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Verma modules and preprojective algebras

Christof GEISS *, Bernard LECLERC † and Jan SCHRÖER ‡

Abstract

We give a geometric construction of the Verma modules of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ in terms of constructible functions on the varieties of nilpotent finite-dimensional modules of the corresponding preprojective algebra $\Lambda$.

1 Introduction

Let $\mathfrak{g}$ be the symmetric Kac-Moody Lie algebra associated to a finite unoriented graph $\Gamma$ without loop. Let $n_-$ denote a maximal nilpotent subalgebra of $\mathfrak{g}$. In [Lu1, §12], Lusztig has given a geometric construction of $U(n_-)$ in terms of certain Lagrangian varieties. These varieties can be interpreted as module varieties for the preprojective algebra $\Lambda$ attached to the graph $\Gamma$ by Gelfand and Ponomarev [GP]. In Lusztig’s construction, $U(n_-)$ gets identified with an algebra $(\mathcal{M}, *)$ of constructible functions on these varieties, where $*$ is a convolution product inspired by Ringel’s multiplication for Hall algebras.

Later, Nakajima gave a similar construction of the highest weight irreducible integrable $\mathfrak{g}$-modules $L(\lambda)$ in terms of some new Lagrangian varieties which differ from Lusztig’s ones by the introduction of some extra vector spaces $W_k$ for each vertex $k$ of $\Gamma$, and by considering only stable points instead of the whole variety [Na, §10].

The aim of this paper is to extend Lusztig’s original construction and to endow $\mathcal{M}$ with the structure of a Verma module $M(\lambda)$.

To do this we first give a variant of the geometrical construction of the integrable $\mathfrak{g}$-modules $L(\lambda)$, using functions on some natural open subvarieties of Lusztig’s varieties instead of functions on Nakajima’s varieties (Theorem 1). These varieties have a simple description in terms of the preprojective algebra $\Lambda$ and of certain injective $\Lambda$-modules $q_\lambda$.

Having realized the integrable modules $L(\lambda)$ as quotients of $\mathcal{M}$, it is possible, using the comultiplication of $U(n_-)$, to construct geometrically the raising operators $E_i^\lambda \in \text{End}(\mathcal{M})$ which make $\mathcal{M}$ into the Verma module $M(\lambda)$ (Theorem 2). Note that we manage in this way to realize Verma modules with arbitrary highest weight (not necessarily dominant).

Finally, we dualize this setting and give a geometric construction of the dual Verma module $M(\lambda)^*$ in terms of the delta functions $\delta_x \in \mathcal{M}^*$ attached to the finite-dimensional nilpotent $\Lambda$-modules $x$ (Theorem 3).

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2 Verma modules

2.1 Let \( g \) be the symmetric Kac-Moody Lie algebra associated with a finite unoriented graph \( \Gamma \) without loop. The set of vertices of the graph is denoted by \( I \). The (generalized) Cartan matrix of \( g \) is \( A = (a_{ij})_{i,j \in I} \), where \( a_{ii} = 2 \) and, for \( i \neq j \), \( -a_{ij} \) is the number of edges between \( i \) and \( j \).

2.2 Let \( g = n \oplus h \oplus n_- \) be a Cartan decomposition of \( g \), where \( h \) is a Cartan subalgebra and \((n, n_-)\) a pair of opposite maximal nilpotent subalgebras. Let \( b = n \oplus h \). The Chevalley generators of \( n \) (resp. \( n_- \)) are denoted by \( e_i \) \((i \in I)\) (resp. \( f_i \)) and we set \( h_i = [e_i, f_i] \).

2.3 Let \( \alpha_i \) denote the simple root of \( g \) associated with \( i \in I \). Let \((-; -)\) be a symmetric bilinear form on \( h^* \) such that \( (\alpha_i; \alpha_j) = a_{ij} \). The lattice of integral weights in \( h^* \) is denoted by \( P \), and the sublattice spanned by the simple roots is denoted by \( Q \). We put \( P_+ = \{ \lambda \in P \mid (\lambda; \alpha_i) \geq 0, \ (i \in I) \} \), \( Q_+ = Q \cap P_+ \).

2.4 Let \( \lambda \in P \) and let \( M(\lambda) \) be the Verma module with highest weight \( \lambda \). This is the induced \( g \)-module defined by \( M(\lambda) = U(g) \otimes_{U(b)} \mathbb{C} u_\lambda \), where \( u_\lambda \) is a basis of the one-dimensional representation of \( b \) given by \( h u_\lambda = \lambda(h) u_\lambda, \ n u_\lambda = 0, \ (h \in h, \ n \in n) \).

As a \( P \)-graded vector space \( M(\lambda) \cong U(n_-) \) (up to a degree shift by \( \lambda \)). \( M(\lambda) \) has a unique simple quotient denoted by \( L(\lambda) \), which is integrable if and only if \( \lambda \in P_+ \). In this case, the kernel of the \( g \)-homomorphism \( M(\lambda) \to L(\lambda) \) is the \( g \)-module \( I(\lambda) \) generated by the vectors \( f_i^{(\lambda; \alpha_i)+1} \otimes u_\lambda, \ (i \in I) \).

3 Constructible functions

3.1 Let \( X \) be an algebraic variety over \( \mathbb{C} \) endowed with its Zariski topology. A map \( f \) from \( X \) to a vector space \( V \) is said to be constructible if its image \( f(X) \) is finite, and for each \( v \in f(X) \) the preimage \( f^{-1}(v) \) is a constructible subset of \( X \).

3.2 By \( \chi(A) \) we denote the Euler characteristic of a constructible subset \( A \) of \( X \). For a constructible map \( f : X \to V \) one defines
\[
\int_{x \in X} f(x) = \sum_{v \in V} \chi(f^{-1}(v)) v \in V.
\]

More generally, for a constructible subset \( A \) of \( X \) we write
\[
\int_{x \in A} f(x) = \sum_{v \in V} \chi(f^{-1}(v) \cap A) v.
\]
4 Preprojective algebras

4.1 Let $\Lambda$ be the preprojective algebra associated to the graph $\Gamma$ (see for example [Ri, GLS]). This is an associative $\mathbb{C}$-algebra, which is finite-dimensional if and only if $\Gamma$ is a graph of type $A, D, E$. Let $s_i$ denote the simple one-dimensional $\Lambda$-module associated with $i \in I$, and let $p_i$ be its projective cover and $q_i$ its injective hull. Again, $p_i$ and $q_i$ are finite-dimensional if and only if $\Gamma$ is a graph of type $A, D, E$.

4.2 A finite-dimensional $\Lambda$-module $x$ is nilpotent if and only if it has a composition series with all factors of the form $s_i$ ($i \in I$). We will identify the dimension vector of $x$ with an element $\beta \in Q^+$ by setting $\text{dim}(s_i) = \alpha_i$.

4.3 Let $q$ be an injective $\Lambda$-module of the form

$$q = \bigoplus_{i \in I} q_i^{a_i}$$

for some nonnegative integers $a_i$ ($i \in I$).

Lemma 1 Let $x$ be a finite-dimensional $\Lambda$-module isomorphic to a submodule of $q$. If $f_1 : x \to q$ and $f_2 : x \to q$ are two monomorphisms, then there exists an automorphism $g : q \to q$ such that $f_2 = gf_1$.

Proof — Indeed, $q$ is the injective hull of its socle $b = \bigoplus_{i \in I} s_i^{b_i}$. Let $c_j$ ($j = 1, 2$) be a complement of $f_j(\text{socle}(x))$ in $b$. Then $c_1 \cong c_2$ and the maps

$$h_j := f_j \oplus \text{id} : x \oplus c_j \to q, \quad (j = 1, 2)$$

are injective hulls. The result then follows from the unicity of the injective hull. \qed

Hence, up to isomorphism, there is a unique way to embed $x$ into $q$.

4.4 Let $\mathcal{M}$ be the algebra of constructible functions on the varieties of finite-dimensional nilpotent $\Lambda$-modules defined by Lusztig [Lu2] to give a geometric realization of $U(n_-)$. We recall its definition.

For $\beta = \sum_{i \in I} b_i \alpha_i \in Q^+$, let $\Lambda_\beta$ denote the variety of nilpotent $\Lambda$-modules with dimension vector $\beta$. Recall that $\Lambda_\beta$ is endowed with an action of the algebraic group $G_\beta = \prod_{i \in I} GL_{b_i}(\mathbb{C})$, so that two points of $\Lambda_\beta$ are isomorphic as $\Lambda$-modules if and only if they belong to the same $G_\beta$-orbit. Let $\mathcal{M}_\beta$ denote the vector space of constructible functions from $\Lambda_\beta$ to $\mathbb{C}$ which are constant on $G_\beta$-orbits. Let

$$\widetilde{\mathcal{M}} = \bigoplus_{\beta \in Q^+} \mathcal{M}_\beta.$$  

One defines a multiplication $*$ on $\widetilde{\mathcal{M}}$ as follows. For $f \in \mathcal{M}_\beta$, $g \in \mathcal{M}_\gamma$ and $x \in \Lambda_{\beta+\gamma}$, we have

$$(f * g)(x) = \int_U f(x')g(x'')$$

where the integral is over the variety of $x$-stable subspaces $U$ of $x$ of dimension $\gamma$, $x''$ is the $\Lambda$-submodule of $x$ obtained by restriction to $U$ and $x' = x/x''$. In the sequel in order to simplify
notation, we will not distinguish between the subspace $U$ and the submodule $x''$ of $x$ carried by $U$. Thus we shall rather write
\[(f \ast g)(x) = \int_{x''} f(x/x'')g(x''),\] where the integral is over the variety of submodules $x''$ of $x$ of dimension $\gamma$.

For $i \in I$, the variety $\Lambda_{\alpha_i}$ is reduced to a single point: the simple module $s_i$. Denote by $1_i$ the function mapping this point to 1. Let $\mathcal{G}(i, x)$ denote the variety of all submodules $y$ of $x$ such that $x/y \cong s_i$. Then by (2) we have
\[(1_i \ast g)(x) = \int_{y \in \mathcal{G}(i, x)} g(y).\] (3)

Let $M$ denote the subalgebra of $\widetilde{M}$ generated by the functions $1_i$ ($i \in I$). By Lusztig [Lu2], $(M, \ast)$ is isomorphic to $U(n_-)$ by mapping $1_i$ to the Chevalley generator $f_i$.

4.5 In the identification of $U(n_-)$ with $M$, formula (3) represents the left multiplication by $f_i$.

In order to endow $M$ with the structure of a Verma module we need to introduce the following important definition. For $\nu \in P_+$, let
\[q_\nu = \bigoplus_{i \in I} q_{i, \widehat{\nu} + \alpha_i}.\]

Lusztig has shown [Lu3, §2.1] that Nakajima’s Lagrangian varieties for the geometric realization of $L(\nu)$ are isomorphic to the Grassmann varieties of $\Lambda$-submodules of $q_\nu$ with a given dimension vector.

Let $x$ be a finite-dimensional nilpotent $\Lambda$-module isomorphic to a submodule of the injective module $q_\nu$. Let us fix an embedding $F : x \rightarrow q_\nu$ and identify $x$ with a submodule of $q_\nu$ via $F$.

**Definition 1** For $i \in I$ let $\mathcal{G}(x, \nu, i)$ be the variety of submodules $y$ of $q_\nu$ containing $x$ and such that $y/x$ is isomorphic to $s_i$.

This is a projective variety which, by 4.3, depends only (up to isomorphism) on $i$, $\nu$ and the isoclass of $x$.

5 Geometric realization of integrable irreducible g-modules

5.1 For $\lambda \in P_+$ and $\beta \in Q_+$, let $\Lambda^{\lambda}_\beta$ denote the variety of nilpotent $\Lambda$-modules of dimension vector $\beta$ which are isomorphic to a submodule of $q_\lambda$. Equivalently $\Lambda^{\lambda}_\beta$ consists of the nilpotent modules of dimension vector $\beta$ whose socle contains $s_i$ with multiplicity at most $(\lambda; \alpha_i)$ ($i \in I$). This variety has been considered by Lusztig [Lu4, §1.5]. In particular it is known that $\Lambda^{\lambda}_\beta$ is an open subset of $\Lambda_\beta$, and that the number of its irreducible components is equal to the dimension of the $(\lambda - \beta)$-weight space of $L(\lambda)$.

5.2 Define $\widetilde{M}^{\lambda}_\beta$ to be the vector space of constructible functions on $\Lambda^{\lambda}_\beta$ which are constant on $G_\beta$-orbits. Let $M^{\lambda}_\beta$ denote the subspace of $\widetilde{M}^{\lambda}_\beta$ obtained by restricting elements of $M_\beta$ to $\Lambda^{\lambda}_\beta$. 


Put $\tilde{M}^\lambda = \bigoplus_\beta \tilde{M}_\beta^\lambda$ and $M^\lambda = \bigoplus_\beta M_\beta^\lambda$. For $i \in I$ define endomorphisms $E_i, F_i, H_i$ of $\tilde{M}^\lambda$ as follows:

\begin{equation}
(E_if)(x) = \int_{y \in G(x, \lambda, i)} f(y), \quad (f \in \tilde{M}_\beta^\lambda, x \in \Lambda^\lambda_{\beta-\alpha_i}),
\end{equation}

\begin{equation}
(F_if)(x) = \int_{y \in G(i, x)} f(y), \quad (f \in \tilde{M}_\beta^\lambda, x \in \Lambda^\lambda_{\beta+\alpha_i}),
\end{equation}

\begin{equation}
(H_if)(x) = (\lambda - \beta; \alpha_i) f(x), \quad (f \in \tilde{M}_\beta^\lambda, x \in \Lambda^\lambda_\beta).
\end{equation}

**Theorem 1** The endomorphisms $E_i, F_i, H_i$ of $\tilde{M}^\lambda$ leave stable the subspace $M_i^\lambda$. Denote again by $E_i, F_i, H_i$ the induced endomorphisms of $M_i^\lambda$. Then the assignments $e_i \mapsto E_i$, $f_i \mapsto F_i$, $h_i \mapsto H_i$, give a representation of $\mathfrak{g}$ on $M_i^\lambda$ isomorphic to the irreducible representation $L(\lambda)$.

5.3 The proof of Theorem 1 will involve a series of lemmas.

5.3.1 For $i = (i_1, \ldots, i_r) \in I^r$ and $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$, define the variety $G(x, \lambda, (i, a))$ of flags of $\Lambda$-modules

$$\mathfrak{f} = (x = y_0 \subset y_1 \subset \cdots \subset y_r \subset q_\lambda)$$

with $y_k/y_{k-1} \cong s_{i_k}^{\oplus a_k} (1 \leq k \leq r)$. As in Definition 1, this is a projective variety depending (up to isomorphism) only on $(i, a)$, $\lambda$ and the isoclass of $x$ and not on the choice of a specific embedding of $x$ into $q_\lambda$.

**Lemma 2** Let $f \in \tilde{M}_\beta^\lambda$ and $x \in \Lambda^\lambda_{\beta-\alpha_1-\cdots-\alpha_r \alpha_i}$. Put $E_i^{(a)} = (1/a!)(a! f_i)$. We have

$$\left(\prod_{j=r}^1 E_i^{(a_j)}\right) f(x) = \int_{\mathfrak{f} \in G(x, \lambda, (i, a))} f(y_r).$$

The proof is standard and will be omitted.

5.3.2 By [Lu1] 12.11 the endomorphisms $F_i$ satisfy the Serre relations

$$\sum_{p=0}^{1-a_{ij}} (-1)^p F_j^{(p)} F_i F_j^{(1-a_{ij}-p)} = 0$$

for every $i \neq j$. A similar argument shows that

**Lemma 3** The endomorphisms $E_i$ satisfy the Serre relations

$$\sum_{p=0}^{1-a_{ij}} (-1)^p E_j^{(p)} E_i E_j^{(1-a_{ij})} = 0$$

for every $i \neq j$.

**Proof** — Let $f \in \tilde{M}_\beta^\lambda$ and $x \in \Lambda^\lambda_{\beta-\alpha_i.(1-a_{ij}) \alpha_j}$. By Lemma 2

$$(E_j^{(p)} E_i E_j^{(1-a_{ij})}) f(x) = \int_{\mathfrak{f}} f(y_3)$$
Lemma 4

Let \( \mathcal{F} = (x \subset y_1 \subset y_2 \subset y_3 \subset q\lambda) \)

with \( y_1/x \cong s_j^{\oplus 1-a_{ij}} \cdot p \), \( y_2/y_1 \cong s_i \) and \( y_3/y_2 \cong s_j^{\oplus p} \). This integral can be rewritten as

\[
\int_{y_3} f(y_3) \chi(\mathcal{F}[y_3;p])
\]

where the integral is now over all submodules \( y_3 \) of \( q\lambda \) of dimension \( \beta \) containing \( x \) and \( \mathcal{F}[y_3;p] \) is the variety of flags \( \mathcal{F} \) as above with fixed last step \( y_3 \). Now, by moding out the submodule \( x \) at each step of the flag, we are reduced to the same situation as in [Lu1, 12.11], and the same argument allows to show that

\[
\sum_{p=0}^{1} \chi(\mathcal{F}[y_3;p]) = 0,
\]

which proves the Lemma. \( \square \)

5.3.3 Let \( x \in \Lambda^A_{\beta} \). Let \( \varepsilon_i(x) \) denote the multiplicity of \( s_i \) in the head of \( x \). Let \( \varphi_i(x) \) denote the multiplicity of \( s_i \) in the socle of \( q\lambda/x \).

Lemma 4 Let \( i, j \in I \) (not necessarily distinct). Let \( y \) be a submodule of \( q\lambda \) containing \( x \) and such that \( y/x \cong s_j \). Then

\[
\varphi_i(y) - \varepsilon_i(y) = \varphi_i(x) - \varepsilon_i(x) - a_{ij}.
\]

Proof — We have short exact sequences

\[
\begin{align*}
0 & \rightarrow x \rightarrow q\lambda \rightarrow q\lambda/x \rightarrow 0, \quad (7) \\
0 & \rightarrow y \rightarrow q\lambda \rightarrow q\lambda/y \rightarrow 0, \quad (8) \\
0 & \rightarrow x \rightarrow y \rightarrow s_j \rightarrow 0, \quad (9) \\
0 & \rightarrow s_j \rightarrow q\lambda/x \rightarrow q\lambda/y \rightarrow 0. \quad (10)
\end{align*}
\]

Clearly, \( \varepsilon_i(x) = |\text{Hom}_A(x, s_i)| \), the dimension of \( \text{Hom}_A(x, s_i) \). Similarly \( \varepsilon_i(y) = |\text{Hom}_A(y, s_i)| \), \( \varphi_i(x) = |\text{Hom}_A(s_i, q\lambda/x)| \), \( \varphi_i(y) = |\text{Hom}_A(s_i, q\lambda/y)| \). Hence we have to show that

\[
|\text{Hom}_A(x, s_i)| - |\text{Hom}_A(y, s_i)| = |\text{Hom}_A(s_i, q\lambda/x)| - |\text{Hom}_A(s_i, q\lambda/y)| - a_{ij}. \quad (11)
\]

In our proof, we will use a property of preprojective algebras proved in [CB §1], namely, for any finite-dimensional \( \Lambda \)-modules \( m \) and \( n \) there holds

\[
|\text{Ext}^1_\Lambda(m, n)| = |\text{Ext}^1_\Lambda(n, m)|. \quad (12)
\]

(a) If \( i = j \) then \( a_{ij} = 2 \), \( |\text{Hom}_A(s_j, s_i)| = 1 \) and \( |\text{Ext}^1_\Lambda(s_j, s_i)| = 0 \) since \( \Gamma \) has no loops. Applying \( \text{Hom}_A(-, s_i) \) to (3) we get the exact sequence

\[
0 \rightarrow \text{Hom}_A(s_j, s_i) \rightarrow \text{Hom}_A(y, s_i) \rightarrow \text{Hom}_A(x, s_i) \rightarrow 0,
\]

hence

\[
|\text{Hom}_A(x, s_i)| - |\text{Hom}_A(y, s_i)| = -1.
\]
Similarly applying $\text{Hom}_A(s_i, -)$ to (10) we get an exact sequence

$$0 \to \text{Hom}_A(s_i, s_j) \to \text{Hom}_A(s_i, q_\lambda/x) \to \text{Hom}_A(s_i, q_\lambda/y) \to 0,$$

hence

$$|\text{Hom}_A(s_i, q_\lambda/x)| - |\text{Hom}_A(s_i, q_\lambda/y)| = 1,$$

and (11) follows.

(b) If $i \neq j$, we have $|\text{Hom}_A(s_i, s_j)| = 0$ and $|\text{Ext}_A^1(s_i, s_j)| = |\text{Ext}_A^1(s_j, s_i)| = -a_{ij}$. Applying $\text{Hom}_A(s_i, -)$ to (8) we get an exact sequence

$$0 \to \text{Hom}_A(s_i, x) \to \text{Hom}_A(s_i, y) \to 0,$$

hence

$$|\text{Hom}_A(s_i, x)| - |\text{Hom}_A(s_i, y)| = 0. \quad (13)$$

Moreover, by [Bo, §1.1], $|\text{Ext}_A^2(s_i, s_j)| = 0$ because there are no relations from $i$ to $j$ in the defining relations of $\Lambda$. (Note that the proof of this result in [Bo] only requires that $I \subseteq J^2$ (here we use the notation of [Bo]). One does not need the additional assumption $J^n \subseteq I$ for some $n$. Compare also the discussion in [BK].)

Since $q_\lambda$ is injective $|\text{Ext}_A^1(s_i, q_\lambda)| = 0$, thus applying $\text{Hom}_A(s_i, -)$ to (7) we get an exact sequence

$$0 \to \text{Hom}_A(s_i, x) \to \text{Hom}_A(s_i, q_\lambda) \to \text{Hom}_A(s_i, q_\lambda/x) \to \text{Ext}_A^1(s_i, x) \to 0,$$

hence

$$|\text{Hom}_A(s_i, x)| - |\text{Hom}_A(s_i, q_\lambda)| + |\text{Hom}_A(s_i, q_\lambda/x)| - |\text{Ext}_A^1(s_i, x)| = 0. \quad (14)$$

Similarly, applying $\text{Hom}_A(s_i, -)$ to (8) we get

$$|\text{Hom}_A(s_i, y)| - |\text{Hom}_A(s_i, q_\lambda)| + |\text{Hom}_A(s_i, q_\lambda/y)| - |\text{Ext}_A^1(s_i, y)| = 0. \quad (15)$$

Subtracting (14) from (15) and taking into account (12) and (13) we obtain

$$|\text{Ext}_A^1(x, s_i)| - |\text{Ext}_A^1(y, s_i)| = |\text{Hom}_A(s_i, q_\lambda/x)| - |\text{Hom}_A(s_i, q_\lambda/y)|. \quad (16)$$

Now applying $\text{Hom}_A(-, s_i)$ to (7) we get the long exact sequence

$$0 \to \text{Hom}_A(y, s_i) \to \text{Hom}_A(x, s_i) \to \text{Ext}_A^1(s_j, s_i) \to \text{Ext}_A^1(y, s_i) \to \text{Ext}_A^1(x, s_i) \to 0,$$

hence

$$|\text{Hom}_A(y, s_i)| - |\text{Hom}_A(x, s_i)| - a_{ij} - |\text{Ext}_A^1(y, s_i)| + |\text{Ext}_A^1(x, s_i)| = 0,$$

thus, taking into account (16), we have proved (11). \hfill \Box

**Lemma 5** With the same notation we have

$$\varphi_i(x) - \varepsilon_i(x) = (\lambda - \beta; \alpha_i).$$
The verification is left to the reader. Hence, using Lemmas 3 and 6, we have proved that the following relations for the endomorphisms $v$ of the variety of submodules $\Lambda_\beta$ and let $y \in \Lambda_{\beta+\alpha_j}$ be a submodule of $q\Lambda$ containing $x$. Using Lemma 6 we get that

$$\varphi_i(y) - \varepsilon_i(y) = (\lambda - \beta; \alpha_i) - a_{ij} = (\lambda - \beta - \alpha_j; \alpha_i),$$

as required, and the lemma follows. \hfill \Box

**Lemma 6** Let $f \in \tilde{\mathcal{M}}^\beta$. We have

$$(E_iF_j - F_jE_i)(f) = \delta_{ij}(\lambda - \beta; \alpha_i)f.$$  

**Proof —** Let $x \in \Lambda_{\beta - \alpha_i + \alpha_j}$. By definition of $E_i$ and $F_j$ we have

$$(E_iF_j f)(x) = \int_{p \in \mathfrak{P}} f(y)$$

where $\mathfrak{P}$ denotes the variety of pairs $p = (u, y)$ of submodules of $q\Lambda$ with $x \subset u, y \subset u, u/x \cong s_i$ and $u/y \cong s_j$. Similarly,

$$(F_jE_i f)(x) = \int_{q \in \Omega} f(y)$$

where $\Omega$ denotes the variety of pairs $q = (v, y)$ of submodules of $q\Lambda$ with $v \subset x, v \subset y, x/y \cong s_j$ and $y/v \cong s_i$.

Consider a submodule $y$ such that there exists in $\mathfrak{P}$ (resp. in $\Omega$) at least one pair of the form $(u, y)$ (resp. $(v, y)$). Clearly, the subspaces carrying the submodules $x$ and $y$ have the same dimension $d$ and their intersection has dimension at least $d - 1$. If this intersection has dimension exactly $d - 1$ then there is a unique pair $(u, y)$ (resp. $(v, y)$), namely $(x + y, y)$ (resp. $(x \cap y, y)$). This means that

$$\int_{p \in \mathfrak{P}; y \neq x} f(y) = \int_{q \in \Omega; y \neq x} f(y).$$

In particular, since when $i \neq j$ we cannot have $y = x$, it follows that

$$(E_iF_j - F_jE_i)(f) = 0, \quad (i \neq j).$$

On the other hand if $i = j$ we have

$$((E_iF_i - F_iE_i)(f))(x) = f(x)(\chi(\mathfrak{P}') - \chi(\Omega'))$$

where $\mathfrak{P}'$ is the variety of submodules $u$ of $q\Lambda$ containing $x$ such that $u/x \cong s_i$, and $\Omega'$ is the variety of submodules $v$ of $x$ such that $x/v \cong s_i$. Clearly we have $\chi(\Omega') = \varepsilon_i(x)$ and $\chi(\mathfrak{P}') = \varphi_i(x)$. The result then follows from Lemma 5. \hfill \Box

5.3.4 The following relations for the endomorphisms $E_i, F_i, H_i$ of $\tilde{\mathcal{M}}^\lambda$ are easily checked

$$[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij}E_j, \quad [H_i, F_j] = -a_{ij}F_j.$$

The verification is left to the reader. Hence, using Lemmas 5 and 6, we have proved that the assignments $e_i \mapsto E_i, f_i \mapsto F_i, h_i \mapsto H_i$, give a representation of $\mathfrak{g}$ on $\tilde{\mathcal{M}}^\lambda$. 

8
Lemma 7 The endomorphisms $E_i, F_i, H_i$ leave stable the subspace $M^\lambda$.

Proof — It is obvious for $H_i$, and it follows from the definition of $M^\lambda$ for $F_i$. It remains to prove that if $f \in M^\lambda_\beta$ then $E_i f \in M^\lambda_{\beta - \alpha_i}$. We shall use induction on the height of $\beta$. We can assume that $f$ is of the form $F_j g$ for some $g \in M^\lambda_{\beta - \alpha_j}$. By induction we can also assume that $E_i g \in M^\lambda_{\beta - \alpha_i - \alpha_j}$. We have
\[
E_i f = E_i F_j g = F_j E_i g + \delta_{ij} (\lambda - \beta + \alpha_j; \alpha_i) g,
\]
and the right-hand side clearly belongs to $M^\lambda_{\beta - \alpha_i}$.

Lemma 8 The representation of $\mathfrak{g}$ carried by $M^\lambda$ is isomorphic to $L(\lambda)$.

Proof — For all $f \in M_\beta$ and all $x \in \Lambda^\lambda_{\beta + (\alpha_i + 1) \alpha_i}$ we have $f \ast 1^{\alpha_i+1}_i(x) = 0$. Indeed, by definition of $\Lambda^\lambda$ the socle of $x$ contains $s_i$ with multiplicity at most $\alpha_i$. Therefore the left ideal of $M$ generated by the functions $1^{\alpha_i+1}_i$ is mapped to zero by the linear map $M \to M^\lambda$ sending a function $f$ on $\Lambda_\beta$ to its restriction to $\Lambda^\lambda_\beta$. It follows that for all $\beta$ the dimension of $M^\lambda_\beta$ is at most the dimension of the $(\lambda - \beta)$-weight space of $L(\lambda)$.

On the other hand, the function $1_0$ mapping the zero $\Lambda$-module to 1 is a highest weight vector of $M^\lambda$ of weight $\lambda$. Hence $1_0 \in M^\lambda$ generates a quotient of the Verma module $M(\lambda)$, and since $L(\lambda)$ is the smallest quotient of $M(\lambda)$ we must have $M^\lambda = L(\lambda)$.

This finishes the proof of Theorem 1.

6 Geometric realization of Verma modules

6.1 Let $\beta \in Q_+$ and $x \in \Lambda_{\beta - \alpha_i}$. Let $q = \bigoplus_{i \in I} q_i^{\alpha_i}$ be the injective hull of $x$. For every $\nu \in P_+$ such that $(\nu; \alpha_i) \geq \alpha_i$ the injective module $q_\nu$ contains a submodule isomorphic to $x$. Hence, for such a weight $\nu$ and for any $f \in M_\beta$, the integral
\[
\int_{y \in G(x, \nu, i)} f(y)
\]
is well-defined.

Proposition 1 Let $\lambda \in P$ and choose $\nu \in P_+$ such that $(\nu; \alpha_i) \geq \alpha_i$ for all $i \in I$. The number
\[
\int_{y \in G(x, \lambda, i)} f(y) - (\nu - \lambda; \alpha_i) f(x \oplus s_i)
\]
does not depend on the choice of $\nu$. Denote this number by $(E_i^\lambda f)(x)$. Then, the function
\[
E_i^\lambda f : x \mapsto (E_i^\lambda f)(x)
\]
belongs to $M_{\beta - \alpha_i}$.

Denote by $E_i^\lambda$ the endomorphism of $M$ mapping $f \in M_\beta$ to $E_i^\lambda f$. Notice that Formula (5), which is nothing but (6), also defines an endomorphism of $M$ independent of $\lambda$ which we again denote by $F_i$. Finally Formula (6) makes sense for any $\lambda$, not necessarily dominant, and any $f \in M_\beta$. This gives an endomorphism of $M$ that we shall denote by $H_i^\lambda$.

Theorem 2 The assignments $e_i \mapsto E_i^\lambda$, $f_i \mapsto F_i$, $h_i \mapsto H_i^\lambda$, give a representation of $\mathfrak{g}$ on $M$ isomorphic to the Verma module $M(\lambda)$.

The rest of this section is devoted to the proofs of Proposition 1 and Theorem 2.
6.2 Denote by $e_i^\lambda$ the endomorphism of the Verma module $M(\lambda)$ implementing the action of the Chevalley generator $e_i$. Let $E_i^\lambda$ denote the endomorphism of $U(n_\cdot)$ obtained by transporting $e_i^\lambda$ via the natural identification $M(\lambda) \cong U(n_\cdot)$. Let $\Delta$ be the comultiplication of $U(n_\cdot)$.

Lemma 9 For $\lambda, \mu \in P$ and $u \in U(n_\cdot)$ we have

$$\Delta(E_i^{\lambda+\mu}u) = (E_i^\lambda \otimes 1 + 1 \otimes E_i^\mu)\Delta u.$$

Proof — By linearity it is enough to prove this for $u$ of the form $u = f_1 \cdots f_r$. A simple calculation in $U(g)$ shows that

$$e_i f_1 \cdots f_r = f_1 \cdots f_r e_i + \sum_{k=1}^r \delta_{iik} f_1 \cdots f_{i-1} f_{i+k-1} h_i f_{i+k+1} \cdots f_r,$$

$$= f_1 \cdots f_r e_i + \sum_{k=1}^r \delta_{iik} \left( f_1 \cdots f_{i-1} f_{i+k} f_{i+k+1} \cdots f_r - \left( \sum_{s=k+1}^r a_{iis} \right) f_1 \cdots f_{i-1} f_{i+k+1} \cdots f_r \right).$$

It follows that, for $\nu \in P$,

$$E_i^\nu(f_1 \cdots f_r) = \sum_{k=1}^r \delta_{iik} \left( \nu; \alpha_i - \sum_{s=k+1}^r a_{iis} \right) f_1 \cdots f_{i-1} f_{i+k-1} \cdots f_r.$$

Now, using that $\Delta$ is the algebra homomorphism defined by $\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i$, one can finish the proof of the lemma. Details are omitted.

6.3 We endow $U(n_\cdot)$ with the $Q_+$-grading given by $\deg(f_i) = \alpha_i$. Let $u$ be a homogeneous element of $U(n_\cdot)$. Write $\Delta u = u \otimes 1 + u^{(i)} \otimes f_i + A$, where $A$ is a sum of homogeneous terms of the form $u' \otimes u''$ with $\deg(u'') \neq \alpha_i$. This defines $u^{(i)}$ unambiguously.

Lemma 10 For $\lambda, \mu \in P$ we have

$$E_i^{\lambda+\mu}u = E_i^\lambda u + (\mu; \alpha_i) u^{(i)}.$$

Proof — We calculate in two ways the unique term of the form $E \otimes 1$ in $\Delta(E_i^{\lambda+\mu}u)$. On the one hand, we have obviously $E \otimes 1 = E_i^{\lambda+\mu}u \otimes 1$. On the other hand, using Lemma 9 we have

$$E \otimes 1 = E_i^\lambda u \otimes 1 + (1 \otimes E_i^\mu)(u^{(i)} \otimes f_i) = E_i^\lambda u \otimes 1 + (\mu; \alpha_i) u^{(i)} \otimes 1.$$

Therefore,

$$E = E_i^{\lambda+\mu}u = E_i^\lambda u + (\mu; \alpha_i) u^{(i)}.$$

\qed
6.4 Now let us return to the geometric realization $\mathcal{M}$ of $U(n_-)$. Let $E^\lambda_i$ denote the endomorphism of $\mathcal{M}$ obtained by transporting $e^\lambda_i$ via the identification $M(\lambda) \cong \mathcal{M}.$

**Lemma 11** Let $\lambda \in P_+, f \in \mathcal{M}_\beta$ and $x \in \Lambda^\lambda_{\beta-\alpha_i}$. Then

$$(E^\lambda_i f)(x) = \int_{y \in G(x,\lambda,i)} f(y).$$

**Proof** — Let $r^\lambda : \mathcal{M} \to \mathcal{M}^\lambda$ be the linear map sending $f \in \mathcal{M}_\beta$ to its restriction to $\Lambda^\lambda_{\beta}$. By Theorem 1, this is a homomorphism of $U(n_-)$-modules mapping the highest weight vector of $\mathcal{M} \cong M(\lambda)$ to the highest weight vector of $\mathcal{M}^\lambda \cong L(\lambda)$. It follows that $r^\lambda$ is in fact a homomorphism of $U(g)$-modules, hence the restriction of $E^\lambda_i f$ to $\Lambda^\lambda_{\beta-\alpha_i}$ is given by Formula (4) of Section 3.

Let again $\lambda \in P$ be arbitrary, and pick $f \in \mathcal{M}_\beta$. It follows from Lemma 10 that for any $\mu \in P$

$$E^\lambda_i f - (\mu; \alpha_i) f^{(i)} = E^\lambda_i f.$$ 

Let $x \in \Lambda_{\beta-\alpha_i}$. Choose $\nu = \lambda + \mu$ sufficiently dominant so that $x$ is isomorphic to a submodule of $q_\nu$. Then by Lemma 11, we have

$$(E^\nu_i f)(x) = \int_{y \in G(x,\nu,i)} f(y).$$

On the other hand, by the geometric description of $\Delta$ given in [GLS §6.1], if we write

$$\Delta f = f \otimes 1 + f^{(i)} \otimes 1_i + A$$

where $A$ is a sum of homogeneous terms of the form $f' \otimes f''$ with $\deg(f'') \neq \alpha_i$, we have that $f^{(i)}$ is the function on $\Lambda_{\beta-\alpha_i}$ given by $f^{(i)}(x) = f(x \mp s_i)$. Hence we obtain that for $x \in \Lambda_{\beta-\alpha_i}$

$$(E^\lambda_i f)(x) = \int_{y \in G(x,\nu,i)} f(y) - (\nu - \lambda; \alpha_i) f(x \mp s_i).$$

This proves both Proposition 1 and Theorem 3.

6.5 Let $\lambda \in P_+$. We note the following consequence of Lemma 11.

**Proposition 2** Let $\lambda \in P_+$. The linear map $r^\lambda : \mathcal{M} \to \mathcal{M}^\lambda$ sending $f \in \mathcal{M}_\beta$ to its restriction to $\Lambda^\lambda_{\beta}$ is the geometric realization of the homomorphism of $g$-modules $M(\lambda) \to L(\lambda).$
7.2 Let $\mathcal{M}^* = \bigoplus_{\beta \in \mathcal{Q}^+} \mathcal{M}_\beta^*$ denote the vector space graded dual of $\mathcal{M}$. For $x \in \Lambda_\beta$, we denote by $\delta_x$ the delta function given by

$$\delta_x(f) = f(x), \quad (f \in \mathcal{M}_\beta).$$

Note that the map $\delta : x \mapsto \delta_x$ is a constructible map from $\Lambda_\beta$ to $\mathcal{M}_\beta^*$. Indeed the preimage of $\delta_x$ is the intersection of the constructible subsets

$$\mathcal{M}_{i_1, \ldots, i_r} = \{ y \in \Lambda_\beta \mid (1_{i_1} \ast \cdots \ast 1_{i_r})(y) = (1_{i_1} \ast \cdots \ast 1_{i_r})(x) \}, \quad (\alpha_{i_1} + \cdots + \alpha_{i_r} = \beta).$$

7.3 We can now dualize the results of Sections 5 and 6 as follows. For such a map $\delta$.

Theorem 3

(i) The formulas above define endomorphisms $E_i^x, F_i^x, H_i^x$ of $\mathcal{M}^*$, and the assignments $e_i \mapsto E_i^x$, $f_i \mapsto F_i^x$, $h_i \mapsto H_i^x$, give a representation of $\mathfrak{g}$ on $\mathcal{M}^*$ isomorphic to the dual Verma module $\mathcal{M}(\lambda)^*$. 

(ii) If $\lambda \in P_+$, the subspace $\mathcal{M}_{\lambda}^*$ of $\mathcal{M}^*$ spanned by the delta functions $\delta_x$ of the finite-dimensional nilpotent submodules $x$ of $q_\lambda$ carries the irreducible submodule $L(\lambda)$. For such a module $x$, Formula (19) simplifies as follows

$$\langle F_i^\lambda \rangle(x) = \int_{y \in \mathcal{G}(x, \lambda, i)} \delta_y.$$

Example 1 Let $\mathfrak{g}$ be of type $A_2$. Take $\lambda = \varpi_1 + \varpi_2$, where $\varpi_i$ is the fundamental weight corresponding to $i \in I$. Thus $L(\lambda)$ is isomorphic to the 8-dimensional adjoint representation of $\mathfrak{g} = \mathfrak{sl}_3$.

A $\Lambda$-module $x$ consists of a pair of linear maps $x_{21} : V_1 \to V_2$ and $x_{12} : V_2 \to V_1$ such that $x_{12}x_{21} = x_{21}x_{12} = 0$. The injective $\Lambda$-module $q = q_\lambda$ has the following form:

$$q = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

This diagram means that $(u_1, v_1)$ is a basis of $V_1$, that $(u_2, v_2)$ is a basis of $V_2$, and that

$q_{21}(u_1) = u_2, \quad q_{21}(v_1) = 0, \quad q_{12}(v_2) = v_1, \quad q_{12}(u_2) = 0$.

Using the same type of notation, we can exhibit the following submodules of $q$:

$$x_1 = (v_1), \quad x_2 = (u_2), \quad x_3 = (v_1 \quad u_2), \quad x_4 = (u_1 \quad u_2), \quad x_5 = (v_1 \quad v_2).$$
\[ x_6 = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}, \quad x_7 = \begin{pmatrix} u_2 \\ v_1 & v_2 \end{pmatrix}. \]

This is not an exhaustive list. For example, \( x'_4 = \begin{pmatrix} (u_1 + v_1) & u_2 \\ v_1 & v_2 \end{pmatrix} \) is another submodule, isomorphic to \( x_4 \). Denoting by \( 0 \) the zero submodule, we see that \( \delta_0 \) is the highest weight vector of \( L(\lambda) \subset M(\lambda)^* \). Next, writing for simplicity \( \delta_i \) instead of \( \delta_{x_i} \) and \( F_i \) instead of \( F_i^\lambda \), Theorem 3(ii) gives the following formulas for the action of the \( F_i \)’s on \( L(\lambda) \).

\[
F_1 \delta_0 = \delta_1, \quad F_2 \delta_0 = \delta_2, \quad F_1 \delta_2 = \delta_3 + \delta_4, \quad F_2 \delta_1 = \delta_3 + \delta_5,
\]

\[
F_1 \delta_3 = F_1 \delta_4 = \delta_6, \quad F_2 \delta_3 = F_2 \delta_5 = \delta_7, \quad F_2 \delta_3 = F_1 \delta_6 = \delta_4, \quad F_1 \delta_q = F_2 \delta_q = 0.
\]

Now consider the \( \Lambda \)-module \( x = s_1 \oplus s_1 \). Since \( x \) is not isomorphic to a submodule of \( q_\lambda \), the vector \( \delta_x \) does not belong to \( L(\lambda) \). Let us calculate \( F_i \delta_x \) \( (i = 1, 2) \) by means of Formula (19). We can take \( \nu = 2\varpi_1 \). The injective \( \Lambda \)-module \( q_\nu \) has the following form:

\[
q_\nu = \begin{pmatrix} w_1 & & w_2 \\ v_1 & & v_2 \end{pmatrix}
\]

It is easy to see that the variety \( G(x, \nu, 2) \) is isomorphic to a projective line \( \mathbb{P}_1 \), and that all points on this line are isomorphic to

\[
y = \begin{pmatrix} w_1 \\ v_1 & v_2 \end{pmatrix}
\]

as \( \Lambda \)-modules. Hence,

\[
F_2 \delta_x = \chi(\mathbb{P}_1) \delta_y - (\nu - \lambda; \alpha_2) \delta_{x \oplus s_2} = 2 \delta_y + \delta_{s_1 \oplus s_1 \oplus s_2}.
\]

On the other hand, \( G(x, \nu, 1) = \emptyset \), so that

\[
F_1 \delta_x = -(\nu - \lambda; \alpha_1) \delta_{x \oplus s_1} = -\delta_{s_1 \oplus s_1 \oplus s_1}.
\]

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