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Transport properties in a spin polarized gas, I

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Résumé. — On étudie l'influence des effets d'indiscernabilité des atomes sur les propriétés de transport dans un gaz orienté à basse température, $^3\text{He}\uparrow$ par exemple, le gaz étant supposé dilué et donc non dégénéré. Dans ce but, on commence par étudier en détail ces effets dans une collision binaire; bien que l'on néglige toute interaction agissant sur les spins pendant la collision, on constate que des effets d'interférence quantique peuvent conduire à des changements de leur orientation (effets de rotation de spins identiques). Dans une seconde étape, on écrit une équation de Boltzmann satisfaite par un opérateur densité de spin $\rho_s(\mathbf{r}, \mathbf{p})$, qui joue un rôle analogue à la fonction de distribution classique de Boltzmann $f(\mathbf{r}, \mathbf{p})$, et peut être obtenu par transformation de Wigner de l'opérateur densité à une particule; dans cette équation, les effets d'indiscernabilité introduisent trois nouvelles « sections efficaces » indépendantes, alors que le terme classique ne dépend que d'une seule. L'application de la méthode de Chapman-Enskog à l'ordre le plus bas introduit alors un opérateur linéaire de collision Γ_c qui dépend explicitement de l'orientation $\mathbf{M}$ du gaz; en particulier, si $\mathbf{M}$ n'est pas nul, Γ_c n'est pas hermitique. Enfin, les résultats précédents sont appliqués à deux cas particuliers, conduction de la chaleur et viscosité, et l'on obtient la dépendance des coefficients correspondants en fonction de $\mathbf{M}$. Cette dépendance peut être très marquée à basse température. Dans l'article qui suit celui-ci, on étudie le problème de la diffusion de spin et l'on prédit un certain nombre d'effets quantiques dus à l'indiscernabilité des particules : équation de diffusion anisotrope, non linéaire, couplage avec un gradient thermique.

Abstract. — The transport properties of a spin polarized gas at low temperature ($^3\text{He}\uparrow$ for instance) are studied, the emphasis being put on the particle indistinguishability effects during collisions; the gas is dilute and no degeneracy effects occur. We first study in detail the particle indistinguishability effects in a binary collision and, although all interactions involving the spins are completely ignored during the collision time, we note that interference effects can lead to a change of the spin directions (« identical spin rotation effect »). From this study, we obtain the collision term of a Boltzmann equation for a spin operator $\rho_s(\mathbf{r}, \mathbf{p})$, which is the analogue of a classical Boltzmann distribution function $f(\mathbf{r}, \mathbf{p})$, and can be obtained by a Wigner transform — with respect to the orbital variables — of the one particle density operator; in this Boltzmann equation, in addition to the classical term which depends on one cross section, several terms arise from particle indistinguishability effects and introduce 3 other, independent, « cross sections ». The classical Chapman-Enskog approximation is then used and a linearized collision operator Γ_c is obtained; Γ_c depends on the spin polarization $\mathbf{M}$ in the gas and, if $\mathbf{M}$ is not zero, Γ_c is not a Hermitian operator. Finally, the preceding results are applied to two particular cases, heat conduction and viscosity of the gas, and the $\mathbf{M}$ dependence of the corresponding coefficient is obtained; such a dependence may be very marked at low temperatures, when quantum effects are dominant.

In the next article, the spin diffusion in the polarized gas is studied and some particle indistinguishability effects are predicted: the spin diffusion equation becomes non-linear and anisotropic, and is coupled to a temperature gradient.

Introduction. — Spin polarized quantum fluids are interesting physical systems, which are predicted to exhibit a series of unusual properties, arising from the enhanced role of particle indistinguishability effects when only one spin state is occupied [1]. This is true, not only for dense systems where degeneracy effects occur, but also for dilute non-degenerate gases, because particle indistinguishability effects may be

essential during collisions and consequently strongly affect the non-equilibrium properties of the system. In this article, we shall more specifically discuss the transport properties (heat conduction and viscosity) of spin polarized ^3He , often denoted by $^3\text{He}\uparrow$, where the spin orientation is purely nuclear. Actually, our study can also be applied to spin polarized H and D since, when the electronic spins of these atoms are

polarized in a high magnetic field ⁽¹⁾, and when the temperature is low enough, the only levels which remain accessible correspond to the various possible orientations of the nucleus (proton or deuteron). Obtaining a nuclear polarization in these systems by creating a population difference between the nuclear sublevels may be possible in the near future ^(*), since several laboratories have recently succeeded in performing measurements on atomic H and D at low temperatures [2, 3, 4].

In a gas, since atomic collision times are relatively short, a good approximation in most practical situations is to consider that the nuclear spins are completely unaffected during collisions [5]; this inertia of the nuclear spins is sometimes called a « spin Franck-Condon effect » [6] or the « nuclear flywheel effect ». Under these circumstances, it seems at first sight that the nuclear variables will be completely decoupled from the transport properties of the gas (except of course in the case of spin diffusion), and that the problem is particularly simple. A trivial method to include indistinguishability effects would then be to use the classical expression for the heat conduction coefficient and just replace the classical cross section by a quantum cross section including symmetry effects, that is a cross section where the square modulus of the scattering amplitude

$$|f_{sc}(\theta)|^2 \quad (\text{distinguishable particles})$$

has been changed into the symmetric form :

$$\frac{1}{2} |f_{sc}(\theta) + \varepsilon f_{sc}(\pi - \theta)|^2 \quad (\text{identical particles})$$

($\varepsilon = +1$ for bosons, -1 for fermions). Actually two atoms entering a collision are perfectly identical only if they are in the same nuclear spin state; thus, the relevant cross section is rather a « spin average » of two cross sections, one symmetrized and one not, with weights given by the probability that any pair of atoms will be in the same spin state; in this way, one obtains expressions of the transport coefficients which depend on the average nuclear polarization M of the gas. This point of view implies that there is no correlation between the velocity and the spin direction of the atoms; it is in fact much too naive, as we shall see below (it does not give correct predictions for the M dependence of the heat conductivity for example).

Another simple approach is to consider that the gas is similar to a mixture of several atomic species, associated with every possible spin orientation along an arbitrary quantization axis (for spin $1/2$ atoms such as ^3He , there would be two different species). Then, the classical theories for studying two (or more) component gaseous mixtures can be used. Such an image is more elaborate than the preceding one (for example, it can predict correlations between the

velocities and the spin state of the atoms). Nevertheless, we will see that it is not adequate in all situations. For instance, it can clearly not include any effect related to non-diagonal elements of the spin density matrix (transverse components of the magnetization); in other words, situations where the average magnetization changes direction in space cannot be described by this simple image.

A good illustration of the non-trivial character of these questions may be found in the literature concerning spin diffusion in gaseous ^3He : even in the simple case where the nuclear polarization is negligible, the correct way to take into account particle indistinguishability effects has not always been perfectly clear. Apparently, for some time, the most popular method was nothing but the first « naive approach » mentioned above (spin averaged cross section), which led to theoretical values of the spin diffusion coefficient D systematically smaller than the experimental results. This situation remained unchanged until Emery [7] pointed out that spin diffusion phenomena were fundamentally different from heat conduction and viscosity, as far as particle indistinguishability effects are concerned : in spin diffusion, due to momentum conservation, collisions between atoms in the same nuclear spin state do not affect directly the spin (magnetization) current J ⁽²⁾. The value of J will then depend only on the cross section for collisions between atoms in opposite spin states, which are in principle distinguishable, so that their cross section is simply proportional to $|f_{sc}(\theta)|^2$. Consequently, one expects that, for spin diffusion phenomena particle indistinguishability effects are not important (the same conclusion is not valid for other transport phenomena). Emery's contribution led to a significantly better agreement between theory and experiment.

The aim of the present article is to develop a detailed theory of transport properties in a dilute spin polarized gas, starting from a more general point of view which includes carefully the particle indistinguishability effects and allows one to see the limits of validity of the preceding models. No assumption will be made on the value of the nuclear polarization M , which is not necessarily small (as would be the case if the spins were near thermal equilibrium). The organization of the article is the following.

We first (§ 1) give a detailed study of a binary collision, recalling the results already obtained in the Appendix II of a preceding article [8]; we discuss the interference effects introduced by particle indistinguishability, and see how they may affect the nuclear spin orientation. In terms of spin density operators, these effects are described by anticommutators and

⁽¹⁾ To simplify, we assume that the hyperfine decoupling is complete.

^(*) Note added in proof : Since the present article was submitted for publication, a strong nuclear polarization in $\text{H}\uparrow$ has been reported by R. W. Cline *et al.* [23].

⁽²⁾ An analogous situation holds for a classical mixture of two gases A and B, where the concentration current is not directly affected by collisions between atoms of the same species; thus, the diffusion coefficient D depends on the cross section σ_{AB} for collision between atoms A and B, but not on σ_{AA} or σ_{BB} (when small corrections are neglected).

commutators, the latter introducing the « identical spin rotation effect ».

We then (§ 2) write a Boltzmann equation for a spin operator $\rho_s(\mathbf{r}, \mathbf{p})$ which replaces the usual Boltzmann distribution function $f(\mathbf{r}, \mathbf{p})$, following the work of Waldman [9] and Snider [10], or calculations concerning quantum fluids (see for example the work of Silin [11]). The « mixed » operator $\rho_s(\mathbf{r}, \mathbf{p})$ is classical with respect to the external variables of the atoms (ρ_s depends on two parameters $\mathbf{r}$ and $\mathbf{p}$), but is still an operator in the spin state space. It can be obtained by a Wigner transform of the one particle density operator with respect to the external variables, and is well suited to any semi-classical situation where the $\mathbf{r}$ and $\mathbf{p}$ dependences are sufficiently slow. We restrict ourselves to dilute gases and the Boltzmann equation we obtain does not include any degeneracy effects. Nevertheless, the structure of the (binary) collision term is more complex than in its classical counterpart : besides a « classical » term (**), several commutators and anticommutators appear which introduce particle indistinguishability effects (« identical spin rotation effect », etc.). Instead of only one collision cross section, this collision term contains 4 « cross sections », which depend in a different way on the interatomic potential. At this stage, it may be seen that, depending on the physical situation of interest (the macroscopic gradients of temperature, or orientation, etc.) and of the value of the magnetization M , different linear combinations of the 4 independent « cross sections » play the dominant role.

In § 3, we briefly discuss the approximation methods for solving the spin Boltzmann equation, and some properties of the linearized collision operator Γ_c . In particular, we show that quantum effects introduce non hermitian terms in Γ_c .

Finally, in §§ 4 and 5, we apply the preceding considerations to the calculation of the heat conduction and viscosity coefficients of a partially spin polarized gas. In the limits where $M = 0$ or $M = 1$, we find again either the classical results or the results obtained in a preceding article [12]. For intermediate values of

the nuclear polarization, we find the value of the coefficients as a function of M and we discuss the correlation between external and spin variables introduced by particle indistinguishability effects.

The following article gives a detailed study of spin diffusion phenomena in a polarized dilute gas. First, a simple approach is used to calculate the M dependence of the spin diffusion coefficient D and, in the limit where M tends to zero, we find again Emery's results. When M is not negligible, the diffusion equation we obtain is somewhat different from the classical equation : it is anisotropic and non-linear, and predicts spin oscillations in some cases. Second, a more elaborate approach is used to study the coupling between heat conduction and spin diffusion, which is introduced by quantum indistinguishability effects. Isotopic mixtures of ^3He and ^4He are also considered.

1. Effects of a binary collision on the spin density operator. — In this section, we study the effect of a collision between two atoms on the density operator associated with their spin variables, including particle indistinguishability effects.

1.1 COLLISION BETWEEN TWO IDENTICAL SPINS. — Let us consider two identical atoms, both in their electronic ground state, entering a collision ; in addition to their external variables, these atoms possess internal variables associated with their nuclear spins. We shall assume that, during the collision, the effects of all interactions on the nuclear spins are completely negligible (this assumption is valid in many practical situations, since the collision times are often very short). No particular assumption is made concerning the state of the internal variables of the atoms before collision : their spins are described by two density operators ρ_1 and ρ_2 (obtained by a trace operation over the external variables), which are in general not equal, or even diagonal in the same basis.

In the centre of mass reference frame, the density operator of the whole system before collision is ⁽³⁾ :

$$\frac{1}{2} [1 + \varepsilon P_{21}] \sigma_{\text{init}} [1 + \varepsilon P_{21}] \quad (1a)$$

with :

$$\sigma_{\text{init}} = \int d^3 k_i u(\mathbf{k}_i) \int d^3 k'_i u^*(\mathbf{k}'_i) |1 : \mathbf{k}_i, 2 : -\mathbf{k}_i\rangle \langle 1 : \mathbf{k}'_i, 2 : -\mathbf{k}'_i| \otimes \rho_1(1) \otimes \rho_2(2). \quad (1b)$$

In this equation, $\hbar \mathbf{k}_i$ is the relative linear momentum of the two atoms and the function $u(\mathbf{k}_i)$ gives (in the interaction representation) the coefficients of a wave packet associated with the relative motion of the particles before collision. We assume that :

$$\int d^3 k_i |u(\mathbf{k}_i)|^2 = 1 ; \quad \text{Tr} \{ \rho_1 \} = \text{Tr} \{ \rho_2 \} = 1 \quad (2a)$$

(**) Throughout the present and the following articles, we shall call « classical terms » the terms of the Boltzmann equation which are not introduced by particle indistinguishability effects. This does not mean that these terms can be obtained from a purely classical calculation : the cross

section $\sigma_k(\theta)$ which appears in this « classical » term is a quantum cross section, expressed in terms of phase shifts and including so-called « diffraction effects ».

⁽³⁾ We use the notation of reference [8].

(normalization condition), and :

$$\int d^3k_i u(\mathbf{k}_i) u(-\mathbf{k}_i) = 0 \quad (2b)$$

[the momentum of the incoming particle is sufficiently well defined to avoid any overlap between $u(\mathbf{k}_i)$ and $u(-\mathbf{k}_i)$]. In equation (1a), we have used the symmetrization operator :

$$\frac{1}{\sqrt{2}} [1 + \varepsilon P_{21}]$$

[due to (2b), the normalization coefficient is simply $1/\sqrt{2}$], where :

$$\varepsilon = \begin{cases} +1 & \text{for bosons} \\ -1 & \text{for fermions.} \end{cases}$$

We note that the P_{21} operator acts, not only on the external variables (it reverses the sign of $\mathbf{k}_i$), but also on the internal variables described by ρ_1 and ρ_2 . To write (1), we have assumed that, before collision, there is no correlation, either between the atoms or between internal and external variables ⁽⁴⁾.

We denote by S the (unitary) operator which, in the interaction representation, gives the evolution of the system between time $-t$ (long before the collision) and $+t$ (long after the collision). Since the collision has no direct effect on the nuclear spins, the S operator acts in the space of the external, but not internal, variables. We shall write :

$$\langle 1 : \mathbf{k}_f ; 2 : -\mathbf{k}_f | S | 1 : \mathbf{k}_i ; 2 : -\mathbf{k}_i \rangle = S(\mathbf{k}_f, \mathbf{k}_i) = \delta(\mathbf{k}_f - \mathbf{k}_i) - 2i\pi \frac{\mu}{\hbar^2 k_i} \delta(k_i - k_f) T(\mathbf{k}_f, \mathbf{k}_i) \quad (3)$$

where μ is the relative mass ($\mu = m/2$ for identical atoms) and T the so called transition operator.

We now want to obtain the spin density operator $\rho_f(\Omega_f)$ of the atoms with final velocity inside the solid angle Ω_f (no energy selection). The matrix elements of this operator are obtained by a partial trace operation :

$$\begin{aligned} \langle m_I | \rho_f(\Omega_f) | m'_I \rangle &= \int_{\Omega_f} d^3k_f \sum_{m''_I} \times \\ &\times \langle 1 : \mathbf{k}_f ; 2 : -\mathbf{k}_f | \langle 1 : m_I ; 2 : m''_I | [1 + \varepsilon P_{21}] S \sigma_{\text{init}} S^+ [1 + \varepsilon P_{21}] | 1 : \mathbf{k}_f ; 2 : -\mathbf{k}_f \rangle | 1 : m'_I ; 2 : m''_I \rangle. \end{aligned} \quad (4)$$

In this equation, the kets $|m_I\rangle$ are any set of states giving an orthonormal basis in the state space of one nuclear spin [since P_{21} and S are commuting operators, and since $(P_{21})^2 = 1$, it has been possible in (4) to suppress the P_{21} operators which act directly on the right or left hand side of σ_{init}]. From (3), we obtain :

$$\begin{aligned} \langle m_I | \rho_f(\Omega_f) | m'_I \rangle &= A(\Omega_f) \langle m_I | \rho_1 | m'_I \rangle + B(\Omega_f) \langle m_I | \rho_2 | m'_I \rangle + \\ &+ \varepsilon C(\Omega_f) \sum_{m''_I} \langle m''_I | \rho_1 | m'_I \rangle \langle m_I | \rho_2 | m''_I \rangle \\ &+ \varepsilon C^*(\Omega_f) \sum_{m''_I} \langle m_I | \rho_1 | m''_I \rangle \langle m''_I | \rho_2 | m'_I \rangle. \end{aligned} \quad (5)$$

where :

$$A(\Omega_f) = \int_{\Omega_f} d^3k_f \int d^3k_i u(\mathbf{k}_i) \int d^3k'_i u^*(\mathbf{k}'_i) S(\mathbf{k}_f, \mathbf{k}_i) S^*(\mathbf{k}_f, \mathbf{k}'_i) \quad (6a)$$

$$C(\Omega_f) = \int_{\Omega_f} d^3k_f \int d^3k_i u(\mathbf{k}_i) \int d^3k'_i u^*(\mathbf{k}'_i) S(-\mathbf{k}_f, \mathbf{k}_i) S^*(\mathbf{k}_f, \mathbf{k}'_i) \quad (6b)$$

⁽⁴⁾ In this section, we study the collision between two atoms and we are not concerned with possible correlations between their internal and external variables. In fact, we may even assume (as is done below) that the momenta of both atoms are very well defined, so that no such correlation can occur.

In the rest of the article, we generalize our study to our ensemble of atoms, described by a Wigner distribution, which includes possible correlations between the velocity and the spin direction of *one* atom. Nevertheless, we shall still assume that the variables of *two* different atoms entering a collision are not correlated (« molecular chaos assumption », valid for dilute gases).

and :

$$B(\Omega_f) = A(-\Omega_f) \quad (6c)$$

$[-\Omega_f]$ symbolizes the solid angle which includes all directions opposed to Ω_f . Equation (5) can also be written in an operator form :

$$\rho_f(\Omega_f) = A(\Omega_f) \rho_1 + A(-\Omega_f) \rho_2 + \varepsilon C(\Omega_f) \rho_2 \rho_1 + \varepsilon C^*(\Omega_f) \rho_1 \rho_2. \quad (7)$$

The first two terms on the right hand side of (7) are easy to understand : when the two atoms are distinguishable, one can detect either of them inside the solid angle Ω_f ; since the nuclear spins are unaffected during the collision, the spin density operator is then either ρ_1 or ρ_2 , with coefficients proportional to the scattering probabilities of atom 1 or 2 into the final directions. The two last terms in the right hand side of (7) arise only when the atoms are indistinguishable ; they are due to interference effects between two different processes for distinguishable atoms (one of the atom is scattered either through a given — or the opposite — direction⁽⁵⁾). These interference terms can be written :

$$\varepsilon C_R(\Omega_f) [\rho_1, \rho_2]_+ - i \varepsilon C_I(\Omega_f) [\rho_1, \rho_2]$$

where C_R and iC_I are the real and imaginary parts of $C(\Omega_f)$, and :

$$[\rho_1, \rho_2]_+ = \rho_1 \rho_2 + \rho_2 \rho_1$$

is the anticommutator of ρ_1 and ρ_2 . Since the trace of a commutator is zero, the number of atoms scattered in any given direction depends only on C_R . The C_I coefficient plays a role when the evolution of the internal variables of the atoms is considered. The corresponding change of the spin variables occurs only if both operators ρ_1 and ρ_2 cannot simultaneously be diagonalized, and is therefore a pure coherence effect (for a 1/2 spin for example, it is well known that changing the relative phase of the coefficients of the state vector on $|+\rangle$ and $|-\rangle$ amounts to changing the transverse spin orientation).

At this point, it is interesting to separate in the equations the effect occurring in the spherically scattered wave from those arising in the transmitted wave (forward scattering). To do this, we can write :

$$\begin{cases} A(\Omega_f) = A_0(\Omega_f) + A_{\text{fwd.}}(\Omega_f) + A_{\text{fwd.}}^*(\Omega_f) + A_{\text{scatt}}(\Omega_f) \\ C(\Omega_f) = C_{\text{fwd.}}(\Omega_f) + C_{\text{fwd.}}^*(-\Omega_f) + C_{\text{scatt}}(\Omega_f) \end{cases} \quad (8)$$

where :

$$A_0(\Omega_f) = \int_{\Omega_f} d^3 k_f u^*(\mathbf{k}_f) u(\mathbf{k}_f) \quad (9a)$$

$$\left. \begin{matrix} A_{\text{fwd.}}(\Omega_f) \\ C_{\text{fwd.}}(\Omega_f) \end{matrix} \right\} = -2i\pi \frac{\mu}{\hbar^2} \int_{\Omega_f} d^3 k_f u^*(\mathbf{k}_f) \int d^3 k_i u(\mathbf{k}_i) k_i^{-1} \delta(k_i - k_f) \begin{cases} T(\mathbf{k}_f, \mathbf{k}_i) \\ T(-\mathbf{k}_f, \mathbf{k}_i) \end{cases} \quad (9b)$$

$$\left. \begin{matrix} A_{\text{scatt}}(\Omega_f) \\ C_{\text{scatt}}(\Omega_f) \end{matrix} \right\} = 4\pi^2 \frac{\mu^2}{\hbar^4} \int_{\Omega_f} d^3 k_f \int d^3 k_i u(\mathbf{k}_i) \int d^3 k'_i u^*(\mathbf{k}'_i) [k_i k'_i]^{-1} \delta(k_i - k_f) \delta(k'_i - k_f) T^*(\mathbf{k}_f, \mathbf{k}'_i) \begin{cases} T(\mathbf{k}_f, \mathbf{k}_i) \\ T(-\mathbf{k}_f, \mathbf{k}_i) \end{cases} \quad (9c)$$

The coefficient A_0 clearly concerns the particles which have not interacted (no collision), since it does not depend on T , and since :

$$\begin{aligned} \text{if } \Omega_i \cap \Omega_f &= 0 & A_0 &= 0 \\ \text{if } \Omega_i &\subset \Omega_f & A_0 &= 1 \end{aligned}$$

where Ω_i is the solid angle in which the momentum of the incident particle can be found [that is where $u(\mathbf{k}_i) \neq 0$]. Condition (2b) implies that there is no C_0 coefficient.

⁽⁵⁾ If $\Omega_f = 4\pi$ (all atoms detected), one can show [Ref. [8], § 2.2.2] that $A(4\pi) = 1$, $C(4\pi) = 0$; all particle indistinguishability effects disappear and $\rho_f(4\pi) = \rho_1 + \rho_2$. Since there is no hamiltonian acting on the spins, it is not surprising that the one atom spin density operator should remain unchanged when the external variables are completely ignored.

One also obtains $\text{Tr} \{ \rho_f(4\pi) \} = 2$, which is related to the fact that the collision involves two atoms.

The coefficients $A_{\text{fwd.}}$ and $C_{\text{fwd.}}$ concern the transmitted particles (interference effects in the forward direction); again, these coefficients vanish if $\Omega_i \cap \Omega_f = 0$.

A_{scatt} and C_{scatt} are related to the particles scattered in all space directions and these coefficients do not vanish in general, when Ω_f is any solid angle.

Until now, we have made no particular assumption concerning the wave packet $u(\mathbf{k}_i)$. We can assume that the function $u(\mathbf{k}_i)$ is strongly peaked around a value $\mathbf{k}_i = \mathbf{k}_0$ (quasi plane wave). Then, the variations over this peak of $T(\mathbf{k}_f, \mathbf{k}_i)$ are negligible, and when $\Omega_f \supset \Omega_i$, $A_{\text{fwd.}}$ and $C_{\text{fwd.}}$ become proportional to :

$$k_0^{-1} T_{k_0}(\pm \hat{k}_0, \hat{k}_0) \int d^3k u^*(\mathbf{k}) \int d^2\hat{k}' k'^2 u(k\hat{k}')$$

where $\hat{k}_0$ and $\hat{k}'$ denote the angular variables of $\mathbf{k}_0$ and $\mathbf{k}'$, and :

$$T_{k_0}(\hat{k}, \hat{k}') = T(k_0 \hat{k}, k_0 \hat{k}'). \quad (10)$$

If the solid angle Ω_i is small enough, the integral over $d^2\hat{k}'$, which is to be calculated (in the $\mathbf{k}'$ space) over a small region of a sphere with radius $k' = k$, can be approximated by an integral over a small portion of a plane (orthogonal to the Oz direction in the same space). The 5-tuple integral then becomes :

$$\int dk_z \iint dk_x dk_y u^*(k_x, k_y, k_z) \iint dk'_x dk'_y u(k'_x, k'_y, k_z) = \int dk_z I^*(k_z) I(k_z) \quad (11a)$$

with

$$I(k_z) = \iint dk_x dk_y u(k_x, k_y, k_z). \quad (11b)$$

If now we introduce the wave function $\psi(\mathbf{r})$ of the incident (relative) particle :

$$\psi(x, y, z) = (2\pi)^{-3/2} \int d^3k e^{i\mathbf{k} \cdot \mathbf{r}} u(\mathbf{k}) \quad (12a)$$

we have :

$$\psi(0, 0, z) = (2\pi)^{-3/2} \int dk_z e^{ik_z z} I(k_z). \quad (12b)$$

But the norm of a function and of its Fourier transform are equal (Parseval-Plancherel equality) :

$$\Phi_i = \int dz |\psi(0, 0, z)|^2 = \left(\frac{1}{2\pi}\right)^2 \int dk_z |I(k_z)|^2. \quad (12c)$$

The integral written in (11a), divided by $4\pi^2$, is thus simply equal to Φ_i . Actually, this number has a simple physical interpretation since, if dS is any infinitesimal surface in the xOy plane (perpendicular to the initial momentum $\hbar k_0$), $\Phi_i dS$ gives the total probability of finding the particle inside a cylinder parallel to Oz and of cross section dS . It is physically not surprising that the effects of a collision are proportional to Φ_i : since we have assumed that $u(\mathbf{k})$ is a very narrow peaked function, $\psi(x, y, z)$ has locally (in the region of the potential) the structure of an almost perfect plane wave, and Φ_i gives the probability flux of this plane wave (more precisely, Φ_i is the time integral of this flux).

The two other coefficients A_{scatt} and C_{scatt} are also proportional to Φ_i since they involve the integral

$$\begin{aligned} \int_{\Omega_f} d^2\hat{k} T_{k_0}^*(\hat{k}, \hat{k}_0) T_{k_0}(\pm \hat{k}, \hat{k}_0) \int dk \int d^2\hat{k}_i k_i^2 u(k\hat{k}_i) \int d^2\hat{k}' k_i'^2 u(k\hat{k}') = \\ = (2\pi)^2 \Phi_i \int_{\Omega_f} d^2\hat{k} T_{k_0}^*(\hat{k}, \hat{k}_0) T_{k_0}(\pm \hat{k}, \hat{k}_0). \end{aligned} \quad (13)$$

The results obtained can be summarized in the following formulas :

Transmitted wave ($\Omega_f \supset \Omega_i$, $|\Omega_f| \ll 4\pi$) :

$$\begin{aligned} \rho_f(\text{fwd.}) - \rho_1 = \Phi_i \left\{ -\frac{8\pi^3\mu}{\hbar^2 k_0} [iT_{k_0}(\hat{k}_0, \hat{k}_0) - iT_{k_0}^*(\hat{k}_0, \hat{k}_0)] \rho_1 - \right. \\ \left. - \varepsilon \frac{8\pi^3\mu}{\hbar^2 k_0} (iT_{k_0}(-\hat{k}_0, \hat{k}_0) \rho_2 \rho_1 - iT_{k_0}^*(-\hat{k}_0, \hat{k}_0) \rho_1 \rho_2) \right\}. \end{aligned} \quad (14)$$

Scattered wave ($\Omega_f = \delta\Omega_f$, no intersection with Ω_i) :

$$\rho_f(\delta\Omega_f) = \delta\Omega_f \Phi_i \frac{16 \pi^4 \mu^2}{\hbar^4} \{ |T_{k_0}(\hat{k}_f, \hat{k}_0)|^2 \rho_1 + |T_{k_0}(-\hat{k}_f, \hat{k}_0)|^2 \rho_2 + \\ + \varepsilon T_{k_0}(-\hat{k}_f, \hat{k}_0) T_{k_0}^*(\hat{k}_f, \hat{k}_0) \rho_2 \rho_1 + \varepsilon T_{k_0}(\hat{k}_f, \hat{k}_0) T_{k_0}^*(-\hat{k}_f, \hat{k}_0) \rho_1 \rho_2 \} . \quad (15)$$

If we choose $\Omega_f = -\Omega_i$, we can obtain ρ_f (backward); in fact, the result is simply given by interchanging ρ_1 and ρ_2 in (14), as one could expect physically.

1.2 PHYSICAL DISCUSSION. — Equation (14) shows that, for distinguishable nuclei, the density operator of the transmitted particles is proportional to ρ_1 and that its variation depends on the imaginary part of $T_{k_0}(\hat{k}_0, \hat{k}_0)$; it is well known that the total cross section σ_k^T depends on the imaginary part of the scattering amplitude in the forward direction :

$$\sigma_T(k) = -\frac{16 \pi^3 \mu}{\hbar^2 k} \text{Im} \{ T_k(\hat{k}, \hat{k}) \} . \quad (16a)$$

Nevertheless, the particle-indistinguishability terms (those proportional to ε) depend, in general (if ρ_1 does not commute with ρ_2), on both the real and imaginary parts of T , calculated when $\hat{k}_f = -\hat{k}_i$ (one of the two interfering processes occurs in the backward direction). It is thus convenient to define the two real quantities $\sigma_{fwd}^{ex.}$ and $\tau_{fwd}^{ex.}$ by :

$$\sigma_{fwd}^{ex.}(k) - i\tau_{fwd}^{ex.}(k) = \frac{16 \pi^3 \mu}{\hbar^2 k} iT_k(-\hat{k}, \hat{k}) . \quad (16b)$$

The particle indistinguishability terms for the transmitted particles are then proportional to :

$$\sigma_{fwd}^{ex.}(k) [\rho_1, \rho_2]_+ + i\tau_{fwd}^{ex.}(k) [\rho_1, \rho_2] .$$

The trace of this expression is related to the total scattering probability (no spin measurement after collision); since only the anticommutator $[\rho_1, \rho_2]_+$ survives a trace operation, the effects of the Pauli principle on this probability are proportional to the coefficient $\sigma_{fwd}^{ex.}$. As for the term in $\tau_{fwd}^{ex.}$, it contains a commutator, and thus corresponds to a change of the internal variable state which can be described in terms of an effective hamiltonian (proportional to $\tau_{fwd}^{ex.} \rho_2$). For spin 1/2 fermions, this introduces a rotation of the spin due to an effective magnetic field; in general, we shall call this effect the « identical spin rotation effect ». One can point out the similarity with the Faraday effect where the spins of photons are rotated in the transmitted beam.

Let us now discuss the spin state of the scattered particles. In equation (15), we see that, for distinguishable atoms, the density operator $\rho_f(\delta\Omega_f)$ depends only on the scattering cross section :

$$\sigma_k(\hat{k}_i, \hat{k}_f) = \sigma_k(\theta) = \frac{16 \pi^4 \mu^2}{\hbar^4} |T_k(\hat{k}_f, \hat{k}_i)|^2 \quad (16c)$$

where θ is the angle between directions $\hat{k}_i$ and $\hat{k}_f$. Atom 1 gives a contribution which is proportional to $\sigma_k(\theta) \rho_1$, atom 2 a contribution proportional to $\sigma_k(\pi - \theta) \rho_2$, which is easy to understand physically. The particle indistinguishability terms are less trivial, and depend on the two real coefficients $\sigma_k^{ex.}$ and $\tau_k^{ex.}$ defined by :

$$\sigma_k^{ex.}(\hat{k}_i, \hat{k}_f) - i\tau_k^{ex.}(\hat{k}_i, \hat{k}_f) = \sigma_k^{ex.}(\theta) - i\tau_k^{ex.}(\theta) = \frac{16 \pi^4 \mu^2}{\hbar^4} T_{k_i}(-\hat{k}_f, \hat{k}_i) T_{k_i}^*(\hat{k}_f, \hat{k}_i) . \quad (16d)$$

Again, $\sigma_k^{ex.}(\theta)$ gives rise to an anticommutator with a non-zero trace in general and $\tau_k^{ex.}(\theta)$ to a commutator describing the « identical spin rotation effect » for the laterally scattered atoms.

With notation (16), equations (14) and (15) become :

$$\rho_f(fwd.) - \rho_1 = \Phi_i \left\{ -\sigma_T(k) \rho_1 - \frac{\varepsilon}{2} \sigma_{fwd}^{ex.}(k) [\rho_1, \rho_2]_+ - i\frac{\varepsilon}{2} \tau_{fwd}^{ex.}(k) [\rho_1, \rho_2] \right\} \quad (17)$$

$$\rho_f(\delta\Omega_f) = \Phi_i \delta\Omega_f \{ \sigma_{k_i}(\hat{k}_i, \hat{k}_f) \rho_1 + \sigma_{k_i}(-\hat{k}_i, \hat{k}_f) \rho_2 + \varepsilon \sigma_{k_i}^{ex.}(\hat{k}_i, \hat{k}_f) [\rho_1, \rho_2]_+ + i\varepsilon \tau_{k_i}^{ex.}(\hat{k}_i, \hat{k}_f) [\rho_1, \rho_2] \} . \quad (18)$$

Let us now assume that both nuclear spins are in two pure, orthogonal, states. Then

$$\rho_1 \rho_2 = \rho_2 \rho_1 = 0$$

and we see that all particle indistinguishability terms in (17) and (18) vanish. In particular, $\rho_f(\delta\Omega_f)$ becomes proportional to :

$$\sigma_k(\theta) \rho_1 + \sigma_k(\pi - \theta) \rho_2 \propto |T_k(\hat{k}_i, \hat{k}_f)|^2 \rho_1 + |T_k(\hat{k}_i, -\hat{k}_f)|^2 \rho_2. \quad (19)$$

The opposite situation occurs when both nuclear spins are in the same pure state, that is when :

$$\rho_1 = \rho_2 = (\rho_1)^2 = (\rho_2)^2.$$

The commutators in (17) and (18) then vanish, and $\rho_f(\delta\Omega_f)$ becomes proportional to :

$$2 \sigma_{\uparrow\uparrow}(\hat{k}_i, \hat{k}_f) \rho_1$$

with :

$$2 \sigma_{\uparrow\uparrow}(\hat{k}_i, \hat{k}_f) = \sigma_k(\theta) + \sigma_k(\pi - \theta) + 2 \varepsilon \sigma_k^{\text{ex}}(\theta) = \frac{16 \pi^4 \mu^2}{\hbar^4} |T_k(\hat{k}_f, \hat{k}_i) + \varepsilon T_k(-\hat{k}_f, \hat{k}_i)|^2. \quad (20)$$

In this case, the atoms are fully indistinguishable and the scattering amplitude : $T_k(\hat{k}_f, \hat{k}_i)$ must be replaced by :

$$T_k(\hat{k}_f, \hat{k}_i) + \varepsilon T_k(-\hat{k}_f, \hat{k}_i)$$

which means that interference effects between scattering processes in opposite directions can occur.

We can now use the unitary character of the S operator to show that the various σ and τ coefficients are not completely independent. From (6) and condition (2b), one easily obtains, if $\Omega_f = 4\pi$:

$$A(4\pi) = 1$$

$$C(4\pi) = 0.$$

Equations (8) then give :

$$1 = 1 + \frac{16 \pi^3 \mu}{\hbar^2 k_0} \Phi_i \text{Im} \{ T_k(k_0, k_0) \} + \frac{16 \pi^4 \mu^2}{\hbar^4} \Phi_i \int_{4\pi} d^2 k_f |T_k(\hat{k}_f, \hat{k}_0)|^2$$

$$0 = \frac{16 \pi^3 \mu}{\hbar^2 k_0} \Phi_i \text{Im} \{ T_k(-\hat{k}_0, \hat{k}_0) \} + \frac{16 \pi^4 \mu^2}{\hbar^4} \Phi_i \int_{4\pi} d^2 \hat{k}_f T_{k_i}^*(\hat{k}_f, \hat{k}_i) T_{k_i}(-\hat{k}_f, \hat{k}_i)$$

or :

$$\sigma_T(k) = \int_{4\pi} d^2 \hat{k}_f \sigma_k(\hat{k}_i, \hat{k}_f) \quad (21a)$$

$$\sigma_{\text{fwd}}^{\text{ex}}(k) = \int_{4\pi} d^2 \hat{k}_f \sigma_k^{\text{ex}}(\hat{k}_i, \hat{k}_f) \quad (21b)$$

$$0 = \int_{4\pi} d^2 \hat{k}_f \tau_k^{\text{ex}}(\hat{k}_i, \hat{k}_f) \quad (21c)$$

($\tau_{\text{fwd}}^{\text{ex}}$ is not the integral of τ_k^{ex} over all possible directions).

The rotational invariance of the interaction hamiltonian can be used to show that :

$$T(\mathbf{k}_f, \mathbf{k}_i) = T(-\mathbf{k}_f, -\mathbf{k}_i) \quad (22a)$$

(invariance under the effect of a rotation of axis perpendicular to $\mathbf{k}_i$ and $\mathbf{k}_f$ and angle π). The time reversal invariance implies that :

$$T(\mathbf{k}_f, \mathbf{k}_i) = T(-\mathbf{k}_i, -\mathbf{k}_f) \quad (22b)$$

which, combined with (22a), gives :

$$T(\mathbf{k}_f, \mathbf{k}_i) = T(\mathbf{k}_i, \mathbf{k}_f) \quad (22c)$$

(equality of the scattering amplitudes for two « inverse collisions »). We therefore have :

$$\sigma(\mathbf{k}_i, \mathbf{k}_f) = \sigma(\mathbf{k}_f, \mathbf{k}_i)$$

$$\sigma^{\text{ex}}(\mathbf{k}_i, \mathbf{k}_f) = \sigma^{\text{ex}}(\mathbf{k}_f, \mathbf{k}_i) \quad (22d)$$

$$\tau^{\text{ex}}(\mathbf{k}_i, \mathbf{k}_f) = \tau^{\text{ex}}(\mathbf{k}_f, \mathbf{k}_i).$$

The expression of the transition matrix T as a function of the phase shifts δ_l is :

$$T_k(\hat{\mathbf{k}}_i, \hat{\mathbf{k}}_f) = -\frac{\hbar^2}{\pi \mu k} \sum_{l,m} e^{i\delta_l} \sin \delta_l Y_l^{m*}(\hat{\mathbf{k}}_f) Y_l^m(\hat{\mathbf{k}}_i). \quad (23a)$$

Then :

$$\left. \begin{aligned} \sigma_T(k) &= \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2 \delta_l \\ \sigma_k(\theta) &= \frac{1}{k^2} \sum_l \sum_{l'} (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin \delta_l \sin \delta_{l'} P_l(\cos \theta) P_{l'}(\cos \theta) \end{aligned} \right\} \quad (23b)$$

$$\left. \begin{aligned} \sigma_{fwd.}^{ex.}(k) - i\tau_{fwd.}^{ex.}(k) &= -i \frac{4\pi}{k^2} \sum_l (-1)^l (2l+1) e^{i\delta_l} \sin \delta_l \\ \sigma_k^{ex.}(\theta) - i\tau_k^{ex.}(\theta) &= \frac{1}{k^2} \sum_l \sum_{l'} (-1)^l (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin \delta_l \sin \delta_{l'} P_l(\cos \theta) P_{l'}(\cos \theta) \end{aligned} \right\} \quad (23c)$$

[the coefficients arising from particle indistinguishability include a $(-1)^l$ in the sommation]. From these equalities, one can obtain again equations (21).

At low energies, when $k \rightarrow 0$, the standard behaviour of the phase shifts δ_l is :

$$\delta_l \sim (ka)^{2l+1}$$

where a is approximately equal to the potential range. Then, in equations (23), all but the $l = 0$ terms become negligible and, if a_0 is the scattering length, we obtain :

$$\begin{aligned} \sigma_T(k) &\sim \sigma_{fwd.}^{ex.}(k) \sim 4\pi(\delta_0/k)^2 = 4\pi a_0^2 \\ \sigma_k(\theta) &\sim \sigma_k^{ex.}(\theta) \sim (\delta_0/k)^2 = a_0^2 \\ \tau_{fwd.}^{ex.}(k) &\sim (4\pi/k^2) \delta_0 \rightarrow \infty \\ \tau_k^{ex.}(\theta) &\sim (\delta_0/k)^2 \delta_1 \rightarrow 0. \end{aligned} \quad (24)$$

It is interesting to remark that, at low energies, the dominant effect of the collision is the « identical spin rotation effect » in the forward direction (effective magnetic field for spin 1/2 particles).

2. A Boltzmann equation for the spin density operator. — In this section, we wish to obtain a Boltzmann equation, valid for particles with internal spin variables, and including all particle indistinguishability effects. Since we are interested only in low density gases (no degeneracy effects), we can treat the external variables of the atoms classically. It is then usual (see for example [7, 9, 10, 11]) to introduce an operator $\rho_S(\mathbf{r}, \mathbf{p})$, acting only in the internal variable state, and depending on the $\mathbf{r}$ and $\mathbf{p}$ parameters, which is the Wigner transform (with respect to the external variables) of the one atom density operator ρ [13]. The $(2I+1) \times (2I+1)$ matrix elements of $\rho_S(\mathbf{r}, \mathbf{p})$ are defined by :

$$\begin{aligned} \langle m_I | \rho_S(\mathbf{r}, \mathbf{p}) | m'_I \rangle &= (2\pi\hbar)^{-3} \int d^3r' e^{i\mathbf{p}\cdot\mathbf{r}'/\hbar} \left\langle m_I, \mathbf{r} - \frac{\mathbf{r}'}{2} \middle| \rho \middle| m'_I, \mathbf{r} + \frac{\mathbf{r}'}{2} \right\rangle \\ &= (2\pi\hbar)^{-3} \int d^3p' e^{i\mathbf{p}'\cdot\mathbf{r}/\hbar} \left\langle m_I, \mathbf{p} + \frac{\mathbf{p}'}{2} \middle| \rho \middle| m'_I, \mathbf{p} - \frac{\mathbf{p}'}{2} \right\rangle. \end{aligned} \quad (25a)$$

The usual classical distribution function $f(\mathbf{r}, \mathbf{p})$ is then given by :

$$f(\mathbf{r}, \mathbf{p}) = \text{Tr}_S \{ \rho_S(\mathbf{r}, \mathbf{p}) \} \quad (25b)$$

and the condition $\text{Tr} \{ \rho \} = 1$ gives :

$$\int d^3r \int d^3p f(\mathbf{r}, \mathbf{p}) = 1. \quad (25c)$$

For spin 1/2 particles, we can write :

$$\rho_S(\mathbf{r}, \mathbf{p}) = \frac{1}{2} f(\mathbf{r}, \mathbf{p}) [1 + \mathbf{M}(\mathbf{r}, \mathbf{p}) \cdot \boldsymbol{\sigma}] \quad (26a)$$

where σ symbolizes the Pauli matrices and M is the (relative) polarization of atoms with position $\mathbf{r}$ and momentum $\mathbf{p}$:

$$0 \leq M \leq 1.$$

We shall also use the notation :

$$\mathcal{M}(\mathbf{r}, \mathbf{p}) = f(\mathbf{r}, \mathbf{p}) \mathbf{M}(\mathbf{r}, \mathbf{p}). \quad (26b)$$

2.1 GENERAL FORM OF THE EQUATION. — For an ideal gas (no particle interactions), the hamiltonian is :

$$H = \sum_i \mathbf{P}_i^2 / 2m$$

where $\mathbf{P}_i$ is the momentum operator of atom i . Using a Wigner transform of the Schrödinger equation, one obtains [13] :

$$\frac{d}{dt} \rho_S(\mathbf{r}, \mathbf{p}) = -\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \rho_S(\mathbf{r}, \mathbf{p}) \quad (27)$$

The right hand side of (27) will be called the drift (or free flight) term.

In order to obtain all terms of a Boltzmann equation, we now have to add the effects of collisions. The usual Boltzmann theory treats the collisions as infinitely short phenomena (impact approximation), so that only the net result of each collision has to be taken into account. This is precisely what we have done for one collision in the preceding section by using the S matrix. The time variation of $\rho_S(\mathbf{r}, \mathbf{p})$ due to the collisions is the sum of two terms which correspond to particles leaving, or entering, the phase space element $d^3r d^3p$:

$$\left. \frac{d}{dt} \right|_{\text{coll}} = \left. \frac{d}{dt} \right|_{\text{coll}}^{\text{out}} + \left. \frac{d}{dt} \right|_{\text{coll}}^{\text{in}}.$$

These two terms will be obtained from equations (17) and (18) respectively.

In section 1, we have assumed that ρ_1 and ρ_2 are normalized density operators with trace one. Here, to describe the statistical ensemble of atoms, we have a different normalization, given by (25b), so that we have the correspondence :

$$\rho_S(\mathbf{r}, \mathbf{p}) \leftrightarrow f(\mathbf{r}, \mathbf{p}) \rho_1.$$

The effect of a collision between one atom with momentum $\mathbf{p}$ and one atom with momentum $\mathbf{p} - \mathbf{q}_i$ can be obtained from equation (17); multiplying this equation by $f(\mathbf{r}, \mathbf{p}) f(\mathbf{r}, \mathbf{p} - \mathbf{q}_i)$ and integrating over d^3q_i , we can obtain the effect of all collisions where one atom leaves the space element $d^3r d^3p$ around $\mathbf{r}$ and $\mathbf{p}$, between time t and $t + dt$:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{\text{coll}}^{\text{out}} \rho_S(\mathbf{r}, \mathbf{p}) \times dt = & - \int d^3q_i \overset{v_r(q_i)}{\leftarrow} dt \left\{ \sigma_T(k_i) f(\mathbf{r}, \mathbf{p} - \mathbf{q}_i) \rho_S(\mathbf{r}, \mathbf{p}) + \right. \\ & \left. + \frac{\varepsilon}{2} \sigma_{\text{fwd.}}^{\text{ex.}}(k_i) [\rho_S(\mathbf{r}, \mathbf{p}), \rho_S(\mathbf{r}, \mathbf{p} - \mathbf{q}_i)] + i \frac{\varepsilon}{2} \tau_{\text{fwd.}}^{\text{ex.}}(k_i) [\rho_S(\mathbf{r}, \mathbf{p}), \rho_S(\mathbf{r}, \mathbf{p} - \mathbf{q}_i)] \right\} \quad (28a) \end{aligned}$$

with :

$$v_r(q) = \frac{q}{m}. \quad (28b)$$

The factor $v_r dt$ gives the flux in the relative particle space, integrated over the time, that is precisely the quantity Φ_i of the preceding section. In (28a), k is the wave vector of the relative particle of mass $\mu = m/2$:

$$\hbar k_i = \mu v_r = \mu \frac{q_i}{m} = \frac{q_i}{2}. \quad (28c)$$

To obtain the effect of collisions transferring atoms into the phase space element $d^3r d^3p$, it is convenient to introduce the notation :

$$\left. \begin{aligned} \mathbf{p}_1 &= \mathbf{p} \\ \mathbf{p}_2 &= \mathbf{p} - \mathbf{q}_i \end{aligned} \right\} \quad \left. \begin{aligned} \mathbf{p}'_1 &= (\mathbf{p} - \frac{1}{2} \mathbf{q}_i) + \frac{1}{2} \mathbf{q}_i \\ \mathbf{p}'_2 &= (\mathbf{p} - \frac{1}{2} \mathbf{q}_i) - \frac{1}{2} \mathbf{q}_i \end{aligned} \right\}. \quad (29)$$

For an elastic collision leaving one atom with momentum $\mathbf{p}$, all possible momenta of the incoming atoms are given by $\mathbf{p}'_1$ and $\mathbf{p}'_2$, where $\mathbf{q}_i$ and $\mathbf{q}_f$ are any vector having the same length.

The number of pairs of atoms in the phase space element $d^3r'_1 d^3p'_1 d^3r'_2 d^3p'_2$ is :

$$d^3r'_1 d^3p'_1 d^3r'_2 d^3p'_2 f(\mathbf{r}'_1, \mathbf{p}'_1) f(\mathbf{r}'_2, \mathbf{p}'_2).$$

Instead of $\mathbf{r}'_1, \mathbf{r}'_2, \mathbf{p}'_1, \mathbf{p}'_2$, we can use the variables $\mathbf{r}, \mathbf{p}, \mathbf{q}_i$ defined by :

$$\left. \begin{aligned} \mathbf{r}'_1 &= \mathbf{r} + \frac{1}{2} \mathbf{p} \\ \mathbf{r}'_2 &= \mathbf{r} - \frac{1}{2} \mathbf{p} \end{aligned} \right\} \quad \left. \begin{aligned} \mathbf{p}'_1 &= \mathbf{p} + \frac{1}{2} \mathbf{q}_i - \frac{1}{2} \mathbf{q}_i \hat{q}_f \\ \mathbf{p}'_2 &= \mathbf{p} - \frac{1}{2} \mathbf{q}_i - \frac{1}{2} \mathbf{q}_i \hat{q}_f \end{aligned} \right\} \quad (30a)$$

where $\hat{q}_f$ is fixed. Then, a simple calculation gives (when $\hat{q}_f$ is considered as fixed, the modulus of the Jacobian is one) :

$$\begin{aligned} d^3r'_1 d^3r'_2 &= d^3r d^3p \\ d^3p'_1 d^3p'_2 &= d^3p d^3q_i \end{aligned} \quad (30b)$$

The reasoning in classical mechanics then goes as follows. During an infinitesimal time dt , the number of pairs of atoms undergoing a collision where one velocity ends up inside the momentum element d^3p , and the final relative momentum has a direction inside the solid angle $d^2\hat{q}_f$, is obtained by integrating d^3p over a cylinder of volume

$$v_r dt \sigma_{k_i}^{cl}(\hat{q}_i, \hat{q}_f) d^2\hat{q}_f$$

where $\sigma_{k_i}^{cl}$ is the « classical » cross section. This number is therefore given by :

$$d^3r d^3p d^3q_i \frac{q_i}{m} dt \sigma_{k_i}(\hat{q}_i, \hat{q}_f) f(\mathbf{r}, \mathbf{p}'_1) f(\mathbf{r}, \mathbf{p}'_2) d^2\hat{q}_f$$

(we have implicitly used the fact that the spatial variations of f are negligible over a distance $v_r dt$). An integration over d^3q_i and $d^2\hat{q}_f$ finally gives the rate of atoms reaching after collision the volume element d^3p .

This classical reasoning can easily be adapted to our case. We can multiply equation (18) by

$$f(\mathbf{r}, \mathbf{p}'_1) f(\mathbf{r}, \mathbf{p}'_2) v_r dt$$

and integrate it over d^3q_i and $d^2\hat{q}_f$. We then obtain :

$$\begin{aligned} \left. \frac{d}{dt} \right|_{\text{coll}} \rho_S(\mathbf{r}, \mathbf{p}) dt &= dt \int d^3q_i \int d^2\hat{q}_f v_r \times \\ &\times \{ \sigma_{k_i}(\hat{q}_i, \hat{q}_f) f(\mathbf{r}, \mathbf{p}'_2) \rho_S(\mathbf{r}, \mathbf{p}'_1) + \sigma_{k_i}(-\hat{q}_i, \hat{q}_f) f(\mathbf{r}, \mathbf{p}'_1) \rho_S(\mathbf{r}, \mathbf{p}'_2) \\ &+ \varepsilon \sigma_{k_i}^{ex}(\hat{q}_i, \hat{q}_f) [\rho_S(\mathbf{r}, \mathbf{p}'_1), \rho_S(\mathbf{r}, \mathbf{p}'_2)]_+ + i \varepsilon \tau_{k_i}^{ex}(\hat{q}_i, \hat{q}_f) [\rho_S(\mathbf{r}, \mathbf{p}'_1), \rho_S(\mathbf{r}, \mathbf{p}'_2)] \} \end{aligned} \quad (31)$$

where the d^3q_i integral is to be calculated only over a half-space, in order not to count the effect of the same collision twice. But this limitation can easily be removed, since changing the sign of $\mathbf{q}_i$ exchanges $\mathbf{p}'_1$ and $\mathbf{p}'_2$, leaves σ_k^{ex} invariant, and changes the sign of τ_k^{ex} . [see definition (16d)], we can therefore extend the integral over all space provided that we divide it by two. Using (27), (28) and (31), we can thus write the spin Boltzmann equation in the form :

$$\begin{aligned} \frac{\partial}{\partial t} \rho_S(\mathbf{r}, \mathbf{p}) &= -\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \rho_S(\mathbf{r}, \mathbf{p}) - \int d^3q_i v_r \left[\left\{ \sigma_T(k_i) f(\mathbf{r}, \mathbf{p} - \mathbf{q}_i) \rho_S(\mathbf{r}, \mathbf{p}) + \right. \right. \\ &+ \frac{\varepsilon}{2} \sigma_{fwd}^{ex}(k_i) \times [\rho_S(\mathbf{r}, \mathbf{p}), \rho_S(\mathbf{r}, \mathbf{p} - \mathbf{q}_i)]_+ + i \frac{\varepsilon}{2} \tau_{fwd}^{ex}(k_i) [\rho_S(\mathbf{r}, \mathbf{p}), \rho_S(\mathbf{r}, \mathbf{p} - \mathbf{q}_i)] \left. \right\} \\ &- \int d^2\hat{q}_f \left\{ \sigma_{k_i}(\hat{q}_i, \hat{q}_f) f(\mathbf{r}, \mathbf{p}'_2) \rho_S(\mathbf{r}, \mathbf{p}'_1) + \frac{\varepsilon}{2} \sigma_{k_i}^{ex}(\hat{q}_i, \hat{q}_f) [\rho_S(\mathbf{r}, \mathbf{p}'_1), \rho_S(\mathbf{r}, \mathbf{p}'_2)]_+ \right. \\ &+ i \frac{\varepsilon}{2} \tau_{k_i}^{ex}(\hat{q}_i, \hat{q}_f) [\rho_S(\mathbf{r}, \mathbf{p}'_1), \rho_S(\mathbf{r}, \mathbf{p}'_2)] \left. \right\} \left. \right]. \end{aligned} \quad (32a)$$

Using equations (21), we can give a similar structure to the terms arising from $\frac{d}{dt}\bigg|_{\text{coll}}^{\text{in}}$ and $\frac{d}{dt}\bigg|_{\text{coll}}^{\text{out}}$, and rewrite (32a) in the form (to simplify the notation, we no longer write explicitly the $\mathbf{r}$ dependence of ρ_s in the collision term) :

$$\begin{aligned} \frac{\partial}{\partial t} \rho_s(\mathbf{r}, \mathbf{p}) + \frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \rho_s(\mathbf{r}, \mathbf{p}) = & \int d^3 q' v_r \int d^2 \hat{q} \left\{ \sigma_k(\theta) \left[f(\mathbf{p}_2') \rho_s(\mathbf{p}_1') - f(\mathbf{p}_2) \rho_s(\mathbf{p}_1) \right] + \right. \\ & + \frac{\varepsilon}{2} \sigma_k^{\text{ex}}(\theta) \left[[\rho_s(\mathbf{p}_1'), \rho_s(\mathbf{p}_2')]_+ - [\rho_s(\mathbf{p}_1), \rho_s(\mathbf{p}_2)]_+ \right] + i \frac{\varepsilon}{2} \tau_k^{\text{ex}}(\theta) [\rho_s(\mathbf{p}_1'), \rho_s(\mathbf{p}_2')] \left. \right\} \\ & - i \frac{\varepsilon}{2} \int d^3 q \mathcal{H}_r \tau_{\text{fwd.}}^{\text{ex}}(k) [\rho_s(\mathbf{p}), \rho_s(\mathbf{p} - \mathbf{q})] \end{aligned} \quad (32b)$$

with [cf. (28b)] :

$$k = \frac{q'}{2\hbar}; \quad v_r = \frac{\hbar k}{\mu} = \frac{q'}{m} \quad (32c)$$

and :

$$\left. \begin{aligned} \mathbf{p}_1 &= \mathbf{p} \\ \mathbf{p}_2 &= \mathbf{p} - q' \hat{q} \end{aligned} \right\} \quad \left. \begin{aligned} \mathbf{p}_1' &= \mathbf{p} + \frac{1}{2} q' - \frac{1}{2} q' \hat{q} \\ \mathbf{p}_2' &= \mathbf{p} - \frac{1}{2} q' - \frac{1}{2} q' \hat{q} \end{aligned} \right\}. \quad (32d)$$

Because $\tau_{\text{fwd.}}^{\text{ex}}$ cannot be expressed as an integral over $d^2 \hat{q}$ of $\tau_k^{\text{ex}}(\hat{q}, \hat{q})$, the term corresponding to the « identical spin rotation effect » in the forward direction is not included in the sum over $d^2 \hat{q}$. If necessary, one can give a more similar form to the terms in $\sigma_q^{\text{ex}}(\theta)$ and $\tau_q^{\text{ex}}(\theta)$ by subtracting a commutator $[\rho_s(\mathbf{p}_1), \rho_s(\mathbf{p}_2)]$ in the latter [relation (21c) shows that the coefficient of this commutator vanishes after integration over $d^2 \hat{q}$]; nevertheless, this operation does not suppress the last commutator in $\tau_{\text{fwd.}}^{\text{ex}}(q)$.

The collision integral in (32b) can be written in several equivalent ways; for example :

$$\begin{aligned} \int d^3 q v_r \int d^2 \hat{q}' &= \int d^3 q' v_r \int d^2 \hat{q} \\ &= \int d^3 p_2 \frac{|\mathbf{p} - \mathbf{p}_2|}{m} \int d^2 \hat{q}'. \end{aligned} \quad (33a)$$

When the integration variables used are $\mathbf{p}_2$ and $\hat{q}$, $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are defined by :

$$\mathbf{p}_{1,2} = \frac{1}{2}(\mathbf{p} + \mathbf{p}_2) \pm \frac{1}{2} |\mathbf{p} - \mathbf{p}_2| \hat{q}. \quad (33b)$$

Since all constants $\sigma_q(\theta)$, $\sigma_q^{\text{ex}}(\theta)$, $\tau_q^{\text{ex}}(\theta)$ and $\tau_{\text{fwd.}}^{\text{ex}}$ are real, as well as the functions f , it can easily be verified on (32b) that the evolution of $\rho_s(\mathbf{r}, \mathbf{p})$ always preserves the hermitian character of this operator.

On the right hand side of equation (32b), we could use the Wigner transform to include terms corresponding to the effect of various forces acting on the atoms. For example, when the atoms interact with an inhomogeneous static or time dependent magnetic field, two additional terms appear. The first one is proportional to :

$$(i\hbar)^{-1} [\rho_s(\mathbf{r}, \mathbf{p}), \mathbf{S} \cdot \mathbf{B}(\mathbf{r}, t)]$$

(where $\mathbf{S}$ is the spin operator) and gives the usual magnetic precession of the spins of the atoms (Bloch equations). The second term corresponds to the « Stern-Gerlach » force and takes the form of an anticommutator [11] :

$$\sum_{i=x,y,z} \left[\frac{\partial}{\partial p_i} \rho_s(\mathbf{r}, \mathbf{p}), \frac{\partial}{\partial x_i} \mathbf{B} \cdot \mathbf{S} \right]_+.$$

In this article, we shall assume that $\mathbf{B}$ is zero and ignore these two terms, but they could easily be included in the theory.

2.2 DISCUSSION OF THE COLLISION TERMS. — Equation (32b) is somewhat similar to the classical Boltzmann equation, although it should be kept in mind that ρ_s is a spin operator and not a ordinary distribution function. As a consequence, the collision term includes terms (commutators and anticommutators) which are more complicated than in classical theory, and depends on four different « generalized cross sections » σ_q , σ_q^{ex} , τ_q^{ex} and $\tau_{\text{fwd.}}^{\text{ex}}$.⁽⁶⁾ The degree of symmetry of equation (32b) is somewhat lower than in classical statistical mechanics, due to the presence of the terms in τ : we have already mentioned that no detailed balance argument makes it possible to include the term in $\tau_{\text{fwd.}}^{\text{ex}}$ inside the sum over $d^2 \hat{q}$. Equations (24) show that these terms may become very important at low energies, and we shall see that they produce significant quantum exchange effects in the transport properties of the gas. On the other hand, at high energies,

⁽⁶⁾ The numbers σ_q^{ex} , τ_q^{ex} and $\tau_{\text{fwd.}}^{\text{ex}}$ are not necessarily positive.

the exchange effects tend to be negligible, since in the summations of equations (23), the series including a $(-1)^l$ are likely to have smaller values than the others.

Equation (32b) is valid for any value of the quantum number I and of the nuclear polarization (we have used no M expansion). All spin coherence effects have been included (the density operator ρ_S is not necessarily diagonal); this last point is particularly crucial for studying spin diffusion problems, where the direction of the average magnetization may change in space. Nevertheless, this equation can only be used for dilute gases, since no degeneracy effects (occupation numbers for the initial and final states) have been taken into account. Also, only binary collisions have been assumed to take place. Equation (32b) contains no phenomenological constants: all coefficients can be calculated from first principles, if the atom-atom potential is known.

It is easy to show that several physical quantities are unaffected by the collisions. The Wigner spin operator $\rho_S(\mathbf{r}, \mathbf{p})$ can be used to define the number density of atoms:

$$\begin{aligned} n(\mathbf{r}) &= \int d^3p \operatorname{Tr} \{ \rho_S(\mathbf{r}, \mathbf{p}) \} \\ &= \int d^3p f(\mathbf{r}, \mathbf{p}) \end{aligned} \quad (34a)$$

the spin orientation density:

$$\begin{aligned} \mathcal{M}(\mathbf{r}) &= \int d^3p \operatorname{Tr} \{ \boldsymbol{\sigma} \rho_S(\mathbf{r}, \mathbf{p}) \} \\ &= \int d^3p f(\mathbf{r}, \mathbf{p}) \mathbf{M}(\mathbf{r}, \mathbf{p}) \end{aligned} \quad (34b)$$

as well as the linear momentum and kinetic energy densities:

$$\mathcal{P}(\mathbf{r}) = \int d^3p \mathbf{p} f(\mathbf{r}, \mathbf{p}) \quad (34c)$$

$$\mathcal{W}(\mathbf{r}) = \frac{1}{2m} \int d^3p \mathbf{p}^2 f(\mathbf{r}, \mathbf{p}). \quad (34d)$$

It can be verified that all these quantities are conserved by the collisions (see Appendix I). Physically, this is because the collisions are perfectly elastic and do not change the total spin orientation.

A very simple case occurs when all atoms are in the same spin state (for example, we can assume that the gas has a 100% spin polarization, all spins being in the level of maximum m_I value, but the results are the same if any other m_I value is selected). We can then write:

$$\rho_S(\mathbf{r}, \mathbf{p}) = \bar{\rho}_S^0 f(\mathbf{r}, \mathbf{p}) \quad (35a)$$

where:

$$\operatorname{Tr} \{ \bar{\rho}_S^0 \} = 1 \quad (35b)$$

and:

$$[\bar{\rho}_S^0]^2 = \rho_S^0 \quad (35c)$$

(this last equation ensures that $\bar{\rho}_S^0$ describes a pure spin state). Equation (32b) then shows that the evolution of $\rho_S(\mathbf{r}, \mathbf{p})$ preserves the form (35a) of the spin density operator (this is not surprising since we have assumed that no hamiltonian acts in the spin state space) and that the evolution of f is given by:

$$\begin{aligned} \frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{p}) &= -\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{p}) + \\ &+ \int d^3q' v_r \int d^2\hat{q} [\sigma_q(\theta) + \varepsilon \sigma_q^{\text{ex}}(\theta)] \\ &\times [f(\mathbf{p}_2') f(\mathbf{p}_1') - f(\mathbf{p}_2) f(\mathbf{p}_1)] \end{aligned} \quad (36a)$$

(all « identical spin rotation » effects disappear in this case). This result is very similar to the classical Boltzmann equation, but the relevant cross section in this case is the fully symmetrized cross section:

$$\sigma_q^{\text{sym}}(\theta) = \sigma_q(\theta) + \varepsilon \sigma_q^{\text{ex}}(\theta). \quad (36b)$$

The opposite case occurs when the atom spins are completely depolarized. Then equations (35a) and (35b) are still valid but (35c) has to be replaced by:

$$[\bar{\rho}_S^0]^2 = \frac{1}{2I+1} \bar{\rho}_S^0. \quad (36c)$$

The preceding calculation can again be done but, this time, the relevant cross section is found to be the « spin averaged cross section »:

$$\sigma_k^{\text{unpol}}(\theta) = \sigma_k(\theta) + \frac{\varepsilon}{2I+1} \sigma_k^{\text{ex}}(\theta). \quad (36d)$$

The particle indistinguishability effects are now divided by $(2I+1)$. Physically, this factor can be interpreted as the probability that any atom will undergo the next collision with a partner in the same spin state.

We therefore see that, when the nuclear polarization is either zero or complete, the simple model mentioned in the introduction, based on the use of spin averaged cross section, is valid. Nevertheless, in the general case where M is neither 0 nor 1, $[\bar{\rho}_S^0]^2$ is not proportional to $\bar{\rho}_S^0$ and the above simplification does not occur, so that it is not possible to condense all relevant cross sections into one linear combination.

There is nevertheless another situation where equation (32b) simplifies, which occurs when all spin density operators ρ_S can be diagonalized in the same spin state basis $\{ |m_S\rangle \}$. We can then write:

$$\rho_S(\mathbf{r}, \mathbf{p}) = \sum_{m_S} |m_S\rangle \langle m_S| \tilde{f}_{m_S}(\mathbf{r}, \mathbf{p}) \quad (37a)$$

with:

$$f(\mathbf{r}, \mathbf{p}) = \sum_{m_S} \tilde{f}_{m_S}(\mathbf{r}, \mathbf{p}) \quad (37b)$$

and equation (32b) becomes:

$$\frac{\partial}{\partial t} \tilde{f}_{ms}(\mathbf{r}, \mathbf{p}) + \frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \tilde{f}_{ms}(\mathbf{r}, \mathbf{p}) = \int d^3 q' v_r \int d^2 \hat{q} \left\{ \sum_{m_s' \neq m_s} \sigma_k(\theta) [\tilde{f}_{m_s'}(\mathbf{p}_2') \tilde{f}_{m_s}(\mathbf{p}_1') - \tilde{f}_{m_s'}(\mathbf{p}_2) \tilde{f}_{m_s}(\mathbf{p}_1)] + \right. \\ \left. + [\sigma_k(\theta) + \varepsilon \sigma_k^{\text{ex}}(\theta)] [\tilde{f}_{m_s}(\mathbf{p}_2') \tilde{f}_{m_s}(\mathbf{p}_1') - \tilde{f}_{m_s}(\mathbf{p}_2) \tilde{f}_{m_s}(\mathbf{p}_1)] \right\} \quad (37c)$$

This is nothing but the classical Boltzmann equation for a mixture of $(2I + 1)$ gases, with distribution functions $\tilde{f}_{ms}(\mathbf{r}, \mathbf{p})$. The atomic masses have the same value m for all gases, but two different cross sections must be used : for collision between atoms belonging to different gases, the cross section is simply $\sigma_k(\theta)$; for collisions between atoms belonging to the same gas, the cross section has a different value, $\sigma_k(\theta) + \varepsilon \sigma_k^{\text{ex}}(\theta)$. This result is physically satisfying since, in a collision where the spins are completely unaffected, only atoms in the same spin state behave in quantum mechanics like identical particles.

In the general case where all operators $\rho_s(\mathbf{r}, \mathbf{p})$ cannot simultaneously be diagonalized, the general equation (32b) must be used to include all coherence effects, and the various terms and cross sections play a role which depends on the physical situation considered.

Let us finally consider the case when $I = 1/2$; using (26), one can easily show that equations (32b) are equivalent to :

$$\frac{\partial}{\partial t} f(\mathbf{r}, \mathbf{p}) = -\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{p}) + \int d^3 p_2 v_r \int d^2 \hat{q} \left\{ \left[\sigma_k(\theta) + \frac{\varepsilon}{2} \sigma_k^{\text{ex}}(\theta) \right] [f(\mathbf{p}_2') f(\mathbf{p}_1') - f(\mathbf{p}_2) f(\mathbf{p}_1)] + \right. \\ \left. + \frac{\varepsilon}{2} \sigma_k^{\text{ex}}(\theta) [\mathcal{M}(\mathbf{p}_1') \cdot \mathcal{M}(\mathbf{p}_2') - \mathcal{M}(\mathbf{p}_1) \cdot \mathcal{M}(\mathbf{p}_2)] \right\} \quad (38a)$$

$$\frac{\partial}{\partial t} \mathcal{M}(\mathbf{r}, \mathbf{p}) = -\frac{1}{m} (\mathbf{p} \cdot \nabla_{\mathbf{r}}) \mathcal{M}(\mathbf{r}, \mathbf{p}) + \int d^3 p_2 v_r \int d^2 \hat{q} \left\{ \sigma_k(\theta) [f(\mathbf{p}_2') \mathcal{M}(\mathbf{p}_1') - f(\mathbf{p}_2) \mathcal{M}(\mathbf{p}_1)] + \right. \\ \left. + \frac{\varepsilon}{2} \sigma_k^{\text{ex}}(\theta) [f(\mathbf{p}_1') \mathcal{M}(\mathbf{p}_2') + f(\mathbf{p}_2') \mathcal{M}(\mathbf{p}_1') - f(\mathbf{p}_1) \mathcal{M}(\mathbf{p}_2) - f(\mathbf{p}_2) \mathcal{M}(\mathbf{p}_1)] \right. \\ \left. - \frac{\varepsilon}{2} \tau_k^{\text{ex}}(\theta) \mathcal{M}(\mathbf{p}_1') \times \mathcal{M}(\mathbf{p}_2') \right\} + \frac{\varepsilon}{2} \int d^3 p_2 v_r \tau_{\text{fwd.}}^{\text{ex}}(k) \mathcal{M}(\mathbf{p}_1) \times \mathcal{M}(\mathbf{p}_2). \quad (38b)$$

The terms in $\sigma_k(\theta)$ are the « classical terms », which are linear in $\mathcal{M}$. The other terms arise from particle indistinguishability and contain several contributions which are quadratic in $\mathcal{M}$; due to these terms, the evolution of $f(\mathbf{r}, \mathbf{p})$ depends on the spin orientation. As we shall see below, this fact may have important physical consequences on the transport properties of the gas.

2.3 COMPARISON WITH THE LITERATURE. — The method we have used to calculate the collision term — namely first using the S matrix for evaluating the effect of each kind of collision, and then summing over all possible collisions — is very similar to Waldmann's approach [9]. Nevertheless, this author does not include in his calculations the particle indistinguishability effects, and therefore does not obtain the commutators and anticommutators of equation (32b). Another method of deriving a quantum Boltzmann equation including spins has been developed by Snider [10] but, again, symmetrization effects are not included. In a subsequent paper [14], Waldmann presents a theory where particle indistinguishability effects are partly taken into account ; the scattering amplitudes are symmetrized [equations (4.7) or (6.6) of this reference] but not the density operators themselves [equation (7.1)] so that a complete symmetrization in the internal variable space is not obtained (the commutators and anticommutators do not appear in this theory). Other references where particle indistinguishability effects are also discussed are the articles by Hunter [15] and Boercker and Dufty [16] but no

particular emphasis is put on the internal degrees of freedom.

3. Approximations for solving the spin Boltzmann equation. — In equation (32b), the collision term is non-linear and has a complicated structure so that an exact solution of this equation is not possible. We shall therefore use some approximations which are common in the transport theory of gases.

3.1 CHAPMAN-ENSKOG EXPANSION. — The right hand side of equation (32b) contains two terms, the drift term :

$$-\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \rho_s(\mathbf{r}, \mathbf{p})$$

and the collision term, which are not of comparable importance in the physical situations of interest to us. If v is a characteristic atomic velocity and L an order of magnitude of the distance over which ρ varies, the drift term is of the order of :

$$\frac{v}{L} \rho_s.$$

On the other hand, the collision term is of the order of :

$$\frac{1}{\tau_{ic}} \rho_s$$

where τ_{ic} is of the order of the time between collisions :

$$\tau_{ic} \simeq \frac{1}{n\sigma}. \quad (39a)$$

As a consequence, the ratio between the drift and collision terms is approximately :

$$\frac{v\tau_{ic}}{L} = \frac{l}{L} \quad (39b)$$

where l is the mean free path in the gas. We shall assume here that l is much smaller than the macroscopic distance over which the properties of the gas change appreciably (hydrodynamic approximation) :

$$\frac{l}{L} \ll 1. \quad (39c)$$

This means that we consider only physical situations where the collision term is much more effective than the drift term, which can then be treated as a small perturbation.

(i) The zero order approximation is obtained when the drift term is simply ignored. Then, equation (32b) shows that no coupling occurs between the evolutions of the density operators $\rho_s(\mathbf{r}, \mathbf{p})$ at different points of space $\mathbf{r}$. Under the effect of collisions, each density operator then tends to the value corresponding to a local equilibrium. Now, collisions cannot change the values of $n(\mathbf{r})$, $\mathcal{P}(\mathbf{r})$, $\mathcal{W}(\mathbf{r})$ and $\mathcal{M}(\mathbf{r})$, which correspond to 8 scalar constants; it is therefore clear that the system reaches locally equilibrium states which depend on the initial values of these constants. We shall admit here that no other quantities are strictly conserved by the collision operator and that, for given values of these 8 constants, only one density operator

$\rho_s^0(\mathbf{r}, \mathbf{p})$ remains invariant under the effect of collisions. This density operator then corresponds to the usual Boltzmann distribution and can be written :

$$\rho_s^0(\mathbf{r}, \mathbf{p}) = f_0(\mathbf{r}, \mathbf{p}) \bar{\rho}_s^0(\mathbf{r}) \quad (40a)$$

with :

$$f_0(\mathbf{r}, \mathbf{p}) = n_0(\mathbf{r}) \left(\frac{\beta_0}{2\pi m} \right)^{3/2} \exp \left\{ -\beta_0 (\mathbf{p} - m\mathbf{u}_0)^2 / 2m \right\} \quad (40b)$$

(k_B is the Boltzmann constant) and (7) for spin 1/2 atoms :

$$\bar{\rho}_s^0(\mathbf{r}) = \frac{1}{2} [1 + \boldsymbol{\sigma} \cdot \mathbf{M}_0(\mathbf{r})]. \quad (40c)$$

It is indeed not difficult to check that, when $\rho_s(\mathbf{r}, \mathbf{p})$ is equal to $\bar{\rho}_s^0(\mathbf{r}, \mathbf{p})$:

$$\left. \frac{d}{dt} \right|_{\text{coll}} \rho_s^0(\mathbf{r}, \mathbf{p}) = 0 \quad (41)$$

by using the collision term of equation (32b) and the fact that the Maxwell distribution satisfies the equality :

$$f_0(\mathbf{r}, \mathbf{p}_1) f_0(\mathbf{r}, \mathbf{p}_2) = f_0(\mathbf{r}, \mathbf{p}'_1) f_0(\mathbf{r}, \mathbf{p}'_2) \quad (42)$$

[the terms in $\sigma_k(\theta)$ or $\sigma_k^{\text{ex}}(\theta)$ give no contribution and all other terms are proportional to the commutator of $\bar{\rho}_s^0$ with itself, which is zero.]

(ii) The next step is to include the first order corrections, which are proportional to the expansion parameter $l/L \ll 1$.

Since the drift term is in essence proportional to L^{-1} , it can simply be approximated by :

$$\left. \frac{d}{dt} \right|_{\text{drift}} \rho_s(\mathbf{r}, \mathbf{p}) = -\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \{ \rho_s^0(\mathbf{r}, \mathbf{p}) \} \quad (43)$$

where ρ_s^0 is given by (40a). This term is then known as a function of the local temperature, magnetization, etc.

To evaluate the first order contribution of the collision term, we set :

$$\rho_s(\mathbf{r}, \mathbf{p}) = \rho_s^0(\mathbf{r}, \mathbf{p}) + f_0(\mathbf{r}, \mathbf{p}) \delta \bar{\rho}_s = f_0(\mathbf{r}, \mathbf{p}) [\bar{\rho}_s^0(\mathbf{r}) + \delta \bar{\rho}_s(\mathbf{r}, \mathbf{p})]. \quad (44)$$

A first order expansion then gives (8) :

$$\begin{aligned} \left. \frac{d}{dt} \right|_{\text{coll}} \delta \bar{\rho}_s(\mathbf{r}, \mathbf{p}) &= I_{\text{coll}}(\delta \bar{\rho}_s) = \int d^3 q' v_r \int d^2 \hat{q} f_0(p_2) \times \\ &\times \left\{ \sigma_k(\theta) [\delta \bar{\rho}_s(\mathbf{p}'_1) - \delta \bar{\rho}_s(\mathbf{p}_1) + \bar{\rho}_s^0 \text{Tr} \{ \delta \bar{\rho}_s(\mathbf{p}'_2) - \delta \bar{\rho}_s(\mathbf{p}_2) \}] \right. \\ &+ \frac{\varepsilon}{2} \sigma_k^{\text{ex}}(\theta) [\bar{\rho}_s^0, \delta \bar{\rho}_s(\mathbf{p}'_2) + \delta \bar{\rho}_s(\mathbf{p}'_1) - \delta \bar{\rho}_s(\mathbf{p}_2) - \delta \bar{\rho}_s(\mathbf{p}_1)]_+ \\ &\left. + i \frac{\varepsilon}{2} \tau_k^{\text{ex}}(\theta) [\bar{\rho}_s^0, \delta \bar{\rho}_s(\mathbf{p}'_2) - \delta \bar{\rho}_s(\mathbf{p}'_1)] \right\} + i \frac{\varepsilon}{2} \int d^3 q v_r \tau_{\text{fwd}}^{\text{ex}}(k) f_0(\mathbf{p} - \mathbf{q}) [\bar{\rho}_s^0, \delta \bar{\rho}_s(\mathbf{p}) - \delta \bar{\rho}_s(\mathbf{p} - \mathbf{q})]. \quad (45) \end{aligned}$$

(7) The local spin orientation is given by $\mathbf{M}_0(\mathbf{r})$, with : $0 \leq M_0 \leq 1$ (40d)

or equivalently by : $\mathcal{M}_0(\mathbf{r}) = n_0(\mathbf{r}) \mathbf{M}_0(\mathbf{r})$. (40e)

Similarly, β_0 (or T_0) and $m\mathbf{u}_0$ define the local energy and linear momentum.

(8) On equation (45), it is immediately obvious that the collision term vanishes if $\delta \bar{\rho}_s$ is independent of $\mathbf{p}$.

The first order Chapman-Enskog approximation consists in using equations (43) and (45), to write the equation :

$$\frac{d}{dt} [f_0(\mathbf{r}, \mathbf{p}) \delta \bar{\rho}_s(\mathbf{r}, \mathbf{p})] + \frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \{ \rho_s^0(\mathbf{r}, \mathbf{p}) \} = f_0(\mathbf{r}, \mathbf{p}) I_{\text{coll}}(\delta \bar{\rho}_s). \quad (46)$$

In this equation, the term $\mathbf{p} \cdot \nabla_{\mathbf{r}} \{ \rho_s^0(\mathbf{r}, \mathbf{p}) \}$ is a known quantity, and the evolution of $\delta \bar{\rho}_s$ at different points of space $\mathbf{r}$ is completely decoupled. From (46), one can in theory obtain $\delta \bar{\rho}_s$, its stationary value gives the currents (energy or linear momentum flux, spin current) which, in turn, lead to the hydrodynamic equations of the gas.

3.2 LINEARIZED COLLISION OPERATOR. — The linearized Boltzmann equation obtained in (46) can conveniently be written with the ket formalism, which is usual in quantum mechanics, in the form :

$$\frac{d}{dt} | \delta \bar{\rho}_s \rangle \rangle = \Gamma_c | \delta \bar{\rho}_s \rangle \rangle + | A \rangle \rangle \quad (47a)$$

where Γ_c is the linear collision operator and $| A \rangle \rangle$ corresponds to the drift term (43). In Appendix II, we discuss in more detail these quantities and various properties of Γ_c . We show that Γ_c has 8 eigenvectors with zero eigenvalues (undamped modes); we shall assume that no other such eigenvector exists. We also assume that all the other eigenvalues of Γ_c have a negative real part. Then, if $| A \rangle \rangle$ is orthogonal to all 8 eigenkets with zero eigenvalues (this is the case in all applications we shall consider), the solution $| \delta \bar{\rho}_s \rangle \rangle$ of equation (46) tends to a well defined limit $| \delta \bar{\rho}_s(\infty) \rangle \rangle$ when the time t increases [to simplify the notation, we shall denote this quantity $| \delta \bar{\rho}_s \rangle \rangle$].

In the following of this article, we shall calculate a flux J (for example a heat or magnetization flux) which is a linear function of $| \delta \bar{\rho}_s \rangle \rangle$ ⁽⁹⁾ :

$$J = \langle \langle B | \delta \bar{\rho}_s \rangle \rangle. \quad (47b)$$

In Appendix II, we discuss the role of the various eigenkets, or eigenmodes, of the collision operator Γ_c : the modes which give the major contributions are the modes with small (modulus of the) eigenvalues, and strong coupling with both the drift ket $| A \rangle \rangle$ and ket $| B \rangle \rangle$. In Appendix III, we rapidly review some approximation methods (in particular the « truncated basis method ») for obtaining $| \delta \bar{\rho}_s \rangle \rangle$.

A spin rotation does not leave the collision operator Γ_c invariant, unless $\bar{\rho}_s^0$ is itself rotation invariant. Nevertheless, as discussed in Appendix II, Γ_c is invariant under the effect of any rotation acting in the orbital space (the $\mathbf{p}$ space) only. This fact will bring important simplifications in our calculations. Finally, in Appendix II, we discuss the hermitian or not hermitian character of the various terms appearing

in Γ_c . We find that the « classical terms » [those depending on $\sigma_k(\theta)$] are indeed hermitian, as well as the particle indistinguishability terms involving anticommutators [those depending on $\sigma_k^{\text{ex}}(\theta)$]. On the other hand, all « identical spin rotation terms » [those depending on $\tau_k^{\text{ex}}(\theta)$ and $\tau_{\text{fwd.}}^{\text{ex.}}$] are not hermitian, but antihermitian. In other words, these terms do not change the damping introduced by the collision operator, but give an oscillatory character to the evolution of $| \delta \bar{\rho}_s \rangle \rangle$.

All properties of the more general non-linear collision term of (32b) are obviously also satisfied by the linear collision operator Γ_c : the hermitian character of $\bar{\rho}_s(\mathbf{r}, \mathbf{p})$ is always conserved, and in the presence of a full or zero spin polarization, the only relevant cross sections are respectively given by (36b) or (36d).

4. Heat conduction and viscosity. — In this section, we apply the preceding considerations to the calculation of the heat conduction coefficient κ in a spin polarized gas.

4.1 CALCULATION OF THE HEAT CONDUCTION COEFFICIENT. — We can use equations (40) to obtain the local equilibrium density operator $\rho_s^0(\mathbf{r}, \mathbf{p})$ adapted to this case, where $\mathbf{u}_0$ is zero, $\mathbf{M}_0$ constant, and β_0 a function of $\mathbf{r}$:

$$\rho_s^0(\mathbf{r}, \mathbf{p}) = f_0(\mathbf{r}, \mathbf{p}) \bar{\rho}_s^0$$

$$f_0(\mathbf{r}, \mathbf{p}) = n_0(\mathbf{r}) \left[\frac{\beta_0(\mathbf{r})}{2\pi m} \right]^{3/2} e^{-\beta_0(\mathbf{r}) p^2/2m}.$$

Since we assume that the pressure in the gas is constant, the ratio n_0/β_0 is independent of $\mathbf{r}$ and we shall write :

$$\frac{n_0(\mathbf{r})}{\beta_0(\mathbf{r})} = \left[\frac{n}{\beta} \right]_0$$

and :

$$f_0(\mathbf{r}, \mathbf{p}) = \left[\frac{n}{\beta} \right]_0 \frac{1}{(2\pi m)^{3/2}} [\beta_0(\mathbf{r})]^{5/2} e^{-\beta_0(\mathbf{r}) p^2/2m} \quad (48a)$$

so that the drift term of (46) becomes :

$$\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \{ \rho_s^0(\mathbf{r}, \mathbf{p}) \} =$$

$$= \frac{1}{m} f_0(\mathbf{r}, \mathbf{p}) \bar{\rho}_s^0 \left[\frac{5}{2\beta_0} - \frac{p^2}{2m} \right] \mathbf{p} \cdot \nabla \beta_0(\mathbf{r}). \quad (48b)$$

We know that $\delta \bar{\rho}_s$ will tend towards a quasi-stationary value only if the drift term is orthogonal to all

⁽⁹⁾ The scalar product notation $\langle \langle | \rangle \rangle$ is defined in Appendix II.

eigenkets of Γ_c^+ with zero eigenvalues (Appendix II). Actually, the drift term has a very simple $\hat{p}$ dependence ($\hat{p}$ refers to the angular variables of $\mathbf{p}$), entirely contained in the function $\mathbf{p} \cdot \nabla \beta_0$ in (48b), which is a linear combination of the three spherical harmonics $v_1^m(\hat{p})$. As a consequence, all collisional invariants which involve only $Y_0(\hat{p})$ [i.e. $n(\mathbf{r})$, $\mathcal{M}(\mathbf{r})$ and $\mathcal{W}(\mathbf{r})$] are indeed orthogonal to the drift term. The scalar product of this term with $\mathcal{F}(\mathbf{r})$ includes the radial integral :

$$\int_0^\infty p^2 dp e^{-\beta p^2/2m} \left[\frac{5}{2} - \frac{\beta p^2}{2m} \right] p^2 = 0. \quad (49)$$

Since the collision operator Γ_c is invariant in any rotation in the $\mathbf{p}$ space, we know the angular dependence of $\delta\bar{\rho}_S$ which is a linear combination of the spherical harmonics Y_1^m . In addition, Γ_c is clearly invariant in any transformation in the spin space which leaves $\bar{\rho}_S^0$ invariant. For spin 1/2 particles, $\bar{\rho}_S^0$ is diagonal if one chooses a quantization axis Oz parallel to the local orientation $\mathbf{M}_0(\mathbf{r})$; with this choice, $\delta\bar{\rho}_S$ is necessarily invariant in any spin rotation about Oz. We can then write :

$$\delta\bar{\rho}_S = \frac{1}{2} \{ \varphi_0(p) \mathbf{g}_0 \cdot \mathbf{p} + \varphi_3(p) \mathbf{g}_3 \cdot \mathbf{p} \sigma_3 \} \quad (50a)$$

where σ_3 is the third Pauli matrix. The functions φ_0 and φ_3 are any functions of the modulus p of $\mathbf{p}$, but φ_0 has to obey one orthogonality constraint (see Appendix II) associated with the linear momentum :

$$\int_0^\infty p^2 dp e^{-\beta p^2/2m} p^2 \varphi_0(p) = 0 \quad (50b)$$

[the other constraints are automatically satisfied by (50a)]. The operator $\delta\bar{\rho}_S$ written in (50a) commutes

with $\bar{\rho}_S^0$ so that all commutators (identical spin rotation effect) automatically disappear from the collision term.

Expression (50a) is general but does not allow a simple solution of equation (46). We shall therefore use an approximation and write a trial spin operator in the form :

$$\delta\bar{\rho}_S = \frac{1}{2} \varphi_0(p) [(\mathbf{g}_0 \cdot \mathbf{p}) + (\mathbf{g}_3 \cdot \mathbf{p}) \sigma_3] \quad (51a)$$

with :

$$\varphi_0(p) = \left[\frac{\beta p^2}{2m} - \frac{5}{2} \right]. \quad (51b)$$

This choice ensures that (50b) is satisfied (it corresponds to a truncation of $\delta\bar{\rho}_S$ to one Sonine — or Laguerre — polynomial). Then, $\delta\bar{\rho}_S$ depends on 6 unknown scalar quantities, the components of $\mathbf{g}_0$ and $\mathbf{g}_3$, which will be determined by a projection method analogous to the method of moments (Appendix III). This will be done by imposing that the scalar product of the drift term by the six operators :

$$\left. \begin{aligned} &\varphi_0(p) p_{x_i} \\ &\varphi_0(p) p_{x_i} \sigma_3 \end{aligned} \right\} \quad x_i = x, y, z \quad (52)$$

is equal to the scalar product of $\frac{d}{dt} \bigg|_{\text{coll}} \delta\bar{\rho}_S$ [where $\delta\bar{\rho}_S$ is given by (51a)] by the same six operators. Introducing the notation ⁽¹⁰⁾ :

$$\mathbf{G}_0 = -\frac{5}{2} \frac{n}{\beta} \nabla \text{Log } \beta \quad (53)$$

and using some simple properties of the Pauli matrices, we obtain the system of equations :

$$\begin{aligned} \mathbf{G}_0 &= \mathbf{g}_0 n^2 [W_1 + W_2 + \varepsilon W_3] + \mathbf{g}_3 n^2 [\varepsilon M W_3] \\ M \mathbf{G}_0 &= \mathbf{g}_0 M n^2 [W_2 + \varepsilon W_3] + \mathbf{g}_3 n^2 [W_1 + \varepsilon W_3] \end{aligned} \quad (54)$$

where the coefficients W_1 , W_2 , W_3 are defined by :

$$\begin{aligned} n^2 \delta_{ij} W_1 &= \int d^3 p \int d^3 q' v_r \int d^2 \hat{q} f_0(\mathbf{p}') f_0(\mathbf{p}_2) \varphi_0(p) p_i \sigma_k(\theta) \{ p'_{1j} \varphi_0(\mathbf{p}_1) - p_{1j} \varphi_0(\mathbf{p}_1) \} \\ n^2 \delta_{ij} W_2 &= \int d^3 p \int d^3 q v_r \int d^2 \hat{q} f_0(\mathbf{p}') f_0(\mathbf{p}_2) \varphi_0(p) p_i \sigma_k(\theta) \{ p'_{2j} \varphi_0(\mathbf{p}_2) - p_{2j} \varphi_0(\mathbf{p}_2) \} \\ n^2 \delta_{ij} W_3 &= \int d^3 p \int d^3 q' v_r \int d^2 \hat{q} f_0(\mathbf{p}') f_0(\mathbf{p}_2) \varphi_0(p) p_i \frac{\sigma_{ex}(\theta)}{2} \times \\ &\quad \times \{ p'_{1j} \varphi_0(\mathbf{p}_1) + p'_{2j} \varphi_0(\mathbf{p}_2) - p_{1j} \varphi_0(\mathbf{p}_1) - p_{2j} \varphi_0(\mathbf{p}_2) \}. \end{aligned} \quad (55)$$

More details about the calculation of these integrals are given in Appendix V. One obtains :

$$\begin{aligned} W_2^1 &= -\frac{m}{\beta} \left\{ \frac{4}{3} \Omega_{[\sigma_k]}^{(2,2)} \pm \frac{55}{6} \Omega_{[\sigma_k]}^{(1,1)} \mp \frac{10}{3} \Omega_{[\sigma_k]}^{(1,2)} \pm \frac{2}{3} \Omega_{[\sigma_k]}^{(1,3)} \right\} \\ W_3 &= -\frac{m}{\beta} \frac{4}{3} \Omega_{[\sigma_k]}^{(2,2)}. \end{aligned} \quad (56)$$

⁽¹⁰⁾ We use the simpler notation n , β , M instead of $n_0(\mathbf{r})$, $\beta_0(\mathbf{r})$, $M_0(\mathbf{r})$.

The definition of the Ω collision integrals is, as usual :

$$\Omega_{[\sigma]}^{(t,s)} = \left(\frac{1}{\pi m \beta} \right)^{1/2} \int_0^\infty d\gamma e^{-\gamma^2} \gamma^{2s+3} Q_{[\sigma]}^{(t)} \left[k = \left(\frac{m}{\beta} \right)^{1/2} \frac{\gamma}{\hbar} \right] \quad (57a)$$

where $Q^{(t)}(k)$ is the angle integrated cross section :

$$Q_{[\sigma]}^{(t)}(k) = 2\pi \int_0^\pi \sin \theta d\theta [1 - \cos^t \theta] \sigma(\theta). \quad (57b)$$

The linear system (54) can easily be solved and gives :

$$\left. \begin{aligned} \mathbf{g}_0 &= \frac{1}{n^2 \Delta_w} [W_1 + \varepsilon W_3 (1 - M^2)] \mathbf{G}_0 \\ \mathbf{g}_3 &= \frac{1}{n^2 \Delta_w} W_1 M \mathbf{G}_0 \end{aligned} \right\} \quad (58a)$$

with :

$$\Delta_w = [W_1 + \varepsilon W_3] [W_1 + W_2 + \varepsilon W_3] - \varepsilon M^2 W_3 [W_2 + \varepsilon W_3]. \quad (58b)$$

On the other hand, the heat current (energy flux) is :

$$\mathbf{J}_w(\mathbf{r}) = \int d^3p f_0(\mathbf{r}, \mathbf{p}) \frac{p^2}{2m} \frac{\mathbf{p}}{m} \text{Tr} \{ \delta \bar{\rho}_s(\mathbf{r}, \mathbf{p}) \}. \quad (59a)$$

For the trial spin operator (51a), we have :

$$\mathbf{J}_w(\mathbf{r}) = \frac{5}{2} \frac{n}{\beta^2} \mathbf{g}_0 \quad (59b)$$

which, by definition of the heat conduction coefficient $\kappa_0(M)$, is also equal to :

$$\mathbf{J}_w(\mathbf{r}) = -\kappa_0(M) \nabla T. \quad (59c)$$

Finally, we obtain the following value for the heat conduction coefficient κ_0 of a gas with spin polarization M :

$$\kappa_0(M) = \kappa_0(0) \frac{1 - \xi_1 M^2}{1 - \xi_2 M^2} \quad (60)$$

where $\kappa_0(0)$ is the coefficient for the unpolarized gas :

$$\kappa_0(0) = \frac{25}{16} \frac{c_v}{m} \frac{kT}{\Omega_{[\sigma]}^{(2,2)} + \frac{\varepsilon}{2} \Omega_{[\sigma]^{ex}}^{(2,2)}} \quad (61a)$$

(c_v is the constant volume heat capacity $c_v = 3k/2$) and the coefficients ξ_1 and ξ_2 are given by :

$$\begin{aligned} \xi_1 &= \frac{\varepsilon W_3}{W_1 + \varepsilon W_3} \\ \xi_2 &= \frac{\varepsilon W_3 (W_2 + \varepsilon W_3)}{(W_1 + W_2 + \varepsilon W_3) (W_1 + \varepsilon W_3)}. \end{aligned} \quad (61b)$$

4.2 PHYSICAL DISCUSSION. — Using (60), we see that the M dependence of the heat conduction coefficient arises from the coefficients ξ_1 and ξ_2 , which are both proportional to ε ; this shows that the M dependence is entirely due to particle indistinguishability effects, and is very different for fermions or bosons. For an unpolarized gas ($M = 0$), formula (61a) simply gives the well known expression of the heat conduction coefficient of a classical gas, with the cross section :

$$\sigma_k(\theta) + \frac{\varepsilon}{2} \sigma_k^{ex}(\theta)$$

which is nothing but the « spin averaged cross section » already written in (36d). Similarly, for a 100 % nuclear polarization, we obtain :

$$\begin{aligned} \kappa_0(M = 1) &= \kappa_0(0) \frac{2 \Omega_{[\sigma k]}^{(2,2)} + \varepsilon \Omega_{[\sigma k^{ex}]}^{(2,2)}}{2 \Omega_{[\sigma k]}^{(2,2)} + 2 \varepsilon \Omega_{[\sigma k^{ex}]}^{(2,2)}} \\ &= \frac{25}{16} \frac{c_v}{m} \frac{kT}{\Omega_{[\sigma k]}^{(2,2)} + \varepsilon \Omega_{[\sigma k^{ex}]}^{(2,2)}} \end{aligned} \quad (62)$$

Now, the relevant cross section is the fully symmetrized cross section (36b). As already discussed in § 2.2, we find that the « naive model » using spin averaged cross sections is valid when $M = 0$ or 1. The comparison between (61a) and (62) shows that a full nuclear polarization enhances the effects of particle indistinguishability (terms in ε); this fact has already been discussed in [12] and the numerical results of this reference can be obtained again from (62).

It is then natural to try to use the « spin averaged cross section » method in the general case, when M is neither 0 nor 1. The probability for any collision to occur between atoms in the same spin state is :

$$\frac{1}{2}(1 + M^2)$$

so that this spin averaged cross section is given by :

$$\begin{aligned} \sigma_{av}(\theta) &= \frac{1}{2}(1 + M^2) [\sigma_k(\theta) + \varepsilon \sigma_k^{ex}(\theta)] + \\ &\quad + \frac{1}{2}(1 - M^2) \sigma_k(\theta) \\ &= \sigma_k(\theta) + \frac{\varepsilon}{2}(1 + M^2) \sigma_k^{ex}(\theta). \end{aligned} \quad (63a)$$

Then the corresponding value for the heat conduction coefficient would be :

$$\frac{25}{16} \frac{c_v}{m} \frac{kT}{\Omega_{[\sigma]}^{(2,2)} + \frac{\varepsilon}{2}(1 + M^2) \Omega_{[\sigma_{\text{ex}}]}^{(2,2)}}. \quad (63b)$$

Clearly, this expression does not coincide with (60), where the values of t and s in the $\Omega^{(t,s)}$ integrals are not limited to $t = s = 2$, and where there is also an M dependence in the numerator. Thus, the simple interpolation (63b) between the two values (61a) and (62) is not justified.

Physically, this is because the spin averaged cross section method totally ignores the correlations between the velocities and the spin orientation of the atoms. Such correlations are in fact predicted by the theory developed in the preceding section since, from (58a), one obtains :

$$\mathbf{g}_3 = M\mathbf{g}_0 + \frac{W_3}{n^2 \Delta} \varepsilon M(M^2 - 1) \mathbf{G}_0 \quad (64a)$$

so that the optimum value of the trial operator (51a) can be written :

$$\delta \bar{\rho}_S = \varphi_0(p) \mathbf{g}_0 \cdot \mathbf{p}$$

$$\left[\bar{\rho}_S^0 + \varepsilon \frac{W_3}{2 n^2 \Delta} M(M^2 - 1) \frac{G_0}{g_0} \sigma_3 \right]. \quad (64b)$$

In the bracket of the right hand side of this equation, the term in σ_3 introduces a spin-velocity correlation into the total density operator :

$$\rho_S(\mathbf{p}) = f_0(\mathbf{p}) [\bar{\rho}_S^0 + \delta \bar{\rho}_S]. \quad (65)$$

This correlation term is due to particle indistinguishability (it is proportional to ε)⁽¹¹⁾ and we can check that it vanishes if either $M = 0$ or 1.

One can then expect that a simpler trial operator $\delta \bar{\rho}_S$, with no velocity-spin correlation, can lead to equation (63b). To see that this is in fact true, let us set :

$$\delta \bar{\rho}_S = \varphi_0(p) \mathbf{g}_0 \cdot \mathbf{p} \bar{\rho}_S^0 \quad (66a)$$

where the only variational parameters are now the three components of $\mathbf{g}_0$. With this new trial operator, all terms in equation (46) become proportional to $\bar{\rho}_S^0$, except the anticommutators in the collision term,

which are proportional to $(\bar{\rho}_S^0)^2$ [the commutators obviously vanish]. Taking the trace of both sides, we then obtain :

$$\begin{aligned} \mathbf{G}_0 &= \mathbf{g}_0 n^2 [W_1 + W_2 + \varepsilon W_3 \text{Tr} \{ (\bar{\rho}_S^0)^2 \}] \\ &= \mathbf{g}_0 n^2 \left[W_1 + W_2 + \frac{\varepsilon}{2} W_3 (1 + M^2) \right] \end{aligned} \quad (66b)$$

which immediately leads to (63b).

The preceding discussion shows that equation (60) takes into account more physical effects than (63b), and is therefore more adequate, although it should not be considered as strictly exact either. For example, by using different functions $\varphi_0(p)$ and $\varphi_3(p)$ in (50a), a different result from (60) would have been obtained. We shall not develop here more elaborate variational calculations, but this will be done in the next article.

Another property of the trial operator (51a) is that no spin current $\mathbf{J}(M_{x,y,z})$ can be obtained; this is a direct consequence of (50b) and, again, of the condition $\varphi_3(p) = \varphi_0(p)$. In the next article, we shall see that, in general, a coupling between a temperature gradient and a spin current is indeed possible.

To conclude this discussion, let us study the low temperature limit. Using equations (24), we obtain the following limits :

$$\begin{aligned} \Omega_{[\sigma_k]}^{(1,1)} &\xrightarrow{T \rightarrow 0} \frac{4 \pi a_0^2}{\sqrt{\pi m \beta}} = \Omega_0 \\ \Omega_{[\sigma_k]}^{(2,2)} \quad \text{and} \quad \Omega_{[\sigma_k]}^{(2,2)} &\xrightarrow{T \rightarrow 0} 2 \Omega_0 \\ \Omega_{[\sigma_k]}^{(1,2)} &\xrightarrow{T \rightarrow 0} 3 \Omega_0 \\ \Omega_{[\sigma_k]}^{(1,3)} &\xrightarrow{T \rightarrow 0} 12 \Omega_0. \end{aligned} \quad (67a)$$

From (56) and (61b), we then obtain when $\varepsilon = -1$:

$$\begin{aligned} \xi_1 &\xrightarrow{T \rightarrow 0} -\frac{16}{43} \\ \xi_2 &\xrightarrow{T \rightarrow 0} +1. \end{aligned} \quad (67b)$$

For fermions, this implies that :

$$\kappa_0(M) \xrightarrow{T \rightarrow 0} \kappa_0(M=0) \frac{1 + \frac{16}{43} M^2}{1 - M^2}. \quad (68)$$

When $M \rightarrow 1$, this result diverges. At low temperatures, a high nuclear polarization therefore increases the heat conduction coefficient dramatically; this phenomenon has already been discussed in [12] and interpreted in terms of a strong increase of the mean free path in the gas.

4.3 VISCOSITY. — Let us now consider a situation where the gas average velocity $\mathbf{u}_0$ is parallel to the x direction and has a gradient $\partial \mathbf{u}_0 / \partial z$ in the z direction (the temperature, pressure and magnetization are

⁽¹¹⁾ Nevertheless, we can use the general discussion of § 2.2 to obtain a classical analogue of these correlations. Since all spin density operators we use here commute with each other, we can use equations (38) and interpret the situation in terms of a mixture of two classical gases a and b . It is then well known [17] that a temperature gradient can induce different distortion of the Maxwell distributions of both gases, even if the two masses m_a and m_b and cross sections σ_{aa} and σ_{bb} are equal, provided that the cross section σ_{ab} is different from σ_{aa} .

constant over the sample). The local equilibrium density operator [equations (40)] is then :

$$\rho_s^0(\mathbf{r}, \mathbf{p}) = f_0(\mathbf{r}, \mathbf{p}) \bar{\rho}_s^0 \quad (69a)$$

with

$$f_0(\mathbf{r}, \mathbf{p}) = n \left(\frac{\beta}{2m\pi} \right)^{3/2} \exp \left\{ -\beta [\mathbf{p} - m\mathbf{u}_0(z)]^2 / 2m \right\} \quad (69b)$$

where the atomic density n and inverse temperature β are now constant. The drift term of equation (46) becomes :

$$\frac{1}{m} \mathbf{p} \cdot \nabla_{\mathbf{r}} \{ \rho_s^0(\mathbf{r}, \mathbf{p}) \} = \frac{\beta}{m} (p_x - mu_0) p_z \frac{\partial u_0}{\partial z} f_0(\mathbf{r}, \mathbf{p}) \bar{\rho}_s^0. \quad (70)$$

This term gives the constant term $|A \gg$ of equation (47a), which has to be solved separately at each point $\mathbf{r}$ of space. To do this, it is convenient to choose a (local) reference frame where u_0 is zero, so that the right hand side of (70) becomes proportional to $p_x p_z$. This simplifies the angular dependence of the drift term which becomes proportional to $(^{12})$

$$[Y_{\frac{1}{2}}(\hat{p}) - Y_{\frac{1}{2}}^{-1}(\hat{p})]$$

it is therefore « orthogonal » to all collision constants $n(r)$, $\mathcal{F}(r)$, $\mathcal{W}(r)$ and $\mathcal{M}(r)$, which ensures that the solution of equation (47a) will tend to a stationary limit when time increases. The rotational invariance of the collision operator Γ_c implies that this limit $\delta\bar{\rho}_s$ ($t = \infty$) has the same angular dependence on $\hat{p}$, and is thus proportional to $p_x p_z$. In addition, the rotational invariance of Γ_c in the spin space around the direction of $\mathbf{M}$ (we assume here that the atoms are spin 1/2 particles) can be used to show that the stationary value of $\delta\bar{\rho}_s$ can be written :

$$\delta\bar{\rho}_s = [\psi_0(p) + \psi_3(p) \sigma_3] p_x p_z \quad (71)$$

where $\psi_0(p)$ and $\psi_3(p)$ are any functions of the modulus p of $\mathbf{p}$. As in the study of heat conduction, it is easily seen that all commutators then vanish (no contribution of the identical spin rotation effect). Here also, to simplify the calculations, we shall use an approximate variational form :

$$\delta\bar{\rho}_s = \frac{1}{2} [g_0 + g_3 \sigma_3] p_x p_z \quad (72)$$

where g_0 and g_3 are two scalar parameters. To determine them, we use the method of moments and project equation (46) onto the operators $p_x p_z$ and $p_x p_z \sigma_3$;

(¹²) More generally, the present discussion is valid, whenever the drift term is proportional to

$$p_i p_j - \delta_{ij} \mathbf{p}^2 / 3$$

in the local reference frame.

a calculation very similar to that of the preceding section gives :

$$g_0 = \frac{n}{m\beta} \frac{\partial u_0}{\partial z} \frac{1}{n^2 \Delta_z} [Z_1 + \varepsilon Z_3 (1 - M^2)]$$

$$g_3 = \frac{n}{m\beta} \frac{\partial u_0}{\partial z} M \frac{Z_1}{n^2 \Delta_z} \quad (73a)$$

where the coefficients Z_1 , Z_2 and Z_3 are given by :

$$Z_2 = -\frac{1}{m^2 \beta^2} \left\{ \frac{4}{5} \Omega_{[\sigma_k]}^{(2,2)} \pm \frac{8}{3} \Omega_{[\sigma_k]}^{(1,1)} \right\}$$

$$Z_3 = -\frac{1}{m^2 \beta^2} \left\{ \frac{4}{5} \Omega_{[\sigma_k^{ex}]}^{(2,2)} \right\} \quad (73b)$$

[Δ_z is simply obtained by replacing the W 's by the Z 's in the right hand side of (58b)].

On the other hand, the xz component of the stress tensor is :

$$p_{xz} = \int d^3p f_0(p) p_x p_z \text{Tr} \{ \delta\bar{\rho}_s \}$$

$$= g_0 \frac{n}{m\beta^2} \quad (74a)$$

which is related to $\partial u_0 / \partial z$ by the relation :

$$p_{xz} = -\eta (\partial u_0 / \partial z) \quad (74b)$$

where η is the tangential viscosity of the gas. A simple calculation then gives :

$$\eta(M_0) = \eta(0) \frac{1 - \zeta_1 M^2}{1 - \zeta_2 M^2} \quad (75a)$$

where $\eta(0)$ is the viscosity of the unpolarized gas :

$$\eta(0) = \frac{5}{8} \frac{kT}{\Omega_{[\sigma_k]}^{(2,2)} + \frac{\varepsilon}{2} \Omega_{[\sigma_k^{ex}]}^{(2,2)}} \quad (75b)$$

and the coefficients ζ_1 and ζ_2 are given by :

$$\zeta_1 = \frac{\varepsilon Z_3}{Z_1 + \varepsilon Z_3}$$

$$\zeta_2 = \frac{\varepsilon Z_3 (Z_2 + \varepsilon Z_3)}{(Z_1 + Z_2 + \varepsilon Z_3) (Z_1 + \varepsilon Z_3)} \quad (75c)$$

Equation (75a) is very similar to (60) and all physical discussion in the preceding section of the variations of the heat conduction can be transposed to the viscosity coefficient. Formula (68), which is valid for fermions, has to be replaced here by :

$$\eta(M) \xrightarrow{T \rightarrow 0} \eta(0) \frac{1 + \frac{3}{5} M^2}{1 - M^2} \quad (76)$$

where we see again that there is a divergence when $M \rightarrow 1$, reinforced by the M dependence in the numerator, arising from spin-velocity correlations in the gas.

5. Conclusion. — The Boltzmann equation we have written for the semi-classical operator $\rho_s(\mathbf{r}, \mathbf{p})$ is well adapted to studying particle indistinguishability effects in dilute systems, when the nuclear polarization is high. This equation has been obtained from a detailed study of the change of spin variables in a binary collision; in addition to the « classical » collision term in $\sigma_k(\theta)$, the equation includes several terms which have the form of commutators and anticommutators, and which give rise to changes in the spin direction (even though the spins do not interact with each other).

The theory we have developed can be adapted, with minor changes, to very dilute solutions of ^3He in superfluid ^4He , when degeneracy effects are negligible. The transport properties of degenerate spin polarized solutions of ^3He in ^4He have been studied by E.P. Bashkin and Meyerovich [18, 19]; these authors use a collision term which includes explicitly the degeneracy effects (occupation numbers), but does not contain all terms obtained in the present article (for example, the « identical spin rotation » terms).

On the other hand, for pure spin polarized ^3He in the normal liquid state, our calculations cannot be directly used: in the Boltzmann equation, that we have written, the simple *drift* term must be replaced by the drift term of the Landau kinetic equation [20], which depends on the $\mathbf{r}$ and $\mathbf{p}$ gradients of the quasi-particle energies (degeneracy effects). On the other hand, our study of the *collision* term in the presence of a high nuclear polarization should apply to quasi-particle interactions in pure liquid ^3He , with some changes (factors involving final state occupation numbers). Actually, the particle indistinguishability effects in collisions, and the degeneracy effects which dominate the drift term in the Landau kinetic theory, are not completely unrelated phenomena, in so far as the former can be physically interpreted as arising from the « transient degeneracy » during collisions. We shall see an illustration of this similarity in the next article, where we study the spin diffusion in a dilute polarized gas, and predict anisotropy effects which are strongly reminiscent of some consequences of the degeneracy of pure liquid ^3He .

A difference between a liquid and a dilute gas is that, for a gas, all transport phenomena can be obtained from a Boltzmann equation where no phenomenological constant enters (they can be obtained from first principles provided that the 2 atom potential is sufficiently well known, which is the case for He); in liquids — pure or dilute ^3He — phenomenological constants (effective mass, etc.) have to be included in the theory. For this reason, experiments on the gas phase may look particularly attractive. On the other hand, the study of very dilute solution of ^3He in superfluid ^4He offers the possibility of in principle

going to arbitrarily low temperatures (in the gas phase, a given density will correspond to a minimum temperature, where liquefaction occurs). Therefore, both systems should be worth studying experimentally.

Appendix I. — In this Appendix, we show that the collision term of equation (32b) does not change the following physical quantities: atomic number density $n(\mathbf{r})$, spin orientation density $\mathcal{M}(\mathbf{r})$, linear momentum density $\mathcal{P}(\mathbf{r})$ and, finally, energy density $\mathcal{W}(\mathbf{r})$ [these quantities are defined by equations (35)].

Let us first consider the value, after integration over d^3p , of the right hand side of equation (32b). Instead of $\mathbf{p}$ and $\mathbf{q}$, a more convenient set of integration variables is here Π and $\mathbf{q}$ defined by:

$$\begin{aligned} \mathbf{p}_1 &= \Pi + \frac{\mathbf{q}}{2} & \mathbf{p}'_1 &= \Pi + \frac{\mathbf{q}'}{2} \\ \mathbf{p}_2 &= \Pi - \frac{\mathbf{q}}{2} & \mathbf{p}'_2 &= \Pi - \frac{\mathbf{q}'}{2}. \end{aligned} \quad (\text{I.1})$$

It is now clear that the two pairs, $\mathbf{p}_1$ and $\mathbf{p}_2$ on one side, $\mathbf{p}'_1$ and $\mathbf{p}'_2$ on the other, play the same role in the integrations. Since $\sigma_k(\theta)$ and $\sigma_k^{\text{ex}}(\theta)$ are not affected when these pairs are interchanged, and since they are multiplied in (32b) by functions which change sign, they give no contribution to the integral. The same cancellation also occurs for the term in $\tau_k^{\text{ex}}(\theta)$ since we can subtract a term in $[\rho_s(\mathbf{p}_1), \rho_s(\mathbf{p}_2)]$ without affecting it (see § 2.1). Finally, the term in $\tau_{\text{fwd}}^{\text{ex}}$ can be written

$$\int d^3\Pi \int d^3q \frac{q}{m} \left[\rho_s\left(\Pi - \frac{\mathbf{q}}{2}\right), \rho_s\left(\Pi + \frac{\mathbf{q}}{2}\right) \right]. \quad (\text{I.2})$$

The commutator being an odd function of $\mathbf{q}$, this integral vanishes. We therefore have

$$\left. \frac{d}{dt} \right|_{\text{coll}} \int d^3p \rho_s(\mathbf{r}, \mathbf{p}) = 0$$

so that $n(\mathbf{r})$ and $\mathcal{M}(\mathbf{r})$ are unchanged under the effect of collisions.

Since

$$\begin{aligned} \mathbf{p}_1 + \mathbf{p}_2 &= \mathbf{p}'_1 + \mathbf{p}'_2 \\ (\mathbf{p}_1)^2 + (\mathbf{p}_2)^2 &= (\mathbf{p}'_1)^2 + (\mathbf{p}'_2)^2 \end{aligned}$$

(energy and momentum conservation in an elastic collision), and since the term in $\tau_{\text{fwd}}^{\text{ex}}(q)$ is a commutator with zero trace, a similar reasoning shows that $\mathcal{P}(\mathbf{r})$ and $\mathcal{W}(\mathbf{r})$ are also unchanged.

Appendix II. — The linearized Boltzmann equation (46) can conveniently be written with the Dirac ket notation:

$$\frac{d}{dt} |\delta \bar{\rho}_s\rangle = \Gamma_c |\delta \bar{\rho}_s\rangle + |A\rangle \quad (\text{II.1})$$

where $|\delta\bar{\rho}_s\rangle\rangle$ is considered as a « ket » belonging to a space $\mathcal{F}$, which is analogous to the state space of a system possessing orbital variables (momentum $\mathbf{p}$) and a spin; the notation $|\rangle\rangle$ emphasizes that $|\delta\bar{\rho}_s\rangle\rangle$ is not a ket in the original state space $\mathcal{E}$ of the particle but a ket of a larger ⁽¹³⁾ space $\mathcal{F}$. The « components » of $|\delta\bar{\rho}_s\rangle\rangle$ are the numbers :

$$\text{Tr} \{ \sigma_i \delta\bar{\rho}_s(\mathbf{p}) \} \quad (\text{II.2})$$

which depend on one discrete index i and one continuous variable $\mathbf{p}$ ⁽¹⁴⁾. Here, the σ_i 's are a series of hermitian operators forming a basis in the space of the spin operators (for spin 1/2 atoms, $\sigma_i = 1, \sigma_x, \sigma_y, \sigma_z$); these quantities may be considered here as the $(2I+1)^2$ components of a spinor. Similarly the « drift ket » $|A\rangle\rangle$ is defined by its components :

$$\frac{1}{mf_0(\mathbf{r}, \mathbf{p})} \mathbf{p} \cdot \nabla_{\mathbf{r}} [\text{Tr} \{ \sigma_i \rho_s^0(\mathbf{r}, \mathbf{p}) \}]. \quad (\text{II.3})$$

In space $\mathcal{F}$, a scalar product $\langle\langle \delta\bar{\rho}_s | \delta\bar{\rho}_s' \rangle\rangle$ can be defined by :

$$\langle\langle \delta\bar{\rho}_s | \delta\bar{\rho}_s' \rangle\rangle = \int d^3p f_0(p) \text{Tr} \{ \delta\bar{\rho}_s^\dagger(\mathbf{p}) \delta\bar{\rho}_s'(\mathbf{p}) \}. \quad (\text{II.4})$$

The linear collision operator Γ_c acts in the space $\mathcal{F}$ and its matrix elements $\langle\langle \mathbf{p}, \sigma | \Gamma_c | \mathbf{p}', \sigma' \rangle\rangle$ are directly obtained from (45). We shall see in Appendix IV that, in opposition to what frequently occurs in classical statistical mechanics, the collision operator Γ_c is not hermitian [this is due to the presence of the commutators in (45), i.e. to the « identical spin rotation effect »]. The eigenvectors of Γ_c are written $|\gamma_n^\tau\rangle\rangle$, with eigenvalues γ_n (the index τ is for possible degeneracies) :

$$\Gamma_c |\gamma_n^\tau\rangle\rangle = \gamma_n |\gamma_n^\tau\rangle\rangle. \quad (\text{II.5a})$$

Similarly, for the hermitian conjugate operator $\Gamma_c^\dagger$, we write

$$\Gamma_c^\dagger |\bar{\gamma}_n^\tau\rangle\rangle = \gamma_n^* |\bar{\gamma}_n^\tau\rangle\rangle. \quad (\text{II.5b})$$

If we can expand $|\delta\bar{\rho}_s\rangle\rangle$ and $|A\rangle\rangle$ on the kets $|\gamma_n^\tau\rangle\rangle$:

$$\begin{aligned} |\delta\bar{\rho}_s(t)\rangle\rangle &= \sum_{n\tau} x_{n\tau}(t) |\gamma_n^\tau\rangle\rangle \\ |A\rangle\rangle &= \sum_{n\tau} a_{n\tau} |\gamma_n^\tau\rangle\rangle \end{aligned} \quad (\text{II.6a})$$

equation (II.1) becomes equivalent to :

$$\frac{d}{dt} x_{n\tau}(t) = \gamma_n x_{n\tau}(t) + a_{n\tau}. \quad (\text{II.6b})$$

⁽¹³⁾ In $\mathcal{F}$, the quantum number associated with the spin can take $(2I+1)^2$ different values [dimension of the Liouville spin state space] instead of $(2I+1)$.

⁽¹⁴⁾ In this formalism, $\mathbf{r}$ is an unessential parameter and we shall not write the corresponding dependence explicitly.

When the real part of γ_n is negative, the stationary solution is given by :

$$x_{n\tau} = -\frac{a_{n\tau}}{\gamma_n}. \quad (\text{II.6c})$$

Similarly, if we write :

$$\begin{aligned} \bar{x}_{n\tau'}(t) &= \langle\langle \bar{\gamma}_{n'}^\tau | \delta\bar{\rho}_s(t) \rangle\rangle \\ \bar{a}_{n\tau'} &= \langle\langle \bar{\gamma}_{n'}^\tau | A \rangle\rangle \end{aligned} \quad (\text{II.7a})$$

we have :

$$\frac{d}{dt} \bar{x}_{n\tau'}(t) = \gamma_n \bar{x}_{n\tau'}(t) + \bar{a}_{n\tau'} \quad (\text{II.7b})$$

and the stationary solution is ⁽¹⁵⁾ :

$$\bar{x}_{n\tau'} = -\frac{\bar{a}_{n\tau'}}{\gamma_n}. \quad (\text{II.7c})$$

We know that Γ_c has 8 eigenkets with zero eigenvalues, which correspond to $n(\mathbf{r})$, $\mathcal{W}(\mathbf{r})$, and the components of $\mathcal{P}(\mathbf{r})$ and $\mathcal{M}(\mathbf{r})$. In section 3.1, we have assumed that Γ_c does not possess any other « undamped » eigenkets, which implies that the local equilibrium density operator $\bar{\rho}_s$ is a well defined function of these 8 conserved quantities. In addition, the stability assumption imposes that all the other eigenvalues have a negative real part.

If both these conditions are realized, two different situations may occur depending on the properties of the « drift ket » $|A\rangle\rangle$:

(i) if some components $a_{n\tau}$ (or equivalently $\bar{a}_{n\tau'}$) of $|A\rangle\rangle$ on the eigenkets $|\gamma_n\rangle\rangle$ with zero eigenvalues are not zero, equation (II.6b) [or (II.7b)] shows that the corresponding components of $|\delta\bar{\rho}_s\rangle\rangle$ change linearly in time. Then, $|\delta\bar{\rho}_s\rangle\rangle$ has no limit when $t \rightarrow \infty$;

(ii) if all components $a_{n\tau}$ (or $\bar{a}_{n\tau'}$) are zero whenever $\gamma_n = 0$, equations (II.6b) [or (II.7b)] show that the corresponding components of $|\delta\bar{\rho}_s\rangle\rangle$ remain constant for all times; the other components tend to limits given by (II.6c) [or (II.7c)] and $|\delta\bar{\rho}_s\rangle\rangle$ tends to a well defined limit when $t \rightarrow \infty$.

In practice, situation (ii) is the only one we consider in this article but, in each particular case, it must be checked that the « drift ket » $|A\rangle\rangle$ has zero components on all « undamped » modes (those with $\gamma_n = 0$).

In transport properties studies, the physical quan-

⁽¹⁵⁾ One has

$$\langle\langle \bar{\gamma}_{n'}^\tau | \Gamma_c | \gamma_n^\tau \rangle\rangle = \gamma_n \langle\langle \bar{\gamma}_{n'}^\tau | \gamma_n^\tau \rangle\rangle = \gamma_{n'} \langle\langle \bar{\gamma}_{n'}^\tau | \gamma_n^\tau \rangle\rangle$$

which shows that the scalar product $\langle\langle \bar{\gamma}_{n'}^\tau | \gamma_n^\tau \rangle\rangle$ is zero whenever $\gamma_n \neq \gamma_{n'}$. This shows that $\bar{x}_{n\tau'}$ [or $\bar{a}_{n\tau'}$] is a linear combination of the $x_{n\tau}$'s [or $a_{n\tau}$'s] with $n = n'$ and, in particular, that $\bar{x}_{n\tau'} = x_{n\tau}$ [and $\bar{a}_{n\tau'} = a_{n\tau}$] when the eigenvalue γ_n is non-degenerate. As a consequence, the equations of evolution (II.6b) and (II.7b) are equivalent.

tity of interest is often a current J which is a linear combination of the components of $|\delta\bar{\rho}_s\rangle\rangle$ ⁽¹⁶⁾ :

$$J = \langle\langle B | \delta\bar{\rho}_s \rangle\rangle \quad (\text{II.8})$$

where the bra $\langle\langle B |$ depends on the problem under study (J is the energy current in the study of heat conduction, the magnetization current in the study of spin diffusion, etc...). The stationary (hydrodynamic) value of J is given by :

$$J_0 = - \sum_{\pi} \frac{1}{\gamma_n} \langle\langle B_J | \gamma_n^{\pi} \rangle\rangle a_n^{\pi}. \quad (\text{II.9})$$

This results shows that the kets which give a large contribution to J are the kets with small $|\gamma_n|$ (slow modes — but $\gamma_n = 0$ is excluded), having a large scalar product with the ket $|B_J\rangle\rangle$ associated to the flux of interest J , and a large « excitation term » a_n^{π} .

It is apparent on (45) that Γ_c depends on $\bar{\rho}_s^0(\mathbf{r})$, which is not rotation invariant, unless $M_0(\mathbf{r})$ is zero [see (40c)]. Thus, Γ_c is not invariant in any rotation in the spin space. On the other hand, the same operator is invariant in any rotation in the $\mathbf{p}$ space (orbital variables). This is because, in the right hand side of (45), all distribution functions f_0 and cross sections σ_q , σ_q^{ex} , τ_q^{ex} and $\tau_{\text{fwd}}^{\text{ex}}$ are invariant when all momenta $\mathbf{p}$ undergo the same rotation. In space $\mathcal{F}$, we therefore have

$$[R_{\text{orb}}, \Gamma_c] = 0 \quad (\text{II.10})$$

where R_{orb} is any rotation operator acting only on the orbital variables.

In addition, if $\bar{\rho}_s^0(\mathbf{r})$ is invariant in all rotations about a particular axis (for spin 1/2 particles, this is always true for rotations with axis parallel to the spin orientation), Γ_c clearly has the same invariance property.

Appendix III. — In practice, the structure of the collision operator is often too complicated to allow an exact solution of the linearized Boltzmann equation (II.1). Nevertheless, the approximation methods which are classical for solving the usual Boltzmann equation (without spin) can be transposed to this case. For example, if one has identified one slow mode which is strongly coupled to J (large value of $\langle\langle B_J | \gamma_n^{\pi} \rangle\rangle$) and also to the drift term, equation (II.9) can be truncated by restricting the $\pi\tau$ summation to this mode.

Another usual approximation is the « method of truncated basis ». On physical grounds, one chooses a family of trial kets $|\delta\bar{\rho}_s(\lambda_1, \dots, \lambda_n)\rangle\rangle$, depending on n parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, and the same number of bras $\langle\langle \varphi_1 |, \langle\langle \varphi_2 |, \dots, \langle\langle \varphi_n |$. If the true stationary solution belongs to this family, the corresponding values of the λ 's can be obtained from the n equations :

$$\langle\langle \varphi_p | \Gamma_c | \delta\bar{\rho}_s(\lambda_1 \dots \lambda_n) \rangle\rangle + \langle\langle \varphi_p | A \rangle\rangle = 0 \quad p = 1, 2, \dots, n. \quad (\text{III.1})$$

The method of truncated basis consists in still using the same set of equations to calculate the values of $\lambda_1 \dots \lambda_n$, when the selected family does not necessarily includes the exact solution (which is nevertheless assumed to be reasonably close to one of its kets). In practice, the family of trial kets is often the kets of a subspace $\mathcal{F}_\varphi$, and the $|\varphi_p\rangle\rangle$ are a basis of this subspace; then the method of truncated basis amounts to replacing the real collision operator by its restriction inside $\mathcal{F}_\varphi$.

Let us for example examine the case where two parameters λ_1 and λ_2 are considered, and where the family of trial kets is given by :

$$|\delta\bar{\rho}_s\rangle\rangle = \lambda_1 |\psi_1\rangle\rangle + \lambda_2 |\psi_2\rangle\rangle.$$

The kets $|\psi_1\rangle\rangle$ and $|\psi_2\rangle\rangle$ are fixed, and they may correspond to the bras $\langle\langle \varphi_1 |$ and $\langle\langle \varphi_2 |$ or not. Then, a simple calculation (2×2 matrix inversion) using the method of truncated basis gives the λ 's, and one finally finds the following value for the current J :

$$J \simeq \frac{1}{A} \left\{ \begin{aligned} &\langle\langle \varphi_1 | \Gamma_c | \psi_1 \rangle\rangle \langle\langle \varphi_2 | A \rangle\rangle \langle\langle B_J | \psi_2 \rangle\rangle \\ &+ \langle\langle \varphi_2 | \Gamma_c | \psi_2 \rangle\rangle \langle\langle \varphi_1 | A \rangle\rangle \langle\langle B_J | \psi_1 \rangle\rangle \\ &- \langle\langle \varphi_1 | \Gamma_c | \psi_2 \rangle\rangle \langle\langle \varphi_2 | A \rangle\rangle \langle\langle B_J | \psi_1 \rangle\rangle \\ &- \langle\langle \varphi_2 | \Gamma_c | \psi_1 \rangle\rangle \langle\langle \varphi_1 | A \rangle\rangle \langle\langle B_J | \psi_2 \rangle\rangle \end{aligned} \right\} \quad (\text{III.2a})$$

with

$$A = \langle\langle \varphi_1 | \Gamma_c | \psi_1 \rangle\rangle \langle\langle \varphi_2 | \Gamma_c | \psi_2 \rangle\rangle - \langle\langle \varphi_2 | \Gamma_c | \psi_1 \rangle\rangle \langle\langle \varphi_1 | \Gamma_c | \psi_2 \rangle\rangle. \quad (\text{III.2b})$$

Variational methods can also be used to obtain approximate solutions. An interesting variational method has been introduced by Roussopoulos [21, 22]; this principle remains valid when the collision operator Γ_c is not hermitian, and the errors on the current J are only second order. In our notation, Roussopoulos considers the two equations

$$\begin{aligned} \Gamma_c | X_0 \rangle\rangle + | A \rangle\rangle &= 0 \\ \Gamma_c | Y_0 \rangle\rangle + | B_J \rangle\rangle &= 0 \end{aligned} \quad (\text{III.3})$$

The ket $|X_0\rangle\rangle$ is the (unknown) limit of $|\delta\bar{\rho}_s(t)\rangle\rangle$ when $t \rightarrow \infty$. Then, if $|X\rangle\rangle$ and $|Y\rangle\rangle$ are any kets, the following function of these kets is introduced

$$F[|X\rangle\rangle, |Y\rangle\rangle] = \langle\langle Y | \Gamma_c | X \rangle\rangle + \langle\langle B_J | X \rangle\rangle + \langle\langle Y | A \rangle\rangle. \quad (\text{III.4})$$

If $|X\rangle\rangle = |X_0\rangle\rangle$, one easily obtains from (III.3) :

$$F[|X_0\rangle\rangle, |Y\rangle\rangle] = \langle\langle B_J | X_0 \rangle\rangle \quad (\text{III.5a})$$

⁽¹⁶⁾ At equilibrium, the flux is zero, which means that $\langle\langle B | \rho_s^0 \rangle\rangle = 0$.

and, if $|Y\rangle = |Y_0\rangle$:

$$\begin{aligned} F[|X\rangle, |Y_0\rangle] &= \langle Y_0 | A \rangle \\ &= - \langle Y_0 | \Gamma_c | X_0 \rangle \\ &= \langle B_J | X_0 \rangle. \end{aligned} \quad (\text{III.5b})$$

Therefore, if either $|X\rangle$ coincides with $|X_0\rangle$ or $|Y\rangle$ with $|Y_0\rangle$ (or both), F becomes independent of the other ket and is simply equal to :

$$\langle B_J | X_0 \rangle = J_0 \quad (\text{III.6})$$

which is nothing but the current we wish to evaluate.

Now, if we set :

$$\begin{aligned} |X\rangle &= |X_0\rangle + |\delta X\rangle \\ |Y\rangle &= |Y_0\rangle + |\delta Y\rangle \end{aligned} \quad (\text{III.7})$$

the variations of F are necessarily given by the crossed term in $|\delta X\rangle$ and $|\delta Y\rangle$. These crossed term can only be introduced by $\langle Y | \Gamma_c | X \rangle$ in (III.4), and one readily obtains :

$$\begin{aligned} \delta F &= F[|X_0\rangle + |\delta X\rangle, |Y_0\rangle + |\delta Y\rangle] - J_0 = \\ &= \langle \delta Y | \Gamma_c | \delta X \rangle. \end{aligned} \quad (\text{III.8})$$

The variational method is then the following : one chooses two families of trial kets $|X(\lambda_1 \dots \lambda_n)\rangle$ and $|Y(\mu_1 \dots \mu_p)\rangle$ and one calculates the quantity :

$$\begin{aligned} F(\lambda_1 \dots \lambda_n; \mu_1 \dots \mu_p) &= \\ &= \langle Y(\mu_1 \dots \mu_p) | \Gamma_c | X(\lambda_1 \dots \lambda_n) \rangle. \end{aligned} \quad (\text{III.9})$$

The « best values » for the λ 's and μ 's are obtained by the equations :

$$\begin{aligned} \frac{\partial F}{\partial \lambda_i} &= 0 \\ \frac{\partial F}{\partial \mu_j} &= 0 \end{aligned} \quad (\text{III.10})$$

(if several sets of λ 's and μ 's are obtained, one of them must be chosen on physical grounds). The corresponding value of F gives an approximate value for J_0 .

To give an example of application, let us choose the two linear ket and bra families :

$$\begin{aligned} |X(\lambda_1, \lambda_2)\rangle &= \lambda_1 |\psi_1\rangle + \lambda_2 |\psi_2\rangle \\ \langle Y(\mu_1, \mu_2)| &= \mu_1 \langle \varphi_1| + \mu_2 \langle \varphi_2|. \end{aligned}$$

One then obtains :

$$\begin{aligned} F(\lambda_1, \lambda_2, \mu_1, \mu_2) &= \lambda_1 \mu_1 \langle \varphi_1 | \Gamma_c | \psi_1 \rangle + \\ &+ \lambda_2 \mu_2 \langle \varphi_2 | \Gamma_c | \psi_2 \rangle + \lambda_1 \mu_2 \langle \varphi_2 | \Gamma_c | \psi_1 \rangle \\ &+ \lambda_2 \mu_1 \langle \varphi_1 | \Gamma_c | \psi_2 \rangle + \lambda_1 \langle B_J | \psi_1 \rangle \\ &+ \lambda_2 \langle B_J | \psi_2 \rangle + \mu_1 \langle \varphi_1 | A \rangle \\ &+ \mu_2 \langle \varphi_2 | A \rangle. \end{aligned}$$

Equation (III.10) then gives only one set of possible values and, after some simple calculations, one obtains the corresponding value of F which is nothing but the right hand side of equation (III.2a). Thus, in this particular case, we find again the result of the method of moments; in addition, the preceding considerations show that, if the kets $|\psi_1\rangle, |\psi_2\rangle, |\varphi_1\rangle$ and $|\varphi_2\rangle$ are well chosen, the error made is only small, since it is a second order correction. The Roussopoulos method also provides a criterion for the choice of $|\varphi_1\rangle$ and $|\varphi_2\rangle$: by linear combination, they must give a good approximation of $|Y_0\rangle$ ⁽¹⁷⁾. Nevertheless, the function F in itself does not give any criterion to choose between several families of trial kets since, in general δF can be positive or negative; no upper or lower limit of J_0 is therefore obtained by this variational method.

Appendix IV. — In this appendix, we study the hermitian or non-hermitian character of the various term of Γ_c which appear in the right hand side of (45). This discussion will be given in term of a new density operator ⁽¹⁸⁾ :

$$\delta \tilde{\rho}_s(\mathbf{p}) = (\bar{\rho}_s^0)^{-1/2} \delta \bar{\rho}_s(\mathbf{p}) (\bar{\rho}_s^0)^{-1/2} \quad (\text{IV.1})$$

(we shall see below why $\delta \tilde{\rho}_s$ is more convenient than $\delta \bar{\rho}_s$ for this discussion). The evolution of $\delta \tilde{\rho}_s$ due to collisions can be obtained by multiplying (45) by $(\bar{\rho}_s^0)^{-1/2}$ on both sides, and replacing $\delta \bar{\rho}_s$ by :

$$(\bar{\rho}_s^0)^{1/2} \delta \tilde{\rho}_s(\mathbf{p}) (\bar{\rho}_s^0)^{1/2}.$$

The effects of these two operations actually cancel for all terms of the right hand side of (45), except for the term in :

$$\sigma_k(\theta) \bar{\rho}_s^0 \text{Tr} \{ \delta \bar{\rho}_s(\mathbf{p}'_2) - \delta \bar{\rho}_s(\mathbf{p}_2) \}$$

which becomes :

$$\sigma_k(\theta) \text{Tr} \{ \bar{\rho}_s^0 [\delta \tilde{\rho}_s(\mathbf{p}'_2) - \delta \tilde{\rho}_s(\mathbf{p}_2)] \}.$$

When this substitution is made, equation (45) directly gives the matrix elements of $\tilde{\Gamma}_c$, which is the linear collision operator for $|\delta \tilde{\rho}_s\rangle$.

We shall use here a definition of the scalar product which differs from the definition in the preceding appendix. Our definition is now :

$$((\delta \tilde{\rho}_s | \delta \tilde{\rho}_s)) = \int d^3 p f_0(p) \text{Tr} \{ \delta \tilde{\rho}_s^\dagger(\mathbf{p}) \bar{\rho}_s^0 \delta \tilde{\rho}_s(\mathbf{p}) \}. \quad (\text{IV.2})$$

⁽¹⁷⁾ In practice, the drift and current kets $|A\rangle$ and $|B_J\rangle$ are often collinear. Therefore, if Γ_c is hermitian it is natural to take the same set of kets for the $|\psi\rangle$'s and $|\varphi\rangle$'s. On the other hand, if Γ_c is not hermitian, it may happen that these sets have to be different in order to give good approximations of the solutions $|X_0\rangle$ and $|Y_0\rangle$.

⁽¹⁸⁾ We assume here that $[\bar{\rho}_s^0]^{-1}$ exists, so that $\delta \tilde{\rho}_s$ can be defined; for spin 1/2 particle, this is true if M_0 is strictly less than one.

One can easily check that, since $\bar{\rho}_s^0$ is a definite positive operator, the number $((\delta\tilde{\rho}_s | \delta\tilde{\rho}_s))$ is always real and ≥ 0 ⁽¹⁹⁾.

Let us now study the contributions to

$$((\delta\tilde{\rho}_s' | \tilde{\Gamma}_c | \delta\tilde{\rho}_s))$$

of the various term of $\tilde{\Gamma}_c$ given by the right hand side of (45). This matrix element is obtained by an integration over d^3p which, like in appendix I, is more conveniently performed in term of the variable Π , defined by (I.1). Let us for example write explicitly the contribution of the term in $\sigma_k(\theta) \delta\tilde{\rho}_s(\mathbf{p}_1)$:

$$\int d^3\Pi \int d^3q' v_r \int d^2\hat{q} \sigma_k(\theta) f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) \text{Tr} \{ \delta\tilde{\rho}_s'^{\dagger}(\mathbf{p}_1) \bar{\rho}_s^0 \delta\tilde{\rho}_s(\mathbf{p}_1) \} \quad (\text{IV.3})$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are given by (I.3). Since:

$$f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) = f_0(\mathbf{p}_1') f_0(\mathbf{p}_2') \quad (\text{IV.4})$$

we can make the corresponding substitution in (IV.3), and then interchange (as in appendix I) $\mathbf{p}_1$ with $\mathbf{p}_1'$, $\mathbf{p}_2$ with $\mathbf{p}_2'$, in order to obtain

$$\int d^3\Pi \int d^3q' v_r \int d^2\hat{q} \sigma_k(\theta) f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) \text{Tr} \{ \delta\tilde{\rho}_s'^{\dagger}(\mathbf{p}_1') \bar{\rho}_s^0 \delta\tilde{\rho}_s(\mathbf{p}_1) \} \quad (\text{IV.5})$$

If, finally, we take the complex conjugate by reversing the order of the operators in the trace, we obtain exactly the contribution of the first term to

$$((\delta\tilde{\rho}_s | \tilde{\Gamma}_c | \delta\tilde{\rho}_s')).$$

This term is therefore hermitian.

The term in $\sigma_k(\theta) \delta\tilde{\rho}_s(\mathbf{p}_1)$ is obviously hermitian; since $\mathbf{p}_1$ is simply equal to $\mathbf{p}$, this term is diagonal in the $\mathbf{p}$ basis, and has real matrix elements.

Let us now consider the two last « classical » terms (they differ in $\tilde{\Gamma}_c$ and $\tilde{\Gamma}_c'$). They introduce integrals which are similar to (IV.2), but where the trace has to be replaced by the product:

$$\text{Tr} \{ \delta\tilde{\rho}_s'^{\dagger}(\mathbf{p}_1) \bar{\rho}_s^0 \} \text{Tr} \{ \bar{\rho}_s^0 \delta\tilde{\rho}_s(\mathbf{p}_2) \}$$

(or the same expression with $\mathbf{p}_2$ replaced by $\mathbf{p}_2'$).

By reversing the order of the operators in the traces and interchanging $\mathbf{p}_1$ and $\mathbf{p}_2$ (which amounts to reversing $\mathbf{q}$), one sees that the first term is hermitian, the second one has the same property, as can easily be seen by the same operations as for the integral (IV.3).

We therefore see that all « classical » terms (those which are not introduced by particle indistinguishability) are hermitian. Let us now study the terms in $\sigma_k^{\text{ex}}(\theta)$ of equation (45). Terms in

$$\bar{\rho}_s^0 \delta\tilde{\rho}_s(\mathbf{p}_i)$$

($i = 1, 2, 1', 2'$) lead to an integral similar to (IV.3) containing

$$\text{Tr} \{ \delta\tilde{\rho}_s'^{\dagger}(\mathbf{p}_i) [\bar{\rho}_s^0]^2 \delta\tilde{\rho}_s(\mathbf{p}_i) \}$$

which, by the same reasoning as above, gives an hermitian contribution. In the same way, for term in:

$$\delta\tilde{\rho}_s(\mathbf{p}_i) \bar{\rho}_s^0$$

the trace becomes

$$\text{Tr} \{ \delta\tilde{\rho}_s'^{\dagger}(\mathbf{p}_i) \bar{\rho}_s^0 \delta\tilde{\rho}_s(\mathbf{p}_i) \bar{\rho}_s^0 \}$$

which, by circular permutation of the operators, can be written as the complex conjugate of:

$$\text{Tr} \{ \delta\tilde{\rho}_s^{\dagger}(\mathbf{p}_i) \bar{\rho}_s^0 \delta\tilde{\rho}_s'(\mathbf{p}_i) \bar{\rho}_s^0 \}.$$

Again, the same reasoning shows that these terms are hermitian.

As for the terms in $\tau_k^{\text{ex}}(\theta)$, they give rise to exactly the same calculations, with only one difference: they include an pure imaginary factor $i\varepsilon$ instead of ε . They are therefore antihermitian instead of hermitian.

The same is true for the last term, proportional to $i\varepsilon\tau_{\text{fwd}}^{\text{ex}}(k)$. This is obvious for the term in $\delta\tilde{\rho}_s(\mathbf{p})$, which is diagonal in the $\mathbf{p}$ basis, with pure imaginary coefficients; as for the term in $\delta\tilde{\rho}_s(\mathbf{p} - \mathbf{q})$, it introduces the integral

$$i \frac{\varepsilon}{2} \int d^3\Pi \int d^3q v_r \tau_{\text{fwd}}^{\text{ex}}(k) \text{Tr} \left\{ \delta\tilde{\rho}_s'^{\dagger} \left(\Pi + \frac{\mathbf{q}}{2} \right) \left[\bar{\rho}_s^0, \delta\tilde{\rho}_s \left(\Pi - \frac{\mathbf{q}}{2} \right) \right] \right\}.$$

A change of sign of $\mathbf{q}$, followed by a complex conjugation, shows that this term is also antihermitian.

Both operators $\tilde{\Gamma}_c$ and $\tilde{\Gamma}_c'$ have the same eigenvalues. Since $\tilde{\Gamma}_c'$ includes non-hermitian terms, the eigenvalues of $\tilde{\Gamma}_c$ are in general not real, but also have an imaginary part. This fact is entirely a consequence of particle indistinguishability effects and more precisely, as shown by the preceding discussion, of the « identical spin rotation effect ». It is physically natural that a spin rotation should give some oscillatory character to the effect of collisions (pure imaginary eigenvalues). In the next article, we shall see that these oscillations may have important consequences on the spin diffusion properties of the gas.

⁽¹⁹⁾ If $[\bar{\rho}_s^0]^{-1}$ exists, $((\delta\tilde{\rho}_s | \delta\tilde{\rho}_s))$ is zero only if $\delta\tilde{\rho}_s = 0$.

Appendix V. — In this appendix we explicit the calculations involved in the determination of the trial wave function by the method of moments for the problem of heat conduction.

As explained in appendix III we replace equation (46) by the weaker requirements :

$$\int d^3p \operatorname{Tr} \left\{ \varphi_0(p) p_{x_i} \frac{\mathbf{p}}{m} \cdot \nabla_r \bar{\rho}_s^0(\mathbf{r}, \mathbf{p}) \right\} = \int d^3p \operatorname{Tr} \left\{ \varphi_0(p) p_{x_i} \Gamma_c \delta \bar{\rho}_s(\mathbf{r}, \mathbf{p}) \right\} \quad (\text{V.1})$$

$$\int d^3p \operatorname{Tr} \left\{ \varphi_0(p) p_{x_i} \sigma_z \frac{\mathbf{p}}{m} \cdot \nabla_r \bar{\rho}_s^0(\mathbf{r}, \mathbf{p}) \right\} = \int d^3p \operatorname{Tr} \left\{ \varphi_0(p) p_{x_i} \sigma_z \Gamma_c \delta \bar{\rho}_s(\mathbf{r}, \mathbf{p}) \right\} \quad (\text{V.2})$$

which leads directly to the two linear equations (54).

The left hand side members are readily evaluated from (48b) :

$$\begin{aligned} L &= \int d^3p \varphi_0(p) p_{x_i} \frac{\mathbf{p}}{m} \cdot \nabla_r \bar{\rho}_s^0(\mathbf{r}, \mathbf{p}) \\ &= \frac{\rho_s^0}{m} \int d^3p f_0(p) \left[\frac{p^2}{2mkT} - \frac{5}{2} \right]^2 p_{x_i} p_{x_j} \nabla_j \{ \operatorname{Log} T \} \end{aligned} \quad (\text{V.3})$$

the integral over the angular variables $d^2\hat{p}$ leads to

$$L = \frac{\rho_s^0}{m} \nabla_i \{ \operatorname{Log} T \} \frac{4\pi}{3} \int dp f_0(p) p^4 \left[\frac{p^2}{2mkT} - \frac{5}{2} \right]^2 \quad (\text{V.4})$$

the last integration can then be done by part or by the use of the normalization conditions of the Sonine (Laguerre) polynomials :

$$S_m^n(x) = \sum_{p=0}^n (-x)^p \frac{(m+p+1)(m+n)}{(n-p)! p!} \quad (\text{V.5})$$

$$\int_0^\infty e^{-x} S_m^p(x) S_m^q(x) x^m dx = \frac{\Gamma(m+p+1)}{p!} \delta(p, q) \quad (\text{V.6})$$

we then obtain

$$L = \frac{5}{2} n \frac{\bar{\rho}_s^0}{\beta} \frac{\partial}{\partial x_i} \operatorname{Log} T. \quad (\text{V.7})$$

The right hand side terms exhibit several different angular behaviours (55). Let us for example give the details of the calculations of W_1 .

The calculations are simplified by the use of the integration variables Π et $\mathbf{g}$. The Jacobian of this transformation, given by :

$$\begin{aligned} \mathbf{p}_1 &= \Pi - \frac{\mathbf{g}_i}{2} & \mathbf{p}'_1 &= \Pi - \frac{\mathbf{g}_f}{2} \\ \mathbf{p}_2 &= \Pi + \frac{\mathbf{g}_i}{2} & \mathbf{p}'_2 &= \Pi + \frac{\mathbf{g}_f}{2} \end{aligned} \quad (\text{V.8})$$

is equal to one, so that W_1 can be written in the form :

$$n^2 \delta(i, j) W_1 = \int d^3\Pi d^3g_i \frac{2g_i}{m} \int d^2\hat{g}_f f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) [\sigma_k(\theta_f)] \varphi_0(p_1) p_{1x_i} \{ \delta \bar{\rho}_s(1') - \delta \bar{\rho}_s(1) \} \quad (\text{V.9})$$

where $\delta \bar{\rho}_s$ is given by (51a).

In the evaluation of equation (V.9) we shall first perform the integration over all scattering angles ($d^2\hat{g}_f \equiv \sin \theta_f d\theta_f d\varphi_f$) and then the integration over Π and $d^2\hat{g}_i$ ($\equiv \sin \theta_i d\theta_i d\varphi_i$). Equation (IV.9) can be written in the form :

$$n^2 \delta(i, j) W_1 = \int d^3\Pi d^3g \frac{g}{m} f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) \varphi_0(p_1) p_{1x_i} J(\Pi, g) \quad (\text{V.10})$$

where

$$J(\Pi, \mathbf{g}_i) = \int d^2 \hat{g}_f \sigma_k(\theta_f) \left[p'_{1x_j} \left(\frac{p_1'^2}{2mkT} - \frac{5}{2} \right) - p_{1x_j} \left(\frac{p_1^2}{2mkT} - \frac{5}{2} \right) \right]. \quad (\text{V.11})$$

This integration leads to two kinds of averages :

$$\int d\varphi_f(\mathbf{g}_f, \Pi) = \int d\varphi_f \cos \theta_f(\mathbf{g}_i, \Pi) \quad (\text{V.12})$$

$$\int d\varphi_f(\mathbf{g}_f, \Pi) (\mathbf{g}_f, \mathbf{k}) = \frac{1}{2} \int d\varphi_f \{ g_i^2(\Pi, \mathbf{k}) - 3(\mathbf{g}_i, \Pi) (\mathbf{g}_i, \mathbf{k}) \} (1 - \cos^2 \theta_f) + \int d\varphi_f(\mathbf{g}_i, \Pi) (\mathbf{g}_i, \mathbf{k}) \quad (\text{V.13})$$

with the help of (V.8, 12, 13) $J(\Pi, \mathbf{g}_i)$ defined in (V.11) can be written in the form :

$$J(\Pi, \mathbf{g}_i) = \frac{\beta}{2m} \int d^2 \hat{g}_f \sigma_k(\theta_f) (1 - \cos \theta_f) \left\{ \Pi_{x_j}(\mathbf{g}_i, \Pi) + \frac{1}{2} g_{ix_j} \left(\Pi^2 + \frac{g_i^2}{4} - \frac{5m}{\beta} \right) \right\} + \\ + \frac{\beta}{2m} \int d^2 \hat{g}_f \sigma_k(\theta_f) (1 - \cos^2 \theta_f) \left\{ g_i^2 \frac{\Pi_{x_i}}{4} - 3 \left(\mathbf{g}_i \cdot \frac{\Pi}{4} \right) g_{ix_j} \right\}.$$

Using definition (57b), $J(\Pi, \mathbf{g}_i)$ becomes :

$$J(\Pi, \mathbf{g}_i) = \frac{\beta}{2m} \left[\Pi_{x_j}(\mathbf{g}_i, \Pi) + \frac{1}{2} g_{ix_j} \left(\Pi^2 + \frac{g_i^2}{4} - \frac{5m}{\beta} \right) \right] Q_{[\sigma]}^1(k) + \frac{\beta}{2m} \left[g_i^2 \frac{\Pi_{x_i}}{4} - 3 g_{ix_j} \left(\mathbf{g}_i \cdot \frac{\Pi}{4} \right) \right] Q_{[\sigma]}^2(k). \quad (\text{V.14})$$

We shall then perform the integrations over $\mathbf{g}_i$ and $\mathbf{P}$; in collecting all the terms from (V.10) and (V.14) we can immediately ignore all terms, odd in Π or $\mathbf{g}_i$, which will disappear on the average; so equation (V.10) can be put in the form :

$$n^2 \delta(i, j) W_1 = \frac{\beta^2}{4m^2} \int d^3 \Pi d^3 g \times \frac{g}{m} \times f_0(\mathbf{p}_1) f_0(\mathbf{p}_2) \times \\ \times \left\{ Q_{[\sigma]}^1(k) \left[\Pi_{x_i}(\mathbf{g}, \Pi) + \frac{1}{2} g_{xi} \left(\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right) \right] \left[\Pi_{x_j}(\mathbf{g}, \Pi) + \frac{1}{2} g_{xj} \left(\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right) \right] \right. \\ \left. + Q_{[\sigma]}^2(k) \left[g^2 \frac{\Pi_{x_i}}{4} - \frac{3}{4} g_{xi}(\mathbf{g}, \Pi) \right] \left[\frac{1}{2} g_{xj}(\mathbf{g}, \Pi) + \Pi_{x_j} \left(\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right) \right] \right\}. \quad (\text{V.15})$$

By rotational invariance this term is non zero only if $i = j$.

The angular integration over $d^2 \hat{g}$ then leads to the averages :

$$\int d^2 \hat{g}(\mathbf{g}, \mathbf{k}) (\mathbf{g}, \Pi) = \int d^2 g g^2 \frac{1}{3} (\Pi, \mathbf{k}) \\ \int d^2 \hat{g}(\mathbf{g}, \mathbf{k})^2 (\mathbf{g}, \Pi)^2 = \int d^2 g g^4 \left\{ \frac{\Pi^2}{15} + \frac{2(\Pi, \mathbf{k})^2}{15} \right\}$$

and equation (V.15) becomes

$$n^2 \delta(i, j) W_1 = - \delta(i, j) \frac{\beta^2}{4m^2} \int d^3 p d^3 g \times \frac{g}{m} f_0(p_1) f_0(p_2) \times \\ \times \left\{ Q_{[\sigma]}^1(k) \left[\frac{1}{3} g^2 \Pi_{x_i}^2 \left(2\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right) + \frac{g^2}{12} \left(\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right)^2 \right] + \frac{1}{8} Q_{[\sigma]}^2(k) \left[\frac{1}{15} g^4 \Pi_{x_i}^2 + \frac{g^4}{5} \Pi^2 \right] \right\}. \quad (\text{V.16})$$

The consideration of angular averages over Π leads then to

$$W_1 = - \frac{\beta^2}{4m^2} \int d^3 \Pi d^3 g \frac{g}{m} f_0(p_1) f_0(p_2) \times \\ \times \left\{ Q_{[\sigma]}^1(k) \left[\frac{1}{9} g^2 \Pi^2 \left(2\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right) + \frac{g^2}{3} \left(\Pi^2 + \frac{g^2}{4} - \frac{5m}{\beta} \right)^2 \right] + Q_{[\sigma]}^2(k) \left[\frac{1}{36} g^4 \Pi^2 \right] \right\}. \quad (\text{V.17})$$

The integration over P can then be done in a straightforward manner by the use of (V.5) or of the recursion formulas :

$$\int d^3p p^{2n} \exp - \frac{\beta p^2}{4m} = (2n+1)(2n-1) \dots 3 \left(\frac{2m}{\beta} \right)^n \left(\frac{4\pi m}{\beta} \right)^{3/2} \quad (\text{V.18})$$

$$n^2 W_1 = - \frac{\beta^2}{4m^2} \int d^3g \frac{g}{m} n^2 \left(\frac{\beta}{4\pi m} \right)^{3/2} \exp - \frac{\beta g^2}{4m} \times$$

$$\times \left\{ Q_{[\sigma]}^1(k) \left[\frac{1}{3} \left(\frac{g}{2} \right)^6 - \frac{5}{3} \frac{m}{\beta} \left(\frac{g}{2} \right)^4 + \frac{55}{12} \left(\frac{g}{2} \right)^2 \frac{m^2}{\beta^2} \right] + Q_{[\sigma]}^2(k) \left[\frac{2}{3} \frac{m}{\beta} \left(\frac{g}{2} \right)^4 \right] \right\}.$$

With the help of definition (57a) this formula can then be written in the form :

$$W_1 = - \frac{m}{\beta} \left[\frac{4}{3} \Omega_{[\sigma]}^{(2,2)} + \frac{55}{6} \Omega_{[\sigma]}^{(1,1)} - \frac{10}{3} \Omega_{[\sigma]}^{(1,2)} + \frac{2}{3} \Omega_{[\sigma]}^{(1,3)} \right].$$

The results for the other integrals are given in the text (equations (56)).

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