Why Delannoy numbers?

Cyril Banderier, Sylviane Schwer

WHY DELANNOY NUMBERS?

CYRIL BANDERIER AND SYLVIANE SCHWER

Abstract. This article is not a research paper, but a little note on the history of combinatorics: We present here a tentative short biography of Henri Delannoy, and a survey of his most notable works. This answers to the question raised in the title, as these works are related to lattice paths enumeration, to the so-called Delannoy numbers, and were the first general way to solve Ballot-like problems.

This version corresponds to an update (May 2002) of the abstract submitted (February 2002) by the first author to the 5th lattice path combinatorics and discrete distributions conference (Athens, June 5-7, 2002). The article will appear in the Journal of Statistical Planning and Inferences and this ArXiV version has some minor typo corrections.

1. Classical lattice paths

Before to tackle the question of Delannoy numbers and Delannoy lattice paths, note that the classical number sequences or lattice paths have the name of a mathematician: the Italian Leonardo Fibonacci (∼1170–∼1250), the French Blaise Pascal (1623–1662), the Swiss Jacob Bernoulli (1654–1705), the Scottish James Stirling (1692–1770), the Swiss Leonhard Euler (1707–1783), the Belgian Eugène Catalan (1814–1894), the German Ernst Schröder (1841–1902), the German Walther von Dyck (1856–1934), the Polish Jan Łukasiewicz (1878–1956), the American Eric Temple Bell (1883–1960), the American Theodore Motzkin (1908-1970), the Indian Tadepalli Venkata Narayana (1930–1987), . . . It is quite amusing that some of them are nowadays more famous in combinatorics for problems which can be explained in terms of lattice paths than in their original field (algebra or logic for Dyck, Schröder, and Łukasiewicz[1]).

Fibonacci numbers appear in his 1202 *Liber abaci* (also spelled *abbaci*) [80]. “Catalan numbers” can be found in various works, including [17, 79]. Catalan called these numbers “Segner numbers”; and the actual terminology is due to Netto who wrote the first classical introduction to combinatorics [68]. The name “Schröder numbers” honors the seminal paper [76] and can be found in Comtet’s “Analyse combinatoire” [21] and also in one of his articles published in 1970. The name “Motzkin numbers” can be found in [37] and is related to Motzkin’s article [66]. The name “Narayana numbers” was given by Kreweras by reference to the article [67] (these numbers were also independently studied by John P. Runyon, a colleague of Riordan. These are called Runyon numbers in Riordan’s book [73], p.17). The name “Dyck paths” comes from the more usual “Dyck words/Dyck Language” which have been widely used for more than fifty years. We strongly recommend the lecture of R. Stanley, which gives some comments about the surprisingly old origin of these names and problems (cf pp. 212–213 of [81]).

Date: November 5, 2004.

[1]See the MacTutor History of Mathematics `http://www-groups.dcs.st-and.ac.uk/~history/`

2. Delannoy numbers

Delannoy is another "famous" name which is associated to an integer sequence related to lattice paths enumeration. Delannoy's numbers indeed correspond to the sequence $(D_{n,k})_{n,k\in\mathbb{N}}$, the number of walks from $(0,0)$ to (n,k), with jumps $(0,1)$, $(1,1)$, or $(1,0)$.

1	19	181	1159	5641	22363	75517	224143	598417	**1462563**
1	17	145	833	3649	13073	40081	108545	**265729**	598417
1	15	113	575	2241	7183	19825	**48639**	108545	224143
1	13	85	377	1289	3653	**8989**	19825	40081	75517
1	11	61	231	681	**1683**	3653	7183	13073	22363
1	9	41	129	**321**	681	1289	2241	3649	5641
1	7	25	**63**	129	231	377	575	833	1159
1	5	**13**	25	41	61	85	113	145	181
1	**3**	5	7	9	11	13	15	17	19
1	1	1	1	1	1	1	1	1	1

In this array, the lower left entry is $D_{0,0} = 1$ and the upper right entry is $D_{10,10} = 8097453$. Entry with coordinates (n,k) gives the number of Delannoy walks from $(0,0)$ to (n,k). The three steps $(0,1)$, $(1,1)$, and $(1,0)$ being respectively encoded by x, y and xy, the generating function of Delannoy walks is

$$F(x,y,t) = \sum_{n\geq 0}(x+y+xy)^n t^n = \frac{1}{1-t(x+y+xy)}\,,$$

where t encodes the length (number of jumps) of the walk.

The *central* Delannoy numbers $D_{n,n}$ (`EIS 1850`[2]) are in bold in the above array. They have appeared for several problems: properties of lattice and posets, number of domino tilings of the Aztec diamond of order n augmented by an additional row of length $2n$ in the middle [75], alignments between DNA sequences [86]... The generating function of the central Delannoy numbers is

$$\begin{aligned} D(z) &:= \sum_{n\geq 0} D_{n,n} z^n = [x^0]\,\frac{1}{1-(zx+z/x+z)} = \frac{1}{\sqrt{1-6z+z^2}} \\ &= 1+3\,z+13\,z^2+63\,z^3+321\,z^4+1683\,z^5+8989\,z^6+48639\,z^7+O(z^8)\,. \end{aligned}$$

The notation $[x^n]F(x)$ stands for the coefficient of x^n in the Taylor expansion of $F(x)$ at $x=0$. The square-root expression is obtained by a resultant or a residue computation (this is classical for the diagonal of rational generating functions). This closed form for $D(z)$ gives, by singularity analysis (see the nice book [43]):

$$\begin{aligned} D_{n,n} &= \frac{(3+2\sqrt{2})^n}{\sqrt{\pi}\sqrt{3\sqrt{2}-4}}\left(\frac{n^{-1/2}}{2} - \frac{23\,n^{-3/2}}{32(8+3\sqrt{2})} + \frac{2401\,n^{-5/2}}{2048(113+72\sqrt{2})} + O(n^{-7/2})\right) \\ &\approx 5.82842709^n\left(.57268163\,n^{-1/2} - .06724283\,n^{-3/2} + .00625063\,n^{-5/2} + \dots\right). \end{aligned}$$

One has also $D_{n,k} = \sum_{i=0}^{n}\binom{n}{i}\binom{k}{i}2^i$. Quite often, people note that there is a link between Legendre polynomials and Delannoy numbers [46, 56, 64], and indeed

[2]This number refers to the wonderful On-Line Encyclopedia of Integer Sequences, see `http://www.research.att.com/~njas/sequences/`

$D_{n,n} = P_n(3)$, but this is not a very relevant link as there is no "natural" combinatorial correspondence between Legendre polynomials and these lattice paths. Comtet [21] showed that the coefficients of any algebraic generating function satisfy a linear recurrence (which allows to compute them in linear time). For $d_n := D_{n,n}$, it leads to $(n+2)d_{n+2} - (6n+9)d_{n+1} + (n+1)d_n = 0$.

For this kind of lattice paths with jumps $-1, 0, +1$, one has links with continuous fraction [42], with determinants [45], with context free grammars [55]... Numerous generalizations have been investigated: walks in the quarter plane [39], multi-dimensional lattice paths [6, 52, 82, 47, 48].

It is classical in probability theory (see [11] for some discussions with a combinatorial flavor) and more precisely in the theory of Brownian motion to consider the following constraints for lattice paths:

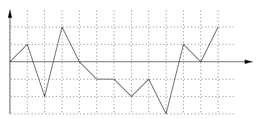
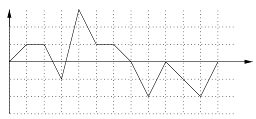
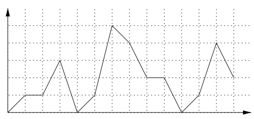
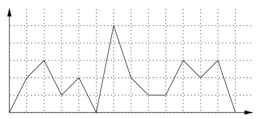
walks	ending anywhere	ending in 0
unconstrained (on $\mathbb{Z}$)	 walk ($\mathcal{W}$)	 bridge ($\mathcal{B}$)
constrained (on $\mathbb{N}$)	 meander ($\mathcal{M}$)	 excursion ($\mathcal{E}$)

FIGURE 1. The four types of paths: walks, bridges, meanders, and excursions.

For these four kinds of walks and for any finite set of jumps, there exists a nice formula for the corresponding generating function, which appears to be algebraic and from which one can derive the asymptotics and limit laws for several parameters of the lattice paths (see [9]).

Delannoy numbers $D_{n,n}$ correspond to bridges with a set of jumps $\{+1, -1, 0\}$ (where the 0 jump is in fact of length 2). Consider now the language $\mathcal{D}$ of central Delannoy paths, encoded via the letters a, b, c (for the jumps $+1, -1, +0$ resp.). Excursions with these jumps are called Schröder paths. We note $\mathcal{S}$ the language of Schröder paths (excursions) and $\bar{\mathcal{S}}$ the set of their mirror with respect to the x-axis. Then, the natural combinatorial decomposition $\mathcal{D} = (c^* a \mathcal{S} b + c^* b \bar{\mathcal{S}} a)^* c^*$ (which means that one sees a Delannoy path [bridge] as a sequence of Schröder paths [excursions] above or below the x-axis) leads to

$$D(z^2) = \frac{1}{1 - 2z^2 S(z)\frac{1}{1-z^2}} \frac{1}{1-z^2} \text{ where } S(z) = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2}$$

is the generating function of Schröder paths. This link between excursions and bridges is always easy to express when the set of jumps is symmetric or with jumps of amplitude at most 1, but there is also a relation between excursions and bridges in a more general case (see [9] for combinatorial and analytical proofs).

Despite all these appearances of Delannoy numbers (see [84] for a list of 29 objects counted by Delannoy numbers!), the classical books in combinatorics or computer

FIGURE 2. Henri Auguste Delannoy (1833-1915). Portrait provided by the Société des Sciences Naturelles et Archéologiques de la Creuse, where it is exhibited.

science which are usually accurate for "redde Caesari quae sunt Caesaris" (e.g., Comtet, Stanley, Knuth) are mute about this mysterious Delannoy.

3. Henri Auguste Delannoy (1833-1915)

Some people suggested that "Delannoy" was related either to the French mathematician Charles Delaunay (like in Delaunay triangulations) or to the Russian mathematician Boris Nikolaevich Delone, but this is not the case, as we shall see.

It is true that "Delannoy" sounds like (and actually is) a French family name `[d∧lanoa]` in approximative phonetic alphabet (∧ like in "duck" and `a` in "have"). There are in fact thousands of Delannoy, mostly in the North of France and in Belgium. This toponym means "de Lannoy", that is to say who originates from the town of Lannoy; "lannoy" meaning a place with a lot of alders; yet how to find "our" Delannoy among all these homonyms? The terminology "Delannoy numbers" became widely used as it can be found in Comtet's book in the footnote from exercise 20, p.93 [21]: "these numbers are often called Delannoy numbers" without any reference. In the English edition "Advanced Combinatorics" [22], the footnote becomes inserted in the text (p.81) but there is still no reference.

In fact, it appears to be a good idea to look in Lucas' books (see [24] for some biographical informations on Édouard Lucas [1842-1891]): Indeed, in the second edition of the first volume of the *Récréations mathématiques* [58], Lucas wrote in the preface "*J'adresse mes plus vifs remerciements à mon ami sincère et dévoué, Henry*[3] *Delannoy,...*" and at page 13 of this introduction we are told that Henri Delannoy

[3]There is no mistake, he was born Henry and asked to change it into Henri. We use in this article the first name Henri, as it was Delannoy's choice and as this was officially approved.

was intendant. In the second volume [63], the fourth recreation was dedicated to *Monsieur Henri Delannoy, ancien élève de l'École Polytechnique, sous-intendant militaire de Première classe*. After the death of his friend Lucas in October 1891, Henri Delannoy contributed with Lemoine and Laisant[4] to the publication of the third and fourth volumes[5] [60, 61] as well as to the book *L'arithmétique amusante* [62].

Like most of the French military intendants, Delannoy was a student from the École Polytechnique (which was the place where military officers received a scientific education). From a database[6] of the former students, one knows that Henri (Auguste) Delannoy is born in 1833 in Bourbonne-les-Bains (Haute-Marne, France). His father was Omère Benjamin Joseph Delannoy (countable officer) and his mother was Françoise Delage; they were living in the city of Bourges. In 1853, he passed the École Polytechnique entrance exam (with rank 62); then he graduated in 1854 with rank 91/106 and finished with rank 67/94 in 1855. It is quite funny that this database also contains, like for any other polytechnician, a physical description of Henri Delannoy: dark brown hair, average brow, average nose, blue eyes, small mouth, round chin, round face, height: 1,68m!

In the archive center of the French Army (in the Château de Vincennes), one can find his record under the number 61241. From this and [1, 7], we know that Delannoy was first in the Artillery corps as sous-lieutenant (with rank 12/37 from the application school), lieutenant (1857), took part in the Italy campaign (27 May-18 August 1859) and in the Solférino battle (24 June 1859). When he came back, he married his dulcinea Olympe-Marguerite Guillon on the 10 November 1859. They had 2 daughters and one boy. Delannoy was promoted captain in 1863. He then became a supplier-administrator: Intendant-Adjoint in 1865, sous-intendant of third class in 1867, of second class in 1872, of first class in 1882 (he was now a widower). He spent three years in Africa (6 Oct. 1866 - 25 Oct. 1869). He was the governor of the military Hospital of Sidi-bel-Abes, Algeria, during the terrible typhus epidemic (he belonged the Supply Corps and they were in charge the sanitary affairs). He translated for himself and perhaps also for his hierarchy several German books/notes about the Supply Corps. He took part in the 1870 war between France and Prussia. It is mentioned without explanation that he was in *Deutschland* on July 26, 1870 (that is, 4 days after the declaration of war...) and on March 7, 1871 (that is, 3 days before the treaty of London...). He was decorated with the *médaille d'Italie*, the *décoration sarde de la Valeur militaire*, the *Croix de la Légion d'Honneur* on July 18, 1868, and the *Rosette d'Officier de la Légion d'Honneur* in December 20, 1886. He could have reached the highest military ranks, but he wanted peace and decided to retire (January 9, 1889) in Guéret (the main city of the French department "la Creuse"), beginning a second life dedicated to science and more particularly to mathematics.

Looking for `Delannoy*` in the Zentralblatt volumes of 1860–1920[7], gives the following nine articles [26, 27, 28, 29, 31, 34, 33, 35, 36]. The references, problems, methods, and solutions used in these articles are similar to the ones often mentioned

[4] The rôle played by each one is explained in [4].

[5] Available at the web site of the French National Library `http://gallica.bnf.fr/`

[6] Available at `http://bibli.polytechnique.fr:4505/ALEPH0/`

[7] The "Jahrbuch über die Fortschritte der Mathematik" (annual review on the progresses of mathematics) is available at `http://www.emis.de/MATH/JFM/`

in Lucas' books. Most of Delannoy's articles are signed by Monsieur (H.) Delannoy, military intendant in the city of Orléans (and later, retired military intendant in the city of Guéret). Delannoy was a quite active member of the French Mathematical Society (SMF) in which he was admitted in 1882, introduced by Lucas and Laisant. He disappeared from the SMF's list in 1905, while he still contributed to l'Intermédiaire des Mathématiciens until 1910. This amateur mathematician then sank into oblivion and we found no obituary in the SMF bulletins at the occasion of his death on February 5, 1915.

In his death certificate, he was referred as president of the *Société des Sciences Naturelles et Archéologiques de la Creuse*. This Society is in fact still very dynamic[8]. It eventually appears that this Society, over which Delannoy presided from 1896 to 1915, has some archives, a part of which was given by Delannoy's family. They include some biographies written when Delannoy was still alive [10, 1], a list of his publications, and also an obituary and a short biography by members of the Society [7, 15]. We shall come back on Delannoy's works in this Society in Section 5 and we now consider Delannoy's contribution to mathematics.

4. Delannoy's mathematical work

Delannoy began his mathematical life reading the mathematical recreations that Lucas began to publish in 1879 in *La Revue Scientifique*. He was in contact with him in 1880 and began immediately to work with him, answering to letters of mathematicians transmitted by Lucas.

The first mention, in a mathematical work, to Delannoy is in an article by Lucas "Figurative arithmetics and permutations (1883)" [57], which deals with enumeration of configurations of 8 queens-like problems (the simplest one being: how to place n tokens on an $n \times n$ array, with no row or column with 2 tokens). Delannoy is there credited for having computed several sequences.

Some years later, in 1886, Delannoy made his first mathematical public appearance in the annual meeting of the "Association Française pour l'Avancements des Sciences". We now give the list of Delannoy's articles.

4.1. **Using a chessboard to solve arithmetical problems(1886)** [25]**.** In this article, Delannoy comes back on Lucas' article mentioned above and explains how he can use a "chessboard" (in modern words: an array) to get the formula $\frac{n-k+1}{n+1}\binom{n}{n+k}$ for the number of Dyck paths of length n ending at altitude k by using something which is not far from what one calls now the Desiré André reflection principle, which was in fact published one year later [3]. Note that Feller says (without references, see pages 72 and 369 of [41], 340 of [40]) that Lord Kelvin "method of images" for solving some partial differential equations is a kind analytic equivalent of the reflection principle in disguise. However, if one has a look on William Thomson's letter to Liouville [85], the link is rather mild and the cleaver combinatorial ideas of André & Delannoy cannot be attributed to Lord Kelvin.

Delannoy makes the link $T_{x,y} = \binom{x}{x+y} - \binom{x-1}{x+y} = \frac{y-x+1}{y+1}\binom{x}{x+y}$ between entries from the rectangular array (our walks on $\mathbb{Z}$, here given by the binomial coefficients) and entries from the triangular array (walks constrained to remain in the upper plane, our Dyck paths). The numbers $T_{x,y}$ are called (in English) "ballot numbers", but they are also called Delannoy–Segner numbers in Albert Sade's review (in the

[8] `htpp://perso.wanadoo.fr/jp-l/SSC23/`

Mathematical Reviews) of Touchard's article [87]. Kreweras [54] and Penaud [71] follow this terminology (quoting Riordan or Errera [38] but none of Delannoy's articles which all sank into oblivion). In conclusion, these "Delannoy–(Segner)" numbers $T_{x,y}$ are *not* the "famous" Delannoy numbers $D_{n,k}$ defined in Section 2.

4.2. **The length of the game (1888)** [26]**.** There are several contributions of Rouché and Bertrand in the Comptes Rendus de l'Académie des Sciences on the following problem that they call *the game*: "two players have n francs and play a game, at each round, the winner gets one franc from his opponent. One stops when one of the two players is ruined." When the game is fair, the probability to be ruined at the beginning of the round m is (with $q = \frac{m-n}{2}$):

$$\begin{aligned} & \frac{(-1)^{m-n}}{n} \sum_{k=1}^{n} (-1)^{k-1} \sin\big(\frac{(2k-1)\pi}{2n}\big) \cos^{m-1}\big(\frac{(2k-1)\pi}{2n}\big) \\ = \quad & \frac{n}{2^{m-1}} \sum_{k=0}^{q/n} (-1)^k \frac{2k+1}{\frac{m+n}{2}+kn} \binom{m-1}{q-kn}. \end{aligned}$$

Rouché proves the left hand part with some determinant computations and Delannoy uses lattice paths to get the right hand part (claiming justly that there was a mistake in Rouché's first formula).

One can see this problem as a Dyck walk in the strip $[-n, n]$, that is why the formula is similar to the formula 14 from [23] in their enumeration of planted plane trees of bounded height (Feller [40] gives also some comments on this).

4.3. **How to use a chessboard to solve various probability theory problems (1889)** [27]**.** This is a potpourri of seven ballot-like or ruin-like problems partially solved by de Moivre, Laplace, Huyghens, Ampère, Rouché, Bertrand, André, ... for which Delannoy presents his simple solutions, obtained by his lattice paths enumeration method. He calls the lattice "chessboard". The different constraints corresponds to different kind of chessboards: triangular for walks in the upper-plane, rectangular for unconstrained walks, pentagonal for walks bounded from above, hexagonal for walks in a strip (modern authors from statistical physics sometimes talk about walks with a wall or two walls [53]). Delannoy numbers (and the two corresponding binomial formulae) appear at page 51. Delannoy says that it corresponds to the directed walk of a queen (sic), and that this problem was suggested to him by Laisant. This (and the further advertisement by Lucas of Delannoy's works, see e.g. p. 174 of [59] on "Delannoy's arithmetical square", which is exactly the array given in Section 2) answers to the question raised in our title. The authors who later wrote about Delannoy numbers/arrays then gave references to Lucas [59], whereas Delannoy's articles sank into oblivion.

4.4. **Various problems about the game (1890)** [29]**.** Using an enumeration argument, simplifying the sum that he obtained and then using the Stirling formula, he gives the asymptotic result $\frac{1}{\sqrt{2\pi}}\sqrt{2n}$ as the difference between the number of won and lost games, after $2n$ games. He also answers to other problems, e.g. what is the probability to have a group of $2, 3, \ldots, 8$ cards of the same color in a packet of 32 cards.

4.5. **Formulae related the binomial coefficients (1890)** [28]**.** He gives several binomial formulae, such as $\sum_{k=0}^{p}(p-2k)^2\binom{p}{k} = p2^p$.

4.6. **On the geometrical trees and their use in the theory of chemical compounds. (1894)** [30, 31]**.** A chemist asked for some explanations of Cayley's results, mentioned in a German review. Delannoy translated this review and corrected a computational mistake, giving his own method, without knowing [18, 19, 20]. This corresponds to the sequences `EIS 22` (centered hydrocarbons with n atoms) and `EIS 200` (bicentered hydrocarbons with n atoms). Application of combinatorics to enumeration of chemical configurations is a subject which will be later revisited by Pólya [72].

4.7. **How to use a chessboard to solve some probability theory problems (1895)** [34]**.** Delannoy makes a summary of 17 applications of his theory of triangular/square/pentagonal/hexagonal chessboard. The array of Delannoy numbers (see our Section 2) appears on the page 76 from this article.

4.8. **On a question of probabilities studied by d'Alembert (1895)** [33]**.** Delannoy corrects some mistakes in Montfort's solution to a problem raised by d'Alembert.

4.9. **A question of undetermined analysis (1897)** [35]**.** A review (by Professor Lampe) of this article can be found in the Jahrbuch über die Fortschritte der Mathematik. However, we were not able to get this article. There were in fact two journals whose name was "Journal de Mathématiques élémentaires" (one edited by Vuibert and the other edited by Bourget/Longchamps), neither of them seems to contain the quoted article.

4.10. **On the probability of simultaneous events (1898)** [36]**.** A priest wrote an article in which he was bravely contesting the "third Laplace principle" $P(A \cap B) = P(A)P(B)$ for two independent events, arguing with three examples. Delannoy shows that they present a misunderstanding of "independent events", which goes back to the original fuzzy definition by de Moivre.

4.11. **Contributions to "L'Intermédiaire des Mathématiciens"** [32]**.** This journal was created in 1894 by C.-A. Laisant and Émile Lemoine. It is quite similar to the actual sci.math newsgroups. This journal was indeed only made of problems/questions/solutions/answers.

During the quoted period, numerous famous mathematicians made some contributions to this journal: Appell, Borel, Brocard, Burali-Forti, Cantor, Catalan, Cayley, Cesàro, Chebyshev, Darboux, Dickson, Goursat, Hadamard, Hermite, Jumbert, Hurwitz, Jensen, Jordan, Kempe, Koenigs, Laisant, Landau, Laurent, Lemoine, Lerch, Lévy, Lindelöf, Lipschitz, Moore, Nobel, Picard, Rouché...

From 1894 until 1908 (date of his last mathematical contribution), Delannoy was an active collaborator: he raised or solved around 70 questions/problems. These are questions number 20, 29, 32, 51, 84, 95, 138, 139, 140, 141, 142, 155, 191, 192, 314, 330, 360, 371, 407, 424, 425, 443, 444, 451, 453, 493, 494, 514, 601, 602, 603, 664, 668, 749, 1090, 1304, 1360, 1459, 1471, 1479, 1551, 1552, 1578, 1659, 1723, 1869, 1875, 1894, 1922, 1925, 1926, 1938, 1939, 2074, 2076, 2077, 2091, 2195, 2212, 2216, 2251, 2305, 2325, 2452, 2455, 2583, 2638, 2648, 2868, 2873, 3326.

These contributions can be classified in three sets: the problems and solutions related to combinatorics (enumeration and applications to probabilistic problems), problems and solutions related to elementary number theory (representations of integers as sum of some powers, Fermat-like problems), and questions/answers related to Lucas' books (so mainly recreative mathematics, but not so trivial problems as it includes, *e.g.*, the four color problem).

To these articles, perhaps one should add some *récréations* of [61] (compare the warning in its preface), and also some problems written by Lucas, but with Delannoy's solutions. The same holds for articles written by Lucas in *La Nature*.

Finally, there are some books [16, 22, 44, 58, 59, 61, 62, 63] (Lucas, Frolow, and Catalan intensively corresponded with Delannoy for their books) or articles [2, 12, 14, 5, 49, 50, 51, 54, 65, 69, 70, 71, 74, 75, 78, 83, 87, 88, 89] which mention either Delannoy numbers or some of Delannoy's results/methods.

5. Other Delannoy's works

Besides mathematics, Delannoy painted watercolors and, perhaps more importantly, studied history. Indeed, from 1897 to 1914, he published 29 accurate archaeological/historical articles in the Mémoires de la Société des Sciences Naturelles et archéologiques de la Creuse.

Let us give a taste of Delannoy's writer talent: here some titles of his articles: "On the signification of word *ieuru*", "One more word about *ieuru*", "A riot in Guéret in 1705", "Aubusson's tapestries", "A bigamist in Guéret", "Grapevines in the Creuse", a lot of studies "Criminal trials in the Marche. The case ...", several studies on abbeys and some "Critical list of the abbots from ...", and last but not least, "An impotence trial in the 18th century". When he died, at the age of 81, about a dozen other articles were still in progress.

Delannoy is surely one of the last "self-made" mathematicians who succeeded in getting a name in this field, rivaling professional mathematicians. What he discovered is nowadays well understood and can be classified as "basic enumerative combinatorics". However, despite the simplicity of his tools, it seems to us that Delannoy's work (and more generally, the underlying combinatorics) is a nice example of what could, but is actually not taught to young students (or even in high-schools), as an introduction to research in mathematics, also allowing the use of computers and computer algebra softwares. This kind of mathematics is only present at the mathematical Olympiads. This attractive bridge between enumeration, geometry, probability theory, analysis, ... deserves a better place.

It appears very clearly, thanks to the archives of the Society of Natural Sciences and Archaeology, that besides his own publications, Delannoy played a great rôle in checking proofs for numerous mathematicians and historians who wrote to have his contribution [4]. The archive from the Society and from Delannoy's family in Guéret reveals a true *honnête homme*, as defined in the seventeenth century.

Acknowledgements.

The first author's interest to Delannoy numbers comes from a talk that Marko Petkovšek gave in the Algorithms Seminar at INRIA in 1999 (a summary of this talk can be found in [8]). As an example, he was dealing with chess king moves (his general result about the nature of different multidimensional recurrences can be found in the article [13]). M. Petkovšek asked the first author what he knew about Delannoy, and C. Banderier then started to conduct an investigation...

The second author's interest to Delannoy numbers comes from her own works. As a researcher in Temporal Representation and Reasoning, she developed a model based on formal languages theory [77] instead of the logical or relational approaches. This framework allowed her to enumerate easily all possible temporal relations between n independent events-chronologies. Then she tried to know if some of these sequences were already known. In the $n = 2$ case, Sloane's On-Line EIS provided her the name of Delannoy. She asked everybody she met who is Delannoy? She already started her own investigation when Philippe Flajolet sent her to the first author.

Many people helped us in our dusty investigations, we want to thank for their technical and friendly help: Jean-Pierre Larduinat (webmaster), Régis Saint-James (secretary) and Étienne Taillemitte (president) from the Société des Sciences Naturelles et Archéologiques de la Creuse (Guéret)[9], Serge Paumier and family Desbaux, from Delannoy's family, Muriel Colombier from the Archives Départementales de la Creuse, Guy Avizou, first maire-adjoint and vice-président of the Conseil Régional de la Creuse, Silke Goebel from the Jahrbuch Project (Electronic Research Archive for Mathematics, Karlsruhe)[10], Anja Becker from the library of the Max-Planck-Institut für Informatik (Saarbrücken)[11], Geneviève Deblock from the Conservatoire numérique des Arts et Métiers (Paris)[12], Brigitte Briot from the library of INRIA (Rocquencourt)[13], Céline Menil from the library of the Maine university (Le Mans)[14], Solange Garnier, Francine Casas and Françoise Thierry from the library of the University of Paris-Nord[15], Nathalie Granottier from the Centre International de Rencontres Mathématiques (Marseilles)[16], and the people from the library of Jussieu (Paris)[17].

We also thank Jean-Michel Autebert, Philippe Flajolet, Nick Goldman, and Peter John for their comments. We appreciated the careful reading by the referees.

During the realisation of this work, Cyril Banderier was partially supported by the Future and Emerging Technologies programme of the EU under contract number IST-1999-14186 (ALCOM-FT), by the INRIA postdoctoral programme and by the Max-Planck-Institut.

---

[9] `http://perso.wanadoo.fr/jp-l/SSC23/`
[10] `http://www.emis.de/projects/JFM/`
[11] `http://www.mpi-sb.mpg.de/services/library/`
[12] `http://cnum.cnam.fr/`
[13] `http://www-rocq.inria.fr/doc/`
[14] `http://bu.univ-lemans.fr/`
[15] http://www.univ-paris13.fr/
[16] `http://www.cirm.univ-mrs.fr/SitBib/Bibli/debut.html`
[17] `http://bleuet.bius.jussieu.fr/`

Although very long, the following bibliography is not a complete list of the so vast literature on lattice paths, but it is mainly a tentative bibliography (by no way exhaustive) about enumeration of lattice paths "related" to Delannoy lattice paths and other Delannoy works.

References

[1] *Dictionnaire Universel du Génie Contemporain*, volume 13. 1893.

[2] A. Aeppli. A propos de l'interprétation géométrique du problème du scrutin. *L'Enseignement Mathématique*, 23:328–329, 1923.

[3] Désiré André. Note: Calcul des probabilités. Solution directe du problème résolu par M. Bertrand. *Comptes Rendus de l'Académie des Sciences. Paris*, 105:436–437, 1887.

[4] Jean-Marie Autebert, Anne-marie Décaillot, and Sylviane R. Schwer. Henri-Auguste Delannoy et la publication des œuvres posthumes d'édouard Lucas. *Gazette des mathématiciens*, 95:51–62, 2004.

[5] Jean-Michel Autebert, Matthieu Latapy, and Sylviane R. Schwer. Le treillis des chemins de Delannoy. *Discrete Mathematics*, 258(1-3):225–234, 2002.

[6] Jean-Michel Autebert and Sylviane R. Schwer. On generalized Delannoy paths. *SIAM Journal on Discrete Mathematics*, 16(2):208–223 (electronic), 2003.

[7] Fernand Autorde and Louis Lacrocq. Nécrologie de M. Henri-Auguste Delannoy. In *Mémoires de la S.S.N.A.C.*, volume 19-2, pages 552–577. Société des Sciences Naturelles et Archéologiques de la Creuse, Guéret, 1915.

[8] Cyril Banderier. Solving discrete initial- and boundary-value problems. In *Proceedings of the Algorithms Seminar Seminars, INRIA Research Report #4056*, 2000. Talk by Marko Petkovšek.

[9] Cyril Banderier and Philippe Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoretical Computer Science*, 281:37–80, 2002.

[10] Berthelot and co. *La grande Encyclopédie, inventaire raisonné des sciences, des lettres et des arts*, volume 13. H. Lamirault et Cie, 1893.

[11] Jean Bertoin and Jim Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bulletin des Sciences Mathématiques*, 118(2):147–166, 1994.

[12] Joseph Bonin, Louis Shapiro, and Rodica Simion. Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths. *Journal of Statistical Planning and Inference*, 34(1):35–55, 1993.

[13] Mireille Bousquet-Mélou and Marko Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Mathematics*, 225(1-3):51–75, 2000. Formal power series and algebraic combinatorics (Toronto, 1998).

[14] Francesco Brenti. Combinatorics and total positivity. *Journal of Combinatorial Theory. Series A*, 71(2):175–218, 1995.

[15] Amédée Carriat. *Dictionnaire bio-bibliographique des auteurs creusois*, volume fascicule 2: B-D. Société des sciences naturelles et archéologiques de la Creuse, 1965.

[16] Eugène Catalan. Nouvelles notes d'algèbre et d'analyse. *Belg. Mém. XLVIII. 98 S*, 1892.

[17] Eugène Catalan. Note sur une équation aux différences finies. *J. M. Pures Appl.*, 3:508–516, 1838.

[18] Arthur Cayley. On the theory of the analytical forms called tree. *Phil. Mag. XIII. p. 172-176*, 1857.

[19] Arthur Cayley. On the analytical forms called trees, with applications to the theory of chemical combinations. *Rep. Brit. Ass. , 257-305*, 1875.

[20] Arthur Cayley. On the analytical forms called trees. *Sylv., Am. J. IV. 266-268.*, 1881.

[21] Louis Comtet. *Analyse combinatoire. Tomes I, II.* Presses Universitaires de France, Paris, 1970.

[22] Louis Comtet. *Advanced combinatorics.* D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[23] N. G. de Bruijn, D. E. Knuth, and S. O. Rice. The average height of planted plane trees. In *Graph theory and computing*, pages 15–22. Academic Press, New York, 1972.

[24] Anne-Marie Décaillot. L'arithméticien Édouard Lucas (1842-1891) : théorie et instrumentation. *Revue d'histoire des mathématiques*, 4(2):191–236, 1998.

[25] Henri Delannoy. Emploi de l'échiquier pour la résolution de problèmes arithmétiques. *Assoc. Franç. Nancy XV*, 1886.

[26] Henri Delannoy. Sur la durée du jeu. *S. M. F. Bull. XVI. 124-128*, 1888.

[27] Henri Delannoy. Emploi de l'échiquier pour la résolution de divers problèmes de probabilité. *Assoc. Franç. Paris XVIII. 43-52*, 1889.

[28] Henri Delannoy. Formules relatives aux coefficients du binôme. *Assoc. Franç. Limoges XIX. 35-37*, 1890.

[29] Henri Delannoy. Problèmes divers concernant le jeu. *Assoc. Franç. Limoges XIX. 29-35*, 1890.

[30] Henri Delannoy. Sur le nombre d'isomères possibles dans une molécule carbonée. *Bulletin de la Société chimique de Paris, 3ème série, T.XI*, pages 239–248, 1892.

[31] Henri Delannoy. Sur les arbres géométriques et leur emploi dans la théorie des combinaisons chimiques. *Assoc. Franç. Caen XXIII. 102-116.*, 1894.

[32] Henri Delannoy. Diverse problems, solutions, and contributions. *Intermédiaire des Mathématiciens*, 1894–1908.

[33] Henri Delannoy. Emploi de l'échiquier pour la résolution de certains problèmes de probabilités. *Assoc. Franç. Bordeaux XXIV, 70-90*, 1895.

[34] Henri Delannoy. Sur une question de probabilités traitée par d'Alembert. *S. M. F. Bull. XXIII. 262-265*, 1895.

[35] Henri Delannoy. Une question d'analyse indéterminée. *Journal de Mathématiques élémentaires*, 1897.

[36] Henri Delannoy. Sur la probabilité des événements composés. *S. M. F. Bull. 26, 64-70*, 1898.

[37] Robert Donaghey and Louis W. Shapiro. Motzkin numbers. *J. Combinatorial Theory Ser. A*, 23(3):291–301, 1977.

[38] Alfred Errera. *Analysis situs. Un problème d'énumeration.* Académie Royale de Belgique, Brussels, 1931.

[39] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev. *Random walks in the quarter-plane.* Springer-Verlag, Berlin, 1999.

[40] William Feller. *An introduction to probability theory and its applications. Vol. I.* John Wiley & Sons Inc., New York, 1968. 3rd edition (first edition in 1950).

[41] William Feller. *An introduction to probability theory and its applications. Vol. II.* John Wiley & Sons Inc., New York, 1971. 2nd edition (first edition in 1966).

[42] Philippe Flajolet. Combinatorial aspects of continued fractions. *Discrete Mathematics*, 32(2):125–161, 1980.

[43] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics.* Book in preparation available on-line http://algo.inria.fr/flajolet/Publications/books.html, 2004.

[44] M. Frolow. *Les carrés magiques. Nouvelle étude.* Paris. Gauthier-Villars. VI u. 46 S. gr. 8°. VII. Taf, 1886.

[45] Ira Gessel and Gérard Viennot. Binomial determinants, paths, and hook length formulae. *Advances in Mathematics*, 58(3):300–321, 1985.

[46] I. J. Good. Legendre polynomials and trinomial random walks. *Proc. Cambridge Philos. Soc.*, 54:39–42, 1958.

[47] E. Goodman and T. V. Narayana. Lattice paths with diagonal steps. *Canad. Math. Bull.*, 12:847–855, 1969.

[48] B. R. Handa and S. G. Mohanty. Higher dimensional lattice paths with diagonal steps. *Discrete Mathematics*, 15(2):137–140, 1976.

[49] Peter E. John. Note on a modified pascal triangle connected with the dimer problem. *J.Mol.Struct.(Theochem)*, 277:329–332, 1992.

[50] Shashidhar Kaparthi and H. Raghav Rao. Higher-dimensional restricted lattice paths with diagonal steps. *Discrete Applied Mathematics*, 31(3):279–289, 1991.

[51] Clark Kimberling. Enumeration of paths, compositions of integers, and Fibonacci numbers. *The Fibonacci Quarterly*, 39(5):430–435, 2001.

[52] Christian Krattenthaler. Asymptotics for random walks in alcoves of affine Weyl groups. *ArXiV math.CO/0301203*, 2003.

[53] Christian Krattenthaler, Anthony J. Guttmann, and Xavier G. Viennot. Vicious walkers, friendly walkers and Young tableaux. II. With a wall. *Journal of Physics. A. Mathematical and General*, 33(48):8835–8866, 2000.

[54] Germain Kreweras. About Catalan-like lattice paths. *Bulletin of the Institute of Combinatorics and its Applications*, 4:63–64, 1992.

[55] Jacques Labelle and Yeong Nan Yeh. Generalized Dyck paths. *Discrete Mathematics*, 82(1):1–6, 1990.

[56] D. F. Lawden. On the solution of linear difference equations. *The Mathematical Gazette*, 36:193–196, 1952.

[57] Édouard Lucas. Sur l'arithmétique figurative. les permutations. *Assoc. Franç. Rouen XII*, 1883.

[58] Édouard Lucas. *Récréations mathématiques. Tome I.* Paris. Gauthier-Villars et Fils., 1891. Second edition (first edition in 1881).

[59] Édouard Lucas. *Théorie des nombres. Tome I. Le calcul des nombres entiers. Le calcul des nombres rationnels. La divisibilité arithmétique.* Paris. Gauthier-Villars et Fils., Edition Blanchard 1961, 1891.

[60] Édouard Lucas. *Récréations mathématiques. Tome III.* Paris. Gauthier-Villars et Fils, 1893.

[61] Édouard Lucas. *Récréations mathématiques. Tome IV.* Paris. Gauthier-Villars et Fils, 1894.

[62] Édouard Lucas. *L'arithmétique amusante. Introduction aux Récréations mathématiques. Amusements scientifiques pour l'enseignement et la pratique du calcul.* Paris. Gauthier-Villars et Fils., 1895.

[63] Édouard Lucas. *Récréations mathématiques. Tome II.* Paris. Gauthier-Villars et Fils., 1896. Second edition (first edition in 1882).

[64] L. Moser and W. Zayachkowski. Lattice paths with diagonal steps. *Scripta Math.*, 26:223–229, 1963.

[65] Leo Moser. King paths on a chessboard. *The Mathematical Gazette*, 39:54, 1955.

[66] Theodore Motzkin. Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products. *Bull. Amer. Math. Soc.*, 54:352–360, 1948.

[67] Venkata Tadepalli Narayana. Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités. *Comptes Rendus de l'Académie des Sciences. Paris*, 240:1188–1189, 1955.

[68] Eugen Netto. *Lehrbuch der Combinatorik.* Chelsea Publishing Company, New York, 1958. Reprint of the second edition. (first edition in 1901).

[69] Jerome L. Paul. Monochromatic lattice paths with diagonal steps in partitions $\mathbf{Z}^n$. In *Proceedings of the thirteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1982)*, volume 36, pages 137–142, 1982.

[70] Paul Peart. Hankel determinants via Stieltjes matrices. In *Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000)*, volume 144, pages 153–159, 2000.

[71] Jean-Guy Penaud. Une preuve bijective d'une formule de Touchard-Riordan. *Discrete Mathematics*, 139(1-3):347–360, 1995. Formal power series and algebraic combinatorics (Montreal, PQ, 1992).

[72] George Pólya. Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. *Acta Math.*, 68:145–254, 1937.

[73] John Riordan. *Combinatorial identities.* Robert E. Krieger Publishing Co., Huntington, N.Y., 1979. Reprint of the 1968 original.

[74] Émile Rouché. Observations en réponse à une Note de M. Delannoy. *S. M. F. Bull. XVI. 149-150*, 1888.

[75] Horst Sachs and Holger Zernitz. Remark on the dimer problem. *Discrete Applied Mathematics*, 51(1-2):171–179, 1994. 2nd Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1991).

[76] Ernst Schröder. Vier combinatorische Probleme. *Schlömilch Z. XV. 361-376.*, 1870.

[77] Sylviane R. Schwer. Dépendances temporelles : les mots pour le dire. Preprint, 1997.

[78] Sylviane R. Schwer. S-arrangements avec répétitions. *Comptes Rendus de l'Académie des Sciences. Paris*, 334(4):261–266, 2002.

[79] Andr. de Segner. Enumeratio modorum quibus figurae planae rectilineae per diagonales dividuntur in triangula. *Novi commentarii academiae scientiarum imperialis petropolitanae*, 7:203–210, 1759.

[80] Laurence E. Sigler. *Fibonacci's Liber Abaci.* Springer Verlag, 2002.

[81] Richard P. Stanley. *Enumerative combinatorics. Vol. 2.* Cambridge University Press, Cambridge, 1999.

[82] Robert Sulanke and Enrica Duchi. The 2^{n-1} factor for multi-dimensional lattice paths with diagonal steps. *Séminaire Lotharingien de Combinatoire*, 51, 2004.

[83] Robert A. Sulanke. Counting lattice paths by Narayana polynomials. *Electronic Journal of Combinatorics*, 7(1):Research Paper 40, 2000.

[84] Robert A. Sulanke. Objects counted by the central Delannoy numbers. *Journal of Integer Sequences*, 6(1):Article 03.1.5, 19 pp. (electronic), 2003.

[85] William Thomson. Extrait d'une lettre de M. William Thomson à M. Liouville. *Journal de Mathématiques de Liouville*, 16(2):208–223 (electronic), 2003.

[86] Angela Torres, Alberto Cabada, and Juan J. Nieto. An exact formula for the number of alignments between two DNA sequences. *DNA Sequence*, 14:227–430, 2003.

[87] Jacques Touchard. Sur un problème de configurations et sur les fractions continues. *Canadian J. Math.*, 4:2–25, 1952.

[88] N. Traverso. Su alcune tavole di addizione per diagonali di passo 1, dedotto dal quadrato aritmetico di Fermat, ed in particolare su quella dell' esagono aritmetico di Delannoy. *Periodico di Mat. 32 [(3) 14], 1-11* , 1917.

[89] Mladen Vassilev and Krassimir Atanassov. On Delanoy numbers. *Annuaire de l'Université de Sofia "St. Kliment Ohridski".*, 81(1):153–162, 1987 (1994).

E-mail address: `Cyril.Banderier@lipn.univ-paris13.fr`,
`http://www-lipn.univ-paris13.fr/~banderier`

E-mail address: `Sylviane.Schwer@lipn.univ-paris13.fr`,
`http://www-lipn.univ-paris13.fr/~schwer/`

LIPN- UMR 7030 Université Paris Nord.
99, avenue J.-B. Clément. 93430 Villetaneuse (France)