributive lattice is distributive (see [39]), so that the congruence lattice of the algebra $(A, +, \cdot)$ is distributive. We get the conclusion if we show that the congruence lattice of $\mathbf{A}$ is a sub-lattice of the distributive congruence lattice of the algebra $(A, +, \cdot)$. First every congruence over $\mathbf{A}$ is a congruence over $(A, +, \cdot)$, because "+" and "·" are term operations. This implies that the set of congruences over $\mathbf{A}$ is a subset of the set of congruences over $(A, +, \cdot)$. The conclusion is now immediate because the meet and the join in both congruence lattices are the same: they are defined set-theoretically as intersection and least equivalence relation. ■

As a matter of notation, for every lambda theory T, we denote by $[t]_T$ the equivalence class of the pure terms u such that $T \vdash t = u$.

Theorem 44 *The interval sub-lattice $[\mathcal{L}) = \{S \in \lambda T : \mathcal{L} \subseteq S\}$ has a contin-*

uum of elements and is a distributive sub-lattice of the lattice of λ-theories.

Proof. The interval $[\mathcal{L})$ has a continuum of elements by Prop. 17.1.9 and Thm. 17.1.10 in Barendregt's book [6]. We now show that the interval $[\mathcal{L})$ is distributive. By Lemma 42 it is sufficient to prove that the congruence lattice of the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ is distributive. Define the following term operations over $\mathbf{\Lambda}^{\mathcal{L}}$, for every $\tau, \sigma \in \mathbf{\Lambda}^{\mathcal{L}}$:

$$\tau + \sigma = [\Omega T t s]_{\mathcal{L}}; \quad \tau \cdot \sigma = [\Omega F t s]_{\mathcal{L}}, \quad \text{for some } t \in \tau \text{ and } s \in \sigma. \tag{4}$$

Then it is easy to verify by using the axioms defining $\mathcal{L}$ that the term operations "+" and "·" satisfy the axioms of a distributive lattice. For example, the identity $\Omega T x (\Omega F x y) = x$, specified in Lemma 41(4), corresponds to the absorption law $x + (x \cdot y) = x$, while the identity $\Omega T x(\Omega F y z) = \Omega F(\Omega T x y)(\Omega T x z)$, specified in Lemma 41(5), corresponds to the distributive law $x + (y \cdot z) = (x + y) \cdot (x + z)$. Then the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ satisfies the hypothesis of Lemma 43, so that it admits a distributive congruence lattice. ■

6.2 Lambda abstraction algebras and combinatory algebras

Another application of the main results of the paper that we develop here concerns lambda abstraction algebras and combinatory algebras. Lambda abstraction algebras (**LAA**'s) were introduced by Pigozzi and Salibra in [40,41] as a purely algebraic theory of the untyped lambda calculus alternative to Curry's highly combinatorial models. Combinatory algebras (**CA**'s) and lambda abstraction algebras are both defined by universally quantified equations and thus form varieties in the universal algebraic sense. There are important differences however that result in theories of very different character. Functional application is taken as a fundamental operation in both **CA**'s and **LAA**'s. Lambda (i.e., functional) abstraction is also fundamental in **LAA**'s but in **CA**'s is defined in terms of the combinators k and s. A more important difference is connected with the role variables play in the lambda calculus as place holders. In a **LAA** this is also abstracted. It takes the form of a system of fundamental elements (nullary operations) of the algebra. This is a crucial feature of **LAA**'s that has no direct analogue in **CA**'s.

The equational theory of **LAA**'s is axiomatized by the equations that hold between contexts of the lambda calculus (i.e., λ-terms with 'holes' [6, Def. 14.4.1]), as opposed to lambda terms with free variables. The essential feature of a context is that a free variable in a λ-term may become bound when we substitute it for a 'hole' within the context. Thus, 'holes' play the role of algebraic variables, and the contexts are the algebraic terms in the similarity type of lambda abstraction algebras. There is a rather peculiar relation between the lattice λT of lambda theories and the variety **LAA**. In [44] Salibra has shown that the lattice λT is isomorphic to the lattice of the equational theories of **LAA**'s. In fact, the correspondence, which maps an arbitrary λ-theory T into the equational theory of the variety generated by the term algebra of T, is an isomorphism of complete lattices. Thus, the properties of an arbitrary lambda theory can be studied by means of the variety of **LAA**'s generated by its term algebra. As we have specified in the introduction, many longstanding open problems of lambda calculus can be restated in terms of algebraic properties of varieties of **LAA**'s.

In this section we show that there exist a congruence distributive variety of **LAA**'s (i.e., a variety $\mathcal{V}$ of **LAA**'s such that every algebra in $\mathcal{V}$ has a distributive congruence lattice) and a congruence distributive variety of **CA**'s. The existence of varieties of **LAA**'s or **CA**'s satisfying strong algebraic properties, such as congruence distributivity, was an open problem since Salibra [45] proved that the variety **LAA** is not congruence modular and Lusin-Salibra [38] proved that every variety $\mathcal{V}$ of **LAA**'s generated by the term algebra of a semi-sensible λ-theory does not satisfy any lattice identity.

Theorem 45 *There exists a congruence distributive variety of lambda abstraction algebras.*

Proof. Let $\mathcal{V}$ be the variety of **LAA**'s generated by the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ of the lambda theory $\mathcal{L}$ defined in Lemma 41. We claim that $\mathcal{V}$ is congruence distributive, that is, every algebra $\mathbf{A} \in \mathcal{V}$ has a distributive congruence lattice. We have shown in the proof of Theorem 44 that the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ has two term operations $+$ and $\cdot$ (defined in (4)), which satisfy the axioms of a distributive lattice. Since $\mathbf{\Lambda}^{\mathcal{L}}$ generates the variety $\mathcal{V}$ and $+, \cdot$ are term operations, then every algebra $\mathbf{A} \in \mathcal{V}$ has also two term operations satisfying the axioms of a distributive lattice. The conclusion is obtained from Lemma 43. ■

Theorem 46 *There exists a congruence distributive variety of combinatory algebras.*

Proof. We recall from [6] that the models of lambda calculus, and in particular the graph models, are combinatory algebras. By Corollary 24 there exists a graph model (D, p), where the set-theoretical union and intersection are λ-represented by the closed pure λ-terms ΩT and ΩF. We claim that the variety $\mathcal{V}$ of **CA**'s generated by the graph model (D, p) is congruence distributive. The conclusion is obtained from Lemma 43 by the following facts.

(i) There exist two combinatory terms t and u such that the interpretations in (D, p) of ΩT and ΩF are equal to those of t and u respectively (see Section

7.3 in Barendregt's book [6]).

(ii) The term operations txy and uxy satisfy the axioms of a distributive lattice in the combinatory algebra (D, p).

(iii) The term operations txy and uxy satisfy the axioms of a distributive lattice in every algebra belonging to the variety generated by (D, p).

■

7 Conclusions and future work

We have generalized Baeten and Boerboom's method of forcing first to generalized terms involving all the continuous functions on a given power set $\mathcal{P}(D)$, and, second, to all weakly continuous operators. This approach allows us to prove very directly results about the lambda-representability of continuous functions on power sets, and also to generalize these results to countable sequences of continuous functions.

Related works are only the very recent papers by Alessi et al. [3] and Dezani-Lusin [22], where the authors use intersection type systems (see [3,7,20]) for synthesizing filter models of lambda calculus in which the interpretation of a simple easy term can be any filter described by a continuous predicate. This result can be interpreted as a generalization of Baeten and Boerboom's method of forcing via the use of intersection type systems. We believe that the framework we have developed in this paper is however more direct than the one used in [3]. We illustrate this with two examples, concerning the λ-representability of the minimal fixed point operator (Corollary 21) and of the pair union/intersection (Corollary 24).

As an application of the existence of (finite/infinite) sequences of terms that are functionally graph easy, we get strong results concerning the structure of the lattice of lambda theories and the existence of varieties of lambda abstraction algebras with very strong algebraic properties. More precisely, we show the existence of a distributive sub-lattice of the lattice of lambda theories and of a congruence distributive variety of lambda abstraction algebras.

In the present paper we only consider domains $\mathcal{P}(D)$, and, correlatively, graph models of lambda-calculus, and we concentrate on a limited number of applications. A further application, which is not treated here, is the question of the lambda-representability of first-order, say, structures in graph models. This will be the subject of another paper.

We would like to extend the results of the present paper to more sophisticated Scott-domains and webs. Various interesting classes of webbed models of lambda-calculus, concerning the main semantics of lambda-calculus, were surveyed in [14]. For the continuous semantics they range from graph models to filter models, with a clear preference for the models whose underlying domain is prime-algebraic (which excludes some filter models), since they can be represented via feasible webs. All are accessible to Baeten and Boerboom's technique (see Section 2.3), but with less facility than for graph models. Compatibility conditions have to be met, depending on the class we consider, which do not occur when dealing with graph models. However no systematic study has been made so far if one excepts filter models [3,22]. Our intention is hence to extend the methods and results presented in this paper to more general Scott-domains and webs.

Appendix

Proposition 47 *Forcing is not Scott-continuous.*

By this we mean that for all infinite sets D and all terms t, the application $H : Q \to \mathcal{P}(\mathcal{D})$ defined by $H(q) = \{ \alpha : q \Vdash \alpha \in t \}$ is not Scott-continuous.

Proof. Let α be a fixed element of D and q be a bijection between $D^* \times D - \{(\{\alpha\}, \alpha)\}$ and $D - \{\alpha\}$. It is clear that $q \Vdash \alpha \in \lambda x.x$, since the only total injection p which extends q satisfies $p(\{\alpha\}, \alpha) = \alpha$. Let $r \subseteq q$ be the partial sub-injection of q such that $dom(r) = \{ (b, \beta) \ / \ \beta \in b \} \cap dom(q)$. Since $q - r$ is infinite and countable there is a countable strictly increasing sequence q_n starting from r and whose union is q.

We claim now that no q' such that $r \subseteq q' \subsetneq q$ can force $\alpha \in \lambda x.x$. Let indeed $(c, \gamma) \in dom(q) - dom(q')$ and let p be a total injection which extends $q\prime$ and satisfies $\alpha = p(c, \gamma)$. From the hypothesis on q, q', p we have that $\alpha \notin (\lambda x.x)^p$. Hence $q' \nVdash \alpha \in \lambda x.x$. ■

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